ALMOST EVERYWHERE CONVERGENCE OF FEJÉR MEANS OF L^1 FUNCTIONS ON RARELY UNBOUNDED VILENKIN GROUPS

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Abstract. It is a highly celebrated problem in dyadic harmonic analysis the pointwise convergence of the Fejér (or $(C,1)$) means of functions on unbounded Vilenkin groups. There are several papers of the author of this paper concerning this. That is, we know the a.e. convergence $\sigma_n f \to f$ $(n \to \infty)$ for functions $f \in L^p$, where $p > 1$ (*Journal of Approx. Theory*, 101(1):1–36, 1999), and also the a.e. convergence $\sigma_{M_n} f \to f$ $(n \to \infty)$ for functions $f \in L^1$ (*Journal of Approx. Theory*, 124(1):25–43, 2003). The aim of this paper is to prove the a.e. relation $\lim_{n\to\infty} \sigma_n f = f$ for each integrable function f on any rarely unbounded Vilenkin group. The concept of the rarely unbounded Vilenkin group is discussed in the paper. Basically, it means that the generating sequence m may be an unbounded one, but its „big elements" are not „too dense".

One of the most celebrated problems in dyadic harmonic analysis is the pointwise convergence of the Fejér (or $(C,1)$) means of functions on unbounded Vilenkin groups.

Fine [1] proved every Walsh-Fourier series (in the Walsh case $m_j = 2$ for all $j \in \mathbb{N}$) is a.e. (C,α) summable for $\alpha > 0$. His argument is an adaptation of the older trigonometric analogue due to Marcinkiewicz [2]. Schipp [3] gave a simpler proof for the case $\alpha = 1$, i.e. $\sigma_n f \to f$ a.e. $(f \in L^1(G_m))$. He proved that σ^* is of weak type $(1,1)$. That σ^* is bounded from H^1 to L^1 was discovered by Fujii [4].

The theorem of Schipp are generalized to the p-series fields ($m_j = p$ for all $j \in \mathbb{N}$) by Taibleson [5] and later to bounded Vilenkin systems by Pál and Simon [6].

The methods known in the trigonometric or in the Walsh, bounded Vilenkin case are not powerful enough. One of the main problems is that the proofs on the bounded Vilenkin groups (or in the trigonometric case) heavily use the fact that the L^1 norm of the Fejér kernels are uniformly bounded. This is not the case if the group G_m is an unbounded one [7]. From this it follows that the original theorem of Fejér does not hold on unbounded Vilenkin groups. Namely, Price proved [7] that for an arbitrary sequence m $(\sup_n m_n = \infty)$ and $a \in G_m$ there exists a function f continuous on G_m and $\sigma_n f(a)$ does not converge to $f(a)$. Moreover, he proved [7] that if $\frac{\log m_n}{M_n} \to \infty$ (the definition of sequences (m_n) and (M_n) will be discussed later), then there exist a function f continuous on G_m whose Fourier series are not $(C,1)$ summable on a set $S \subset G_m$ which is non-denumerable. That is, only, a.e. convergence can be stated for unbounded Vilenkin groups. The almost everywhere convergence of the full partial sums for functions in $L^p, p > 1$, is known in the bounded case [8] but not in the unbounded case.

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In 1999 the author [9] proved that if $f \in L^p(G_m)$, where $p > 1$, then $\sigma_n f \to f$ almost everywhere. This was the very first "positive" result with respect to the a.e. convergence of the Fejér means of functions on unbounded Vilenkin groups. In 2003 the author of this paper gave a partial answer for the L^1 case [10]. Namely, let $f \in L^1(G_m)$. Then we have $\sigma_{M_n} f \to f$ almost everywhere. In this paper we introduce the notion of the rarely unbounded Vilenkin group. The aim of this paper is to prove the Fejér-Lebesgue theorem on rarely unbounded Vilenkin groups. That is, that the a.e. relation $\sigma_n f \to f$ holds for all integrable f.

This result is interesting also in the following point of view. The mean convergence of the full partial sums for functions in $L^p, p > 1$ in norm, that is, $\|S_n f - f\|_p \to 0$ is known even in the unbounded case. See [11]. It trivially follows the norm convergence $\|\sigma_n f - f\|_p \to 0$ for $L^p, p > 1$. It is also interesting that from the result of Price above it follows the existence of an integrable function f on any unbounded Vilenkin group such that $\|\sigma_n f - f\|_1 \not\to 0$. This is a sharp contrast between the bounded and the unbounded case. So, the main result of this paper shows that convergence in norm, and convergence almost everywhere are quite different. I would like to draw the attention of the reader to some results known in the two-dimensional unbounded case due to Wade and the author [12, 13, 14]. For more on the theory of Vilenkin groups see the book [15]. Finally, I would like to to draw the attention of the reader to the problem of almost everywhere convergence of Fejér means of functions on locally compact Vilenkin groups. For some details with respect to this issue see the paper [16].

First we give a brief introduction to the theory of Vilenkin systems. These orthonormal systems were introduced by N. Ja. Vilenkin in 1947 (see e.g. [17, 15]) as follows.

Let $m := (m_k, k \in \mathbb{N})$ $(\mathbb{N} := \{0, 1, \dots\})$ be a sequence of integers each of them not less than 2. Let Z_{m_k} denote the discrete cyclic group of order m_k. That is, Z_{m_k} can be represented by the set $\{0, 1, ..., m_k - 1\}$, with the group operation mod m_k addition. Since the groups are discrete, then every subset is open. The normalized Haar measure on Z_{m_k}, μ_k is defined by $\mu_k(\{j\}) := 1/m_k$ $(j \in \{0, 1, ..., m_k - 1\})$. Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

Then every $x \in G_m$ can be represented by a sequence $x = (x_i, i \in \mathbb{N})$, where $x_i \in Z_{m_i}$ $(i \in \mathbb{N})$. The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ), which is the normalized Haar measure, and the topology are the product measure and topology. Consequently, G_m is a compact Abelian group. If $\sup_{n \in \mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. If there exists a constant C and $L \in \mathbb{P}$ such that for all $i, j \in \mathbb{P}$ we have

$$\frac{\min(m_i, m_{i+j})}{(m_{i+1} \cdot \dots \cdot m_{i+j-1})^L} \le C, \tag{1}$$

(the empty product is defined to be 1, and the constant C may depend on the sequence m - of course), then we call the Vilenkin group G_m a rarely unbounded Vilenkin group. Every bounded Vilenkin group is a rarely unbounded Vilenkin group. Unfortunately, not all unbounded ones are rarely unbounded, since for instance the rarely unboundedness implies the inequality $\min(m_i, m_{i+1}) \le C$. So, e.g. if (m_n) tends to plus infinity, then G_m is not rarely unbounded. On the other hand, there are many unbounded Vilenkin groups, which are rarely unbounded ones.

Let $M_0 := 1, M_{n+1} := m_n M_n \, (n \in \mathbb{N})$ be the so-called generalized powers. The Vilenkin group is metrizable in the following way:

$$d(x,y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{M_{i+1}} \quad (x, y \in G_m).$$

The topology induced by this metric, the product topology, and the topology given below are the same. A base for the neighborhoods of G_m can be given by the intervals:

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \,\text{for}\, i < n\}$$

for $x \in G_m, n \in \mathbb{P} := \mathbb{N} \setminus \{0\}$. Let $0 = (0, i \in \mathbb{N}) \in G_m$ denote the nullelement of $G_m, I_n := I_n(0) \, (n \in \mathbb{N})$. Sometimes we also use the notation

$$I_n(x,k) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \,\text{for}\, i < n, \, x_n = k\}.$$

Let e_n denote the element of G_m whose nth coordinate is one, and the others are zeros $(n \in \mathbb{N})$. Furthermore, let $L^p(G_m) \, (1 \leq p \leq \infty)$ denote the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) on G_m, $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x) \, (x \in G_m)$, and E_n the conditional expectation operator with respect to $\mathcal{A}_n \, (n \in \mathbb{N})$ $(E_{-1} f := 0 \, (f \in L^1))$.

The concept of the maximal Hardy space ([18]) $H^1(G_m)$ is defined by the maximal function $f^* := \sup_n |E_n f| \; (f \in L^1(G_m))$, saying that f belongs to the Hardy space $H^1(G_m)$ if $f^* \in L^1(G_m)$. $H^1(G_m)$ is a Banach space with the norm

$$\|f\|_{H^1} := \|f^*\|_1.$$

Let f be an element of either $H^1(G_m)$ or $L^p(G_m)$ for some $1 \leq p \leq \infty$. We say that operator T is of type (p,p) if there exist an absolute constant $C > 0$ for which $\|Tf\|_p \leq C\|f\|_p$ for all $f \in L^p(G_m)$. T is of weak type $(1,1)$ if there exist an absolute constant $C > 0$ for which $\mu(Tf > \lambda) \leq C\|f\|_1/\lambda$ for all $\lambda > 0$ and $f \in L^1(G_m)$. It is known that the operator which maps function f to the maximal function f^* is of weak type $(1,1)$, and of type (p,p) for all $1 < p \leq \infty$ (see e.g. [19, 9]).

Each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0, 1, ..., m_i - 1\}, \; i \in \mathbb{N}),$$

where only a finite number of n_i 's differ from zero. The order of the positive integer n is defined as $|n| := A$, where $M_A \leq n < M_{A+1}$. The generalized Rademacher functions are defined as

$$r_n(x) := \exp(2\pi\imath \frac{x_n}{m_n}) \quad (x \in G_m, n \in \mathbb{N}, \imath := \sqrt{-1}).$$

It is known that $\sum_{i=0}^{m_n-1} r_n^i(x) = \begin{cases} 0, & \text{if } x_n \neq 0, \\ m_n, & \text{if } x_n = 0 \end{cases}$ $(x \in G_m, \; n \in \mathbb{N})$. The n^{th} Vilenkin function is

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \quad (n \in \mathbb{N}).$$

The system $\psi := (\psi_n : n \in \mathbb{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m, and all the characters of G_m are of this form. Define the m -adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j (\bmod m_j)) M_j \quad (k, n \in \mathbb{N}).$$

Then , $\psi_{k\oplus n} = \psi_k\psi_n,\ \psi_n(x+y) = \psi_n(x)\psi_n(y),\ \psi_n(-x) = \bar{\psi}_n(x),\ |\psi_n| = 1\ (k, n \in \mathbb{N}, x, y \in G_m)$.

Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, and the Fejér kernels with respect to the Vilenkin system ψ as follows

$$\hat{f}(n) := \int_{G_m} f\bar{\psi}_n d\mu,$$

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k,$$

$$D_n(y,x) = D_n(y-x) := \sum_{k=0}^{n-1} \psi_k(y)\bar{\psi}_k(x),$$

$$\sigma_n f := \frac{1}{n}\sum_{k=1}^{n-1} S_k f,$$

$$K_n(y,x) = K_n(y-x) := \frac{1}{n}\sum_{k=1}^{n-1} D_k(y-x),$$

$$(n \in \mathbb{P}, y, x \in G_m,\ \hat{f}(0) := \int_{G_m} f d\mu,\ S_0 f = D_0 = 0,\ f \in L^1(G_m)).$$

It is well-known that

$$S_n f(y) = \int_{G_m} f(x) D_n(y-x) d\mu(x),$$

$$\sigma_n f(y) = \int_{G_m} f(x) K_n(y-x) d\mu(x)$$

$$(n \in \mathbb{P},\ y \in G_m,\ f \in L^1(G_m)).$$

It is also well-known that

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n(0), \\ 0, & \text{if } x \notin I_n(0) \end{cases},$$

$$S_{M_n} f(x) = M_n \int_{I_n(x)} f d\mu = E_n f(x) \quad (f \in L^1(G_m), n \in \mathbb{N}).$$

Moreover, [15] for $n \in \mathbb{N}$

$$D_n = \psi_n \sum_{j=0}^{\infty} D_{M_j} \sum_{i=m_j-n_j}^{m_j-1} r_j^i.$$

That is, for $z \in I_t \setminus I_{t+1}\quad (t \in \mathbb{N})$

$$D_n(z) = \psi_n(z)\left(\sum_{j=0}^{t-1} n_j M_j + M_t \sum_{i=m_t-n_t}^{m_t-1} r_t^i(z)\right). \tag{2}$$

Define the maximal operator of the Fejér means of the integrable function f as $\sigma^* f := \sup_{n\in\mathbb{P}} |\sigma_n f|$.

The main result of this paper that we are to prove is

Theorem 1. *Let G_m be a rarely unbounded Vilenkin group. Then the operator σ^* is of weak type $(1,1)$.*

A straightforward consequence of Theorem 1 is the poof of the Fejér-Lebesgue theorem on rarely unbounded Vilenkin groups. That is,

Theorem 2. *Let G_m be a rarely unbounded Vilenkin group, and $f \in L^1(G_m)$. Then we have the a.e. relation $\sigma_n f \to f$.*

In order to prove Theorem 1 we need several lemmas and some results achieved by the author of this paper earlier ([9, 10]).

Besides, we need the so-called Calderon-Zygmund decomposition Lemma [20, 10, 21] on unbounded Vilenkin groups (for the proof see e.g.[21]) which reads as: Let $f \in L^1(G_m)$, and $\lambda > \|f\|_1 > 0$ arbitrary. Then the function f can be decomposed in the following form:

$$f = f_0 + \sum_{j=1}^{\infty} f_j, \quad \|f_0\|_\infty \le C\lambda, \quad \|f_0\|_1 \le C\|f\|_1,$$

$$\operatorname{supp} f_j \subset \bigcup_{l=\alpha_j}^{\beta_j} I_{k_j}(z^j, l) = J_j, \quad \int_{G_m} f_j d\,\mu = 0 \quad (j \in \mathbb{P}),$$

and for

$$F = \bigcup_{j \in \mathbb{P}} J_j, \quad \mu(F) \le C\frac{\|f\|_1}{\lambda}.$$

Moreover, the sets J_j are disjoint $(z^j \in G_m, k_j \in \mathbb{N}, j \in \mathbb{P})$.

Let $f \in L^1(G_m)$ such that

$$\operatorname{supp} f \subset \bigcup_{l=\alpha}^{\beta} I_k(u, l) = J, \quad \mu^{-1}(J)\int_J |f| d\mu \le \lambda.$$

Then we prove:

Lemma 3. *$E_{t+1}|f|(y) \le \lambda\mu(J)M_{t+1}1_{I_{t+1}(u)}(y)$ for any $t < k$ $(t, k \in \mathbb{N})$ and $y \in G_m$.*

Proof. If $y \notin I_{t+1}(u) \supset I_k(u) \supset J$, then $I_{t+1}(y)$ and $I_k(u)$ are disjoint intervals, and consequently so do $I_{t+1}(y)$ and J. That is, in this case we have $E_{t+1}|f|(y) = 0$.

Thus, $y \in I_{t+1}(u)$ can be supposed. Since $I_{t+1}(u) \cap J = J$, then we get

$$\begin{aligned}
&E_{t+1}|f|(y)\\
&= M_{t+1}\int_{I_{t+1}(y)} |f(x)| d\mu(x)\\
&= M_{t+1}\int_J |f(x)| d\mu(x)\\
&= M_{t+1}\sum_{l=\alpha}^{\beta}\int_{I_{k+1}(u_0,\ldots,u_{k-1},l)} |f(x)| d\mu(x)\\
&= M_{t+1}\frac{\beta-\alpha+1}{M_{k+1}}\frac{M_{k+1}}{\beta-\alpha+1}\int_J |f(x)| d\mu(x)\\
&\le M_{t+1}\mu(J)\lambda.
\end{aligned}$$

This completes of the proof of Lemma 3. □

Then consider the well-known Calderon-Zygmund decomposition lemma (see e.g. [10, 21]). We do not apply it at this point, but it will be used later. That is, let

$$f = f_0 + \sum_{i=1}^{\infty} f_i \quad \text{and} \quad \operatorname{supp} f_i \subset \bigcup_{l=\alpha_i}^{\beta_i} I_{k_i}(u^i, l) = J_i \, (i \in \mathbb{P})$$

be disjoint sets. Moreover, let $\mu^{-1}(J_i) \int_{J_i} |f_i| d\mu \leq \lambda$ $(i \in \mathbb{P})$. Then we have:

Lemma 4. *$E_{t+1}|\sum_{\{i\in\mathbb{P}\,:\,k_i>t\}} f_i|(y) \leq \lambda$ for a.e. $y \in G_m$ $(t \in \mathbb{N})$.*

Proof. Basically, the proof is nothing else then an application of Lemma 3. Consequently,

$$E_{t+1}|\sum_{\{i\in\mathbb{P}\,:\,k_i>t\}} f_i|(y) \leq \lambda M_{t+1} \sum_{\{i\in\mathbb{P}\,:\,k_i>t\}} \mu(J_i) 1_{I_{t+1}(u^i)}(y).$$

The addends in the sum on the right hand side can be different from zero only in the case of an i such that $I_{t+1}(u^i) = I_{t+1}(y)$. That is, since the sets J_i are disjoint and $J_i \subset I_{t+1}(u^i) = I_{t+1}(y)$, then this sum is bounded by

$$\lambda M_{t+1} \sum_{\{i\in\mathbb{P}\,:\,J_i\subset I_{t+1}(y)\}} \mu(J_i) \leq \lambda M_{t+1} \mu(I_{t+1}(y)) = \lambda.$$

This completes the proof of Lemma 4. □

Introduce the operator $F_t : L^1(G_m) \to L^1(G_m)$ $(n \in \mathbb{N})$ in the following way.

$$F_t f(y) := \frac{1}{m_t} \sum_{x_t=0}^{m_t-1} f(y + e_t(x_t - y_t)).$$

It is obvious that $F_t E_{t+1} = E_{t+1} F_t = E_t$. Applying the notations of Lemma 4 we have

Lemma 5. *$E_{t+1}|\sum_{\{i\in\mathbb{P}\,:\,k_i=t\}} F_t f_i|(y) \leq \lambda$ for a.e. $y \in G_m$ $(t \in \mathbb{N})$.*

Proof. We apply Lemma 3 and follow the proof of Lemma 4. That is,

$$\begin{aligned}
&E_{t+1}|\sum_{\{i\in\mathbb{P}\,:\,k_i=t\}} F_t f_i|(y) \\
&\leq \sum_{\{i\in\mathbb{P}\,:\,k_i=t\}} E_{t+1} F_t |f_i|(y) \\
&= \sum_{\{i\in\mathbb{P}\,:\,k_i=t\}} E_t |f_i|(y) \\
&\leq \sum_{\{i\in\mathbb{P}\,:\,k_i=t\}} \lambda \mu(J_i) M_t 1_{I_t(u^i)}(y) \\
&\leq \sum_{\{i\in\mathbb{P}\,:\,J_i\subset I_t(y)\}} \lambda \mu(J_i) M_t \\
&\leq \lambda \mu(I_t(y)) M_t = \lambda.
\end{aligned}$$

□

Define the operator $T_n^{s,t} : L^1(G_m) \to L^1(G_m)$, where $n, s, t \in \mathbb{N}$ and $A = |n| \geq s > t$

$$T_n^{s,t} f(y) := \frac{1}{m_t} \left| \sum_{x_t=0, x_t \neq y_t}^{m_t - 1} E_s(f \bar{\psi}_{n^{(s)}})(y + e_t(x_t - y_t)) \frac{1}{1 - r_t(y - x)} \right|.$$

Also set

$$T_{*,A}^{s,t} f := \sup \left\{ T_n^{s,t} f : |n| = A \right\}, \quad T_*^{s,t} f := \sup \left\{ T_{*,A}^{s,t} f : A \in \mathbb{P} \right\}.$$

In the paper [9, Lemma 4.1] Gát proved for every $1 < p \leq 2$ that

$$\|T_*^{s,t} f\|_p \leq C \|f\|_p, \tag{3}$$

where the constant C depends only on p. Next, we prove

Lemma 6. *Let $f \in L^1(G_m)$, $\lambda > 0$, $A, s, t \in \mathbb{N}$, $A > s > t$ so that $E_{t+1}|f| \leq C\lambda$ a.e. Let $0 < \alpha < 1$ be an arbitrary real. Then*

$$\mu \left\{ \left(\frac{M_{t+1} m_s}{M_A} \right)^\alpha T_{*,A}^{s,t} f > \lambda \right\} \leq C \frac{\|f\|_1}{\lambda},$$

where C depends on α (and the constant can be seen at (1)).

Proof. We have $T_{*,A}^{s,t} f = T_{*,A}^{s,t}(E_{A+1} f) \leq T_*^{s,t}(E_{A+1} f)$, then by the help of (3) we get

$$\begin{aligned} &\mu \left\{ \left(\frac{M_{t+1} m_s}{M_A} \right)^\alpha T_{*,A}^{s,t} f > \lambda \right\} \\ &\leq \frac{1}{\lambda^p} \left(\frac{M_{t+1} m_s}{M_A} \right)^{p\alpha} \|T_{*,A}^{s,t} f\|_p^p \\ &\leq \frac{C}{\lambda^p} \left(\frac{M_{t+1} m_s}{M_A} \right)^{p\alpha} \|E_{A+1} f\|_p^p. \end{aligned} \tag{4}$$

In order to go further into the investigation of (4) we give an upper bound for $\|E_{A+1} f\|_p^p$.

$$\begin{aligned} \|E_{A+1} f\|_p^p &= \frac{1}{m_0 \dots m_A} \sum_{x_0=0}^{m_0-1} \cdots \sum_{x_A=0}^{m_A-1} |E_{A+1} f(x_0, \dots x_A)|^p \\ &\leq \frac{1}{m_0 \dots m_A} \sum_{x_0=0}^{m_0-1} \cdots \sum_{x_t=0}^{m_t-1} \left(\sum_{x_{t+1}=0}^{m_{t+1}-1} \cdots \sum_{x_A=0}^{m_A-1} |E_{A+1} f(x_0, \dots x_A)| \right)^p \\ &= \frac{(m_{t+1} \dots m_A)^{p-1}}{m_0 \dots m_t} \sum_{x_0=0}^{m_0-1} \cdots \sum_{x_t=0}^{m_t-1} \left(\frac{1}{m_{t+1} \dots m_A} \sum_{x_{t+1}=0}^{m_{t+1}-1} \cdots \sum_{x_A=0}^{m_A-1} |E_{A+1} f(x_0, \dots x_A)| \right)^p \\ &= \frac{(m_{t+1} \dots m_A)^{p-1}}{m_0 \dots m_t} \sum_{x_0=0}^{m_0-1} \cdots \sum_{x_t=0}^{m_t-1} (E_{t+1}|E_{A+1} f|(x_0, \dots, x_t))^p \\ &\leq \frac{(m_{t+1} \dots m_A)^{p-1}}{m_0 \dots m_t} \sum_{x_0=0}^{m_0-1} \cdots \sum_{x_t=0}^{m_t-1} (E_{t+1}|f|(x_0, \dots, x_t))^p \\ &\leq C\lambda^{p-1} (m_{t+1} \dots m_A)^{p-1} \frac{1}{m_0 \dots m_t} \sum_{x_0=0}^{m_0-1} \cdots \sum_{x_t=0}^{m_t-1} E_{t+1}|f|(x_0, \dots, x_t) \\ &= C\lambda^{p-1} (m_{t+1} \dots m_A)^{p-1} \|f\|_1. \end{aligned} \tag{5}$$

By the help of (4) and (5) we have

$$\mu\left\{\left(\frac{M_{t+1}m_s}{M_A}\right)^{\alpha} T^{s,t}_{*,A}f > \lambda\right\} \leq \frac{C}{\lambda}\left(\frac{M_{t+1}m_s}{M_A}\right)^{p\alpha}(m_{t+1}\dots m_A)^{p-1}\|f\|_1. \tag{6}$$

By a completely different - and more simple - approach we also get another bound for the left hand side of (6).

$$T^{s,t}_{*,A}f(y) \leq \frac{1}{m_t}\sum_{x_t=0, x_t\neq y_t}^{m_t-1} E_s|f|(y+e_t(x_t-y_t))\frac{1}{|\sin(\pi(y_t-x_t)/m_t)|}.$$

This immediately gives

$$\|T^{s,t}_{*,A}f(y)\|_1 \leq C\log(m_t)\|f\|_1$$

and

$$\mu\left\{\left(\frac{M_{t+1}m_s}{M_A}\right)^{\alpha} T^{s,t}_{*,A}f > \lambda\right\} \leq \frac{C}{\lambda}\log(m_t)\left(\frac{M_{t+1}m_s}{M_A}\right)^{\alpha}\|f\|_1. \tag{7}$$

In other words, (6) and (7) give that if could prove

$$\min\left\{\log(m_t)\left(\frac{M_{t+1}m_s}{M_A}\right)^{\alpha}, \left(\frac{M_{t+1}m_s}{M_A}\right)^{p\alpha}(m_{t+1}\dots m_A)^{p-1}\right\} \leq C, \tag{8}$$

then the proof of Lemma 6 would be complete.

Let $\delta > 1, 2\alpha - 1$ be discussed later, and $1 < p \leq 2$ defined as $\frac{p}{p-1}\alpha - 1 = \delta$. It is obvious that such a p exists, since for $p = 2$ the left hand side would be $2\alpha - 1$, and tending $p \to 1$, the left hand side converges to plus infinity. In this case we have $p - 1 \leq \frac{\alpha}{\delta}$. Thus, we get

$$\log(m_t)\left(\frac{M_{t+1}m_s}{M_A}\right)^{\alpha} \leq C\left(\frac{m_t}{(m_{t+1}\dots m_{s-1}m_{s+1}\dots m_{A-1})^{\delta}}\right)^{\frac{\alpha}{\delta}}$$

and

$$\begin{aligned}
&\left(\frac{M_{t+1}m_s}{M_A}\right)^{p\alpha}(m_{t+1}\dots m_A)^{p-1}\\
&=\left(\frac{m_s m_A}{(m_{t+1}\dots m_{s-1}m_{s+1}\dots m_{A-1})^{\frac{p}{p-1}\alpha-1}}\right)^{p-1}\\
&=\left(\frac{m_s m_A}{(m_{t+1}\dots m_{s-1}m_{s+1}\dots m_{A-1})^{\delta}}\right)^{p-1}
\end{aligned}$$

We can suppose that $\frac{m_s m_A}{(m_{t+1}\dots m_{s-1}m_{s+1}\dots m_{A-1})^{\delta}} > 1$, otherwise inequality (8) is verified. In this case

$$\left(\frac{m_s m_A}{(m_{t+1}\dots m_{s-1}m_{s+1}\dots m_{A-1})^{\delta}}\right)^{p-1} \leq \left(\frac{m_s m_A}{(m_{t+1}\dots m_{s-1}m_{s+1}\dots m_{A-1})^{\delta}}\right)^{\frac{\alpha}{\delta}}.$$

This gives that in order to have (8) we have to prove

$$\frac{\min\{m_t, m_s m_A\}}{(m_{t+1}\dots m_{s-1}m_{s+1}\dots m_{A-1})^{\delta}} \leq C$$

for some δ.

Case $m_t \leq m_s$. This by $\frac{m_t}{(m_{t+1}\dots m_{s-1})^L} \leq C$ gives that choosing any $\delta \geq L$ we finished the proof of (8). That is, $m_t > m_s$ can be supposed.

Case $m_A \leq m_s$. This by $\frac{m_A}{(m_{s+1}\ldots m_{A-1})^L} \leq C$ and by $\frac{m_s}{(m_{t+1}\ldots m_{s-1})^L} \leq C$ gives that choosing any $\delta \geq L$ we finished the proof of (8).

Case $m_A, m_t > m_s$. Then by $\frac{m_s}{(m_{t+1}\ldots m_{s-1})^L} \leq C$ we get $m_s \leq C(m_{t+1}\ldots m_{s-1})^L$ and this provides

$$\begin{aligned}
&\frac{\min\{m_t, m_s m_A\}}{(m_{t+1}\ldots m_{s-1}m_{s+1}\ldots m_{A-1})^{L^2+2L}} \\
&\leq \frac{m_s \min\{m_t, m_A\}}{(m_{t+1}\ldots m_{s-1})^{L^2+L}(m_{t+1}\ldots m_{s-1}m_{s+1}\ldots m_{A-1})^L} \\
&\leq C\frac{\min\{m_t, m_A\}}{m_s^L(m_{t+1}\ldots m_{s-1}m_{s+1}\ldots m_{A-1})^L} \leq C.
\end{aligned}$$

This means that choosing any $\delta \geq L^2 + 2L$ we completed the proof of (8). That is, the proof of Lemma 6 is complete. □

Lemma 7. *Let $0 < \alpha < 1$, $A, s, t \in \mathbb{N}$ and $A > s > t$. Then the operator $\left(\frac{M_{t+1}m_s}{M_A}\right)^\alpha T^{s,t}_{*,A}$ is of weak type $(1,1)$.*

Proof. Let $f \in L^1(G_m)$, $\lambda > \|f\|_1$. Apply the Calderon-Zygmund decomposition lemma for function f.

$$\begin{aligned}
f &= f_0 + \sum_{i=1}^{\infty} f_i = f_0 + \sum_{\{i\in\mathbb{P}:k_i>t\}} f_i + \sum_{\{i\in\mathbb{P}:k_i=t\}} F_t f_i + \sum_{\{i\in\mathbb{P}:k_i<t\}} f_i + \sum_{\{i\in\mathbb{P}:k_i=t\}} (f_i - F_t f_i) \\
&=: f_0 + g_1 + g_2 + g_3 + g_4.
\end{aligned} \tag{9}$$

Since $\|f_0\|_\infty \leq C\lambda$, then by $E_{t+1}|f_0|(y) \leq \|f_0\|_\infty \leq C\lambda$ and by Lemma 6 we have

$$\mu\left\{\left(\frac{M_{t+1}m_s}{M_A}\right)^\alpha T^{s,t}_{*,A} f_0 > \lambda\right\} \leq C\frac{\|f_0\|_1}{\lambda} \leq C\frac{\|f\|_1}{\lambda}. \tag{10}$$

Lemma 4 gives that $E_{t+1}|g_1|(y) \leq \lambda$ a.e. Consequently, also by Lemma 6 we get the proof of inequality

$$\mu\left\{\left(\frac{M_{t+1}m_s}{M_A}\right)^\alpha T^{s,t}_{*,A} g_1 > \lambda\right\} \leq C\frac{\|g_1\|_1}{\lambda} \leq C\frac{\|f\|_1}{\lambda}. \tag{11}$$

In the very same way Lemma 5 and 6 give

$$\mu\left\{\left(\frac{M_{t+1}m_s}{M_A}\right)^\alpha T^{s,t}_{*,A} g_2 > \lambda\right\} \leq C\frac{\|g_2\|_1}{\lambda} \leq C\frac{\|f\|_1}{\lambda}. \tag{12}$$

As a consequence of (9), (10),(11) and (12) we have to investigate g_3 and g_4.

First, we discuss g_3. Let $j \in \mathbb{P}$ such that $k_j < t$. Let $y \notin F = \bigcup_{i\in\mathbb{P}} J_i$. Consequently, $y \notin J_j = \bigcup_{l=\alpha_j}^{\beta_j} I_{k_j}(w^j, l)$. Let $n \in \mathbb{P}$, $A = |n|$. Then we have

$$T^{s,t}_n f_j(y) = \frac{M_s}{m_t}\int_{\bigcup_{z=1}^{m_t-1} I_s(y+e_t z)} f_j(x)\bar{\psi}_{n^{(s)}}(x)\frac{1}{1 - r_t(y-x)}d\mu(x) = 0$$

because

$$\bigcup_{z=1}^{m_t-1} I_s(y + e_t z) \subset I_t(y) \subset I_{k_j+1}(y)$$

and

$$I_{k_j+1}(y) \cap \bigcup_{l=\alpha_j}^{\beta_j} I_{k_j}(u^j, l) = \emptyset$$

for $y \notin J_j$. In other words, we integrate f_j on the empty set. This follows $T_n^{s,t} g_3(y) = 0$. That is, $T_{*,A}^{s,t} g_3(y) = 0$ for $y \notin F$. Consequently,

$$\begin{aligned}
&\mu\left\{\left(\frac{M_{t+1}m_s}{M_A}\right)^\alpha T_{*,A}^{s,t} g_3 > \lambda\right\} \\
&\le \mu(F) + \mu\left\{y \in G_m \setminus F : \left(\frac{M_{t+1}m_s}{M_A}\right)^\alpha T_{*,A}^{s,t} g_3(y) > \lambda\right\} \qquad (13)\\
&\le C\frac{\|f\|_1}{\lambda}.
\end{aligned}$$

It is left to investigate g_4. Let $i \in \mathbb{P}$ such that $k_i = t$. Set $\gamma_i = \lfloor(\alpha_i + \beta_i)/2\rfloor$.

Define the distance of $j, k \in \{0, 1, \dots, m_t - 1\} = Z_{m_t}$ as

$$\rho(j,k) := \begin{cases} |j-k|, & \text{if } |j-k| \le \frac{m_t}{2}, \\ m_t - |j-k|, & \text{if } |j-k| > \frac{m_t}{2}. \end{cases}$$

In other words, Z_{m_t} is considered as a circle. Define the set $6J_i$ in the following way:
If $\beta_i - \alpha_i + 1 \ge m_t/6$, then $6[\alpha_i, \beta_i] := \{0, \dots, m_t - 1\}$,

$$6J_i := \bigcup_{j \in 6[\alpha_i,\beta_i]} I_t(u^i, j) = I_t(u^i).$$

On the other hand, if $\beta_i - \alpha_i + 1 < m_t/6$, then $6[\alpha_i, \beta_i] := \{j \in Z_{m_t} : \rho(j, \gamma_i) \le 3(\beta_i - \alpha_i + 1)\}$,

$$6J_i := \bigcup_{j \in 6[\alpha_i,\beta_i]} I_t(u^i, j).$$

It is obvious that $\mu(J_i) \le \mu(6J_i) \le 6\mu(J_i)$.

Next, we choose an $y \notin 6J_i$. Then either there exists a $j < t$ such that $y \in I_j(u^i) \setminus I_{j+1}(u^i)$, or $y \in I_t(u^i)$ and $y_t \notin 6[\alpha_i, \beta_i]$.

In the first case we have

$$\bigcup_{z=1}^{m_t-1} I_s(y + e_t z) \subset I_t(y) \setminus I_{t+1}(y) \subset I_t(y) \subset I_j(u^i) \setminus I_{j+1}(u^i),$$

and this set is disjoint from supp $(f_i - F_t f_i) \subset I_t(u^i)$ (recall that $k_i = t$). That is, we integrate $f_i - F_t f_i$ on the empty set calculating the value of $T_n^{s,t}(f_i - F_t f_i)(y)$. Consequently, $T_{*,A}^{s,t}(f_i - F_t f_i)(y) = 0$ for such an y.

In the second case we have $y \in I_t(u^i)$ and $y_t \notin 6[\alpha_i, \beta_i]$. This gives

$$
\begin{aligned}
&T_n^{s,t}(f_i - F_t f_i)(y) \\
&= \frac{M_s}{m_t} \int_{\bigcup_{z=1}^{m_t-1} I_s(y+e_t z)} (f_i - F_t f_i)(x) \bar{\psi}_{n^{(s)}}(x) \left[\frac{1}{1-r_t(y-x)} - \frac{1}{1-r_t(y-\gamma_i e_t)} \right] d\mu(x) \\
&+ \frac{M_s}{m_t} \frac{1}{1-r_t(y-\gamma_i e_t)} \int_{\bigcup_{z=1}^{m_t-1} I_s(y+e_t z)} (f_i - F_t f_i)(x) \bar{\psi}_{n^{(s)}}(x) d\mu(x) \\
&=: I + II.
\end{aligned} \tag{14}
$$

First, we discuss II.

$$
\begin{aligned}
&\int_{\bigcup_{z=0}^{m_t-1} I_s(y+e_t z)} f_i(x) \bar{\psi}_{n^{(s)}}(x) - F_t f_i(x) \bar{\psi}_{n^{(s)}}(x) d\mu(x) \\
&= \int_{\bigcup_{z=0}^{m_t-1} I_s(y+e_t z)} f_i(x) \bar{\psi}_{n^{(s)}}(x) - F_t(f_i \bar{\psi}_{n^{(s)}})(x) d\mu(x) = 0
\end{aligned} \tag{15}
$$

gives

$$
II \leq C \frac{M_s}{m_t} \frac{m_t}{\rho(y_t, \gamma_i)} \int_{I_s(y)} |f_i|(x) + F_t|f_i|(x) d\mu(x) \leq C(E_s|f_i|(y) + E_s F_t|f_i|(y)). \tag{16}
$$

On the other hand, the investigation of I (that is, the first addend in (14)) is based on

$$
\begin{aligned}
&\left| \frac{1}{1-r_t(y-x)} - \frac{1}{1-r_t(y-\gamma_i e_t)} \right| \\
&\leq C \frac{|\sin(\pi(x_t - \gamma_i)/m_t)|}{|\sin(\pi(y_t - x_t)/m_t)||\sin(\pi(y_t - \gamma_i)/m_t)|} \\
&\leq C \frac{(\beta_i - \alpha_i + 1)/m_t}{\rho^2(y_t, \gamma_i)/m_t^2} \\
&\leq C m_t \frac{\beta_i - \alpha_i + 1}{\rho^2(y_t, \gamma_i)}
\end{aligned} \tag{17}
$$

for $x_t \in [\alpha_i, \beta_i]$ and $y \in I_t(u^i)$, $y_t \notin 6[\alpha_i, \beta_i]$. This follows

$$
\begin{aligned}
|I| &\leq C \frac{M_s}{m_t} \int_{\bigcup_{z=1}^{m_t-1} I_s(y+e_t z)} (|f_i(x)| + |F_t f_i|(x)) m_t \frac{\beta_i - \alpha_i + 1}{\rho^2(y_t, \gamma_i)} d\mu(x) \\
&= C M_s \frac{\beta_i - \alpha_i + 1}{\rho^2(y_t, \gamma_i)} \int_{\bigcup_{z=1}^{m_t-1} I_s(y+e_t z)} (|f_i(x)| + |F_t f_i|(x)) d\mu(x).
\end{aligned} \tag{18}
$$

These assumptions, more exactly, the definition g_4 (see (9)), (14), (16) and (18) give

$$
\begin{aligned}
&\mu\left\{\left(\frac{M_{t+1}m_s}{M_A}\right)^\alpha T^{s,t}_{*,A}g_4>\lambda\right\}\\
&\le \mu(6F)+\mu\left\{y\in G_m\setminus 6F:\left(\frac{M_{t+1}m_s}{M_A}\right)^\alpha T^{s,t}_{*,A}g_4(y)>\lambda\right\}\\
&\le C\frac{\|f\|_1}{\lambda}+\frac{1}{\lambda}\int_{G_m\setminus 6F}\left(\frac{M_{t+1}m_s}{M_A}\right)^\alpha T^{s,t}_{*,A}g_4(y)d\mu(y)\\
&\le C\frac{\|f\|_1}{\lambda}+\frac{1}{\lambda}\sum_{\{i\in\mathbb{P}\,:\,k_i=t\}}\int_{G_m\setminus 6J_i}\left(\frac{M_{t+1}m_s}{M_A}\right)^\alpha T^{s,t}_{*,A}(f_i-F_tf_i)(y)d\mu(y) \qquad (19)\\
&\le C\frac{\|f\|_1}{\lambda}+\frac{1}{\lambda}\sum_{\{i\in\mathbb{P}\,:\,k_i=t\}}\int_{G_m\setminus I_t(u^i)}E_s|f_i|(y)+E_sF_t|f_i|(y)\mu(y)\\
&+C\frac{1}{\lambda}\sum_{\{i\in\mathbb{P}\,:\,k_i=t\}}\int_{\bigcup_{k\notin 6[\alpha_i,\beta_i]}I_t(u^i,k)}M_s\frac{\beta_i-\alpha_i+1}{\rho^2(y_t,\gamma_i)}\int_{\bigcup_{z=1}^{m_t-1}I_s(y+e_tz)}(|f_i(x)|+|F_tf_i|(x))d\mu(x)d\mu(y)
\end{aligned}
$$

It is easy to have that

$$\frac{M_s}{m_t}\int_{\bigcup_{z=0}^{m_t-1}I_s(y+e_tz)}g(x)d\mu(x)=F_tE_sg(y)$$

and consequently

$$\frac{M_s}{m_t}\int_{\bigcup_{z=1}^{m_t-1}I_s(y+e_tz)}g(x)d\mu(x)=F_tE_sg(y)-\frac{1}{m_t}E_sg(y).$$

By the help of this equality we give the estimation of the right hand side of (19).

$$
\begin{aligned}
&\le C\frac{\|f\|_1}{\lambda}+C\frac{1}{\lambda}\sum_{\{i\in\mathbb{P}:k_i=t\}}\|f_i\|_1\\
&+C\frac{1}{\lambda}\sum_{\{i\in\mathbb{P}\,:\,k_i=t\}}\int_{\bigcup_{k\notin 6[\alpha_i,\beta_i]}I_t(u^i,k)}\frac{\beta_i-\alpha_i+1}{\rho^2(y_t,\gamma_i)}\left(m_tF_tE_s|f_i|(y)+E_s|f_i|(y)\right)d\mu(y)\\
&\le C\frac{\|f\|_1}{\lambda}+C\frac{1}{\lambda}\sum_{\{i\in\mathbb{P}\,:\,k_i=t\}}\int_{\bigcup_{k\notin 6[\alpha_i,\beta_i]}I_t(u^i,k)}E_s|f_i|(y)d\mu(y)\\
&+C\frac{1}{\lambda}\sum_{\{i\in\mathbb{P}\,:\,k_i=t\}}\int_{\bigcup_{k\notin 6[\alpha_i,\beta_i]}I_t(u^i,k)}\frac{\beta_i-\alpha_i+1}{\rho^2(y_t,\gamma_i)}m_tF_tE_s|f_i|(y)d\mu(y)=:I+II+III.
\end{aligned}
$$

it is simple to have that $I+II\le C\frac{\|f\|_1}{\lambda}$. For III we get

$$
\begin{aligned}
III&\le C\frac{1}{\lambda}\sum_{\{i\in\mathbb{P}\,:\,k_i=t\}}\sum_{k\notin 6[\alpha_i,\beta_i]}\frac{\beta_i-\alpha_i+1}{\rho^2(y_t,\gamma_i)}\int_{I_t(u^i,k)}m_tF_tE_s|f_i|(y)d\mu(y)\\
&\le C\frac{1}{\lambda}\sum_{\{i\in\mathbb{P}\,:\,k_i=t\}}\sum_{\{k\,:\,k\ge 3(\beta_i-\alpha_i+1)\}}\frac{\beta_i-\alpha_i+1}{k^2}\int_{I_t(u^i)}F_tE_s|f_i|(y)d\mu(y) \qquad (20)\\
&\le C\frac{1}{\lambda}\sum_{\{i\in\mathbb{P}\,:\,k_i=t\}}\|f_i\|_1\le C\frac{1}{\lambda}\|f\|_1,
\end{aligned}
$$

because for any integrable function g

$$\int_{I_t(u^i,0)} F_t g(y) d\mu(y) = \cdots = \int_{I_t(u^i,m_t-1)} F_t g(y) d\mu(y).$$

This completes the proof of Lemma 7. $\square$

It will often be used the following inequality. Define the stopping time ν (see e.g. [22]) in the following way:

$$\nu(x) := \inf\{k \in \mathbb{N} : E_k(|f|)(x) > \lambda\} \quad (\inf \emptyset = +\infty).$$

It is known that $\mu(\nu < \infty) \le \|f\|_1/\lambda$. Denote the characteristic function of the set $B \subset G_m$ by 1_B. Let l, A be natural numbers. Then we prove

$$\|1_{\{\nu=l\}} E_A f\|_1 \le \|1_{\{\nu=l\}} f\|_1.$$

If $l \ge A+1$, then we have $1_{\{\nu=l\}}(x) E_A(|f|)(x) \le 1_{\{\nu=l\}}(x) E_l(|f|)(x)$ because if for some x we $1_{\{\nu=l\}}(x) = 0$, then there is nothing to prove, and in the case of $1_{\{\nu=l\}}(x) = 1$ we have by the definition of ν that $E_l(|f|)(x) > \lambda \ge E_A(|f|)(x)$. Consequently, if $l \ge A+1$, then by the $\mathcal{A}_A$ measurability of the function $1_{\{\nu=l\}}$ we proved

$$\|1_{\{\nu=l\}} E_A f\|_1 \le \|1_{\{\nu=l\}} E_A(|f|)\|_1 \le \|1_{\{\nu=l\}} E_l(|f|)\|_1 \le \|E_l(1_{\{\nu=l\}}|f|)\|_1 \le \|1_{\{\nu=l\}} f\|_1.$$

On the other hand, if $l \le A$, then we have a more simple situation. Namely, the function $1_{\{\nu=l\}}$ is $\mathcal{A}_A$ measurable, and this gives

$$1_{\{\nu=l\}} E_A f = E_A(1_{\{\nu=l\}} f), \quad \|1_{\{\nu=l\}} E_A f\|_1 = \|E_A(1_{\{\nu=l\}} f)\|_1 \le \|1_{\{\nu=l\}} f\|_1.$$

Let $0 < \alpha < 1$, $j, k \in \mathbb{P}$. Define the operator

$$\mathbf{T}_*^{j,k} f := \sup_{A \ge j+k} \left\{ \left(\frac{M_{A-j-k+1} m_{A-j}}{M_A} \right)^\alpha T_{*,A}^{A-j,A-j-k} f \right\}.$$

We prove a weak $(1,1)$-like type inequality concerning this operator. We remark that in Lemma 7 we proved a weak $(1,1)$ type inequality with respect to the operator $\left(\frac{M_{t+1}m_s}{M_A}\right)^\alpha T_{*,A}^{s,t}$. In the forthcoming lemma the role of s will be played by $A-j$, and role of t will be played by $A-j-k$.

Lemma 8. *Let $0 < \alpha < 1$ be any fixed real. Then we have a constant $C > 0$ such that for all $j, k \in \mathbb{P}$, $\lambda > 0$ and $f \in L^1(G_m)$ the inequality*

$$\mu\left(\mathbf{T}_*^{j,k} f > \lambda\right) \le C\|f\|_1 \frac{1}{\lambda}(j+k+2)^2$$

holds.

Proof. The proof is based on Lemma 7. Define the stopping time ν (see e.g. [22]) in the following way:

$$\nu(x) := \inf\{k \in \mathbb{N} : E_k(|f|)(x) > \lambda\} \quad (\inf \emptyset = +\infty).$$

It is known that $\mu(\nu < \infty) \le \|f\|_1/\lambda$. Denote by the characteristic function of the set $B \subset G_m$ by 1_B. Since for any $n, |n| = A$ we have

$$T_n^{s,t} f = T_n^{s,t} E_{A+1} f \quad \text{and} \quad T_n^{s,t} E_A f = 0,$$

then we also have

$$T_{*,A}^{A-j,A-j-k} f = T_{*,A}^{A-j,A-j-k}(E_{A+1} f - E_A f).$$

This equality by

$$1 = 1_{\{\nu<\infty\}} + 1_{\{\nu=\infty\}} = 1_{\{\nu>A+1\}} + 1_{\{\nu<A-j-k\}} + 1_{\{A-j-k\le\nu\le A+1\}}$$

provides

$$\begin{aligned}
&\mu\left(\mathbf{T}_*^{j,k} f > \lambda\right) \le \mu\left(1_{\{\nu<\infty\}}\mathbf{T}_*^{j,k} f > \frac{\lambda}{2}\right)\\
&+ \mu\left(1_{\{\nu=\infty\}} \sup_{A\ge j+k}\left(\frac{M_{A-j-k+1}m_{A-j}}{M_A}\right)^{\alpha} T_{*,A}^{A-j,A-j-k}(1_{\{\nu>A+1\}}(E_{A+1}f - E_A f)) > \frac{\lambda}{6}\right)\\
&+ \mu\left(1_{\{\nu=\infty\}} \sup_{A\ge j+k}\left(\frac{M_{A-j-k+1}m_{A-j}}{M_A}\right)^{\alpha} T_{*,A}^{A-j,A-j-k}(1_{\{\nu<A-j-k\}}(E_{A+1}f - E_A f)) > \frac{\lambda}{6}\right)\\
&+ \mu\left(1_{\{\nu=\infty\}} \sup_{A\ge j+k}\left(\frac{M_{A-j-k+1}m_{A-j}}{M_A}\right)^{\alpha} T_{*,A}^{A-j,A-j-k}(1_{\{A-j-k\le\nu\le A+1\}}(E_{A+1}f - E_A f)) > \frac{\lambda}{6}\right)\\
&=: I + II + III + IV.
\end{aligned}$$

We already have that $I \le C\|f\|_1/\lambda$. The equality $III = 0$ is given by the equality that

$$T_{*,A}^{A-j,A-j-k}(1_{\{\nu<A-j-k\}}f) = 1_{\{\nu<A-j-k\}}T_{*,A}^{A-j,A-j-k}f$$

for any integrable function f and by the fact that the product on the right hand side is zero on the set $\{\nu = \infty\}$. In order to have an appropriate upper bound for II we use Lemma 4.1 in [9] again. In other words, we use

$$\|T_{*,A}^{A-j,A-j-k}g\|_2^2 \le C\|g\|_2^2$$

for any $g \in L^2(G_m)$. (Recall that $\left(\frac{M_{A-j-k+1}m_{A-j}}{M_A}\right)^{\alpha} \le 1$.) This gives

$$\begin{aligned}
II &\le C\frac{1}{\lambda^2}\sum_{A=j+k}^{\infty}\|T_{*,A}^{A-j,A-j-k}(1_{\{\nu>A+1\}}(E_{A+1}f - E_A f))\|_2^2\\
&\le C\frac{1}{\lambda^2}\sum_{A=j+k}^{\infty}\|1_{\{\nu>A+1\}}(E_{A+1}f - E_A f)\|_2^2 \le C\frac{1}{\lambda^2}\|f\|_1\lambda\\
&= C\frac{\|f\|_1}{\lambda}.
\end{aligned}$$

The last inequality is given by the lemma of Burkholder [19, 22]. So, we have to investigate IV. This is the last to do in order to prove Lemma 8. The equality

$$1_{\{A-j-k\le\nu\le A+1\}} = \sum_{l=A-j-k}^{A+1} 1_{\{\nu=l\}}$$

$$\begin{aligned}
&\sup_{A\ge j+k}\left(\frac{M_{A-j-k+1}m_{A-j}}{M_A}\right)^{\alpha} T_{*,A}^{A-j,A-j-k}(1_{\{A-j-k\le\nu\le A+1\}}(E_{A+1}f - E_A f))\\
&\le (j+k+2)\sup_{A\ge j+k}\sup_{l\in[A-j-k,A+1]}\left(\frac{M_{A-j-k+1}m_{A-j}}{M_A}\right)^{\alpha} T_{*,A}^{A-j,A-j-k}(1_{\{\nu=l\}}(E_{A+1}f - E_A f)).
\end{aligned}$$

This by the help of Lemma 7 gives

$$
\begin{aligned}
IV &\leq \sum_{A=j+k}^{\infty} \sum_{l=A-j-k}^{A+1} \mu\Bigg(\left(\frac{M_{A-j-k+1}m_{A-j}}{M_A}\right)^{\alpha} T_{*,A}^{A-j,A-j-k}(1_{\{\nu=l\}}(E_{A+1}f - E_A f)) \\
&> \frac{\lambda}{6(j+k+2)}\Bigg) \\
&\leq C \sum_{A=j+k}^{\infty} \sum_{l=A-j-k}^{A+1} \frac{j+k+2}{\lambda} \|1_{\{\nu=l\}}(E_{A+1}f - E_A f)\|_1 \\
&= C\frac{j+k+2}{\lambda} \sum_{l=0}^{\infty} \sum_{A=l-1}^{l+j+k} \|1_{\{\nu=l\}} f\|_1 \\
&= C\frac{(j+k+2)^2}{\lambda} \sum_{l=0}^{\infty} \|1_{\{\nu=l\}} f\|_1 \\
&= C\frac{(j+k+2)^2}{\lambda} \|f\|_1.
\end{aligned}
$$

This completes the proof of Lemma 8. □

Define the operator $R_n^t : L^1(G_m) \to L^1(G_m)$, where $n, t \in \mathbb{N}$ and $A = |n| > t$

$$
R_n^t f(y) := T_n^{A,t} f(y) = \frac{1}{m_t} \left| \sum_{x_t=0, x_t \neq y_t}^{m_t - 1} E_A(f \bar{r}_A^{n_A})(y + e_t(x_t - y_t)) \frac{1}{1 - r_t(y-x)} \right|.
$$

Also set

$$
R_{*,A}^t f := \sup\left\{R_n^t f : |n| = A\right\} = T_{*,A}^{A,t} f, \quad \text{and} \quad R_*^t f := \sup_{A \geq t} R_{*,A}^t f.
$$

Let $0 < \alpha < 1$, $k \in \mathbb{P}$. Define the operator $\mathbf{R}_*^k f := \sup_{A \geq k} \left(\frac{M_{A-k+1}}{M_A}\right)^{\alpha} R_{*,A}^{A-k} f$. We prove a weak $(1,1)$-like type inequality concerning this operator, just as we proved for $\mathbf{T}_*^{j,k}$ in Lemma 8.

Lemma 9. *Let $0 < \alpha < 1$ be any fixed real. Then we have a constant $C > 0$ such that for all $k \in \mathbb{P}$, $\lambda > 0$ and $f \in L^1(G_m)$ the inequality*

$$
\mu\left(\mathbf{R}_*^k f > \lambda\right) \leq C\|f\|_1 \frac{1}{\lambda}(k+2)^2
$$

holds.

Proof. The proof of this lemma is nothing else but a step by step repeat of the appropriate version of the proof of Lemma 8, including the the proofs of the necessary Lemmas 6 and 7. By the inequality (3) we have for every $1 < p \leq 2$ that $\|R_*^t f\|_p \leq C\|f\|_p$ (C depends on p). The the appropriate version of the statement of Lemma 6 reads as follows: Let $f \in L^1(G_m)$, $\lambda > 0$, $A, t \in \mathbb{N}$, $A > t$ so that $E_{t+1}|f| \leq C\lambda$ a.e. Let $0 < \alpha < 1$ be an arbitrary real. Then

$$
\mu\left\{\left(\frac{M_{t+1}}{M_A}\right)^{\alpha} R_{*,A}^t f > \lambda\right\} \leq C\frac{\|f\|_1}{\lambda}, \tag{21}
$$

where C depends only on α and p. The prove inequality (21) we follow the proof of Lemma 6. For example our version of inequality (6) is

$$\mu\left\{\left(\frac{M_{t+1}}{M_A}\right)^{\alpha} R^t_{*,A} f > \lambda\right\} \leq \frac{C}{\lambda}\left(\frac{M_{t+1}}{M_A}\right)^{p\alpha}(m_{t+1}\dots m_A)^{p-1}\|f\|_1.$$

After we have completed the proof of (21) we turn our attention to our version of Lemma 7.

Namely, let $0<\alpha<1$, $A,s,t\in\mathbb{N}$ and $A>s>t$. Then the operator $\left(\frac{M_{t+1}}{M_A}\right)^{\alpha} R^t_{*,A}$ is of weak type $(1,1)$. Just follow the proof of Lemma 7. The only difference is that the factor m_s is missing, and in this case $s=A$. We do not go further in details, this is left to the reader. Then, in order to prove the appropriate version of Lemma 8 we do the same procedure then we did in in the proof of Lemma 8. We give the sketch of this below.

$$\begin{aligned}
&\mu\left(\mathbf{R}^k_* f > \lambda\right) \leq \mu\left(1_{\{\nu<\infty\}}\mathbf{R}^k_* f > \frac{\lambda}{2}\right)\\
&+\mu\left(1_{\{\nu=\infty\}}\sup_{A\geq k}\left(\frac{M_{A-k+1}}{M_A}\right)^{\alpha} R^{A-k}_{*,A}(1_{\{\nu>A+1\}}(E_{A+1}f-E_Af)) > \frac{\lambda}{6}\right)\\
&+\mu\left(1_{\{\nu=\infty\}}\sup_{A\geq k}\left(\frac{M_{A-k+1}}{M_A}\right)^{\alpha} R^{A-k}_{*,A}(1_{\{\nu<A-k\}}(E_{A+1}f-E_Af)) > \frac{\lambda}{6}\right)\\
&+\mu\left(1_{\{\nu=\infty\}}\sup_{A\geq k}\left(\frac{M_{A-k+1}}{M_A}\right)^{\alpha} R^{A-k}_{*,A}(1_{\{A-k\leq\nu\leq A+1\}}(E_{A+1}f-E_Af)) > \frac{\lambda}{6}\right)\\
&=: I+II+III+IV.
\end{aligned}\tag{22}$$

By the very same way, as we got in the proof of Lemma 8, we get $I\leq C\|f\|_1/\lambda$, and also $III=0$. The latter come from

$$R^{A-k}_{*,A}(1_{\{\nu<A-k\}}f) = 1_{\{\nu<A-k\}}R^{A-k}_{*,A}f$$

for any integrable function f, and by the fact that the product on the right hand side is zero on the set $\{\nu=\infty\}$. In order to have an appropriate upper bound for II we use Lemma 4.1 in [9] again. In other words, we use

$$\|R^{A-k}_{*,A}g\|_2^2 \leq C\|g\|_2^2$$

for any $g\in L^2(G_m)$. (Recall that $\left(\frac{M_{A-j-k+1}}{M_A}\right)^{\alpha}\leq 1$.) This gives

$$\begin{aligned}
II &\leq C\frac{1}{\lambda^2}\sum_{A=k}^{\infty}\|R^{A-k}_{*,A}(1_{\{\nu>A+1\}}(E_{A+1}f-E_Af))\|_2^2\\
&\leq C\frac{1}{\lambda^2}\sum_{A=k}^{\infty}\|1_{\{\nu>A+1\}}(E_{A+1}f-E_Af)\|_2^2 \leq C\frac{1}{\lambda^2}\|f\|_1\lambda\\
&= C\frac{\|f\|_1}{\lambda}.
\end{aligned}$$

(Compare the corresponding lines in the proof of Lemma 8.) So, it is left to investigate IV. Since, there is no (main) difference with respect the proof of the same thing in the proof of Lemma 8, then it is left to the reader. This completes, the proof of Lemma 9. $\square$

Define the operator $V_n^t : L^1(G_m) \to L^0(G_m)$ (and some others), where $t \in \mathbb{N}, n, k \in \mathbb{P}$ and $A = |n| > t$ as

$$V_n^t f(y) := \frac{1}{M_{|n|}} \left| M_t^2 \int_{I_t(y)\setminus I_{t+1}(y)} f(x)\bar{\psi}_{n^{(t+1)}}(x) \frac{r_t^{n_t}(y-x)-1}{(r_t(y-x)-1)^2} d\mu(x) \right|,$$

and

$$V_{*,A}^t f := \sup\left\{V_n^t f : n \in \mathbb{P}, |n| = A\right\}, \quad V_*^t f := \sup\left\{V_{*,A}^t f : A \geq t\right\},$$

$$\mathbf{V}_*^k f := \sup\left\{V_{*,A}^{A-k} f : A \in \mathbb{P}, A \geq k\right\}.$$

We prove a weak $(1,1)$-like type inequality concerning the operator $\mathbf{V}_*^k$, a similar one as we proved for $\mathbf{R}_*^k$ in Lemma 9.

Lemma 10. *We have a constant $C > 0$ such that for all $k \in \mathbb{P}$, $\lambda > 0$ and $f \in L^1(G_m)$ the inequality*

$$\mu\left(\mathbf{V}_*^k f > \lambda\right) \leq C\|f\|_1 \frac{k+2}{2^k \lambda}$$

holds.

Proof. First, we prove that the operator $2^{A-t}V_{*,A}^t f$ is of strong type $(1,1)$. In other words, we prove $\|V_{*,A}^t f\|_1 \leq C\|f\|_1$. It is easy to have that

$$\begin{aligned} V_{*,A}^t f(y) &\leq \frac{1}{M_A} M_t^2 \sup_{n_t \in \{0,\dots,m_t-1\}} \int_{I_t(y)\setminus I_{t+1}(y)} |f(x)| \frac{|r_t^{n_t}(y-x)-1|}{|(r_t(y-x)-1)^2|} d\mu(x) \\ &\leq C \frac{1}{M_A} M_{t+1}^2 \int_{I_t(y)\setminus I_{t+1}(y)} |f(x)| \frac{1}{(y_t - x_t)^2} d\mu(x). \end{aligned}$$

This implies

$$\begin{aligned} &\|V_{*,A}^t f\|_1 \leq \int_{G_m} \frac{1}{M_A} M_{t+1}^2 \int_{I_t(y)\setminus I_{t+1}(y)} |f(x)| \frac{1}{(y_t - x_t)^2} d\mu(x) d\mu(y) \\ &= \frac{1}{M_A} M_{t+1}^2 \int_{G_m} \int_{I_t(y)\setminus I_{t+1}(y)} |f(x)| \frac{1}{(y_t - x_t)^2} d\mu(x) d\mu(y) \\ &= \frac{1}{M_A} M_{t+1}^2 \sum_{y_0=0}^{m_0-1} \cdots \sum_{y_t=0}^{m_t-1} \int_{I_{t+1}(y)} \int_{I_t(y)\setminus I_{t+1}(y)} |f(x)| \frac{1}{(y_t - x_t)^2} d\mu(x) d\mu(y) \\ &= \frac{1}{M_A} M_{t+1}^2 \sum_{y_0=0}^{m_0-1} \cdots \sum_{y_t=0}^{m_t-1} \int_{I_t(y)\setminus I_{t+1}(y)} |f(x)| \int_{I_{t+1}(y)} \frac{1}{(y_t - x_t)^2} d\mu(y) d\mu(x) \\ &= \frac{1}{M_A} M_{t+1} \sum_{y_0=0}^{m_0-1} \cdots \sum_{y_t=0}^{m_t-1} \sum_{j \neq y_t} \int_{I_t(y,j)} |f(x)| \frac{1}{(y_t - j)^2} d\mu(x) \\ &\leq C \frac{1}{M_A} M_{t+1} \sum_{y_0=0}^{m_0-1} \cdots \sum_{y_{t-1}=0}^{m_{t-1}-1} \sum_{j=0}^{m_t-1} \int_{I_t(y,j)} |f(x)| d\mu(x) \\ &\leq C\|f\|_1 \frac{1}{2^{A-t}}. \end{aligned}$$

We also prove that the operator $2^{A-t}V^t_{*,A}f$ is of strong type (∞,∞).

$$\|V^t_{*,A}f\|_\infty \leq \|f\|_\infty \frac{1}{M_A} M^2_{t+1} \int_{I_t(y)\setminus I_{t+1}(y)} \frac{1}{(y_t-x_t)^2} d\mu(x)$$
$$\leq \|f\|_\infty \frac{M_{t+1}}{M_A} \leq C\|f\|_\infty \frac{1}{2^{A-t}}.$$

That is, we have that the operator $2^{A-t}V^t_{*,A}f$ is of strong type $(1,1)$ and (∞,∞). By the interpolation theorem of Marcinkiewicz we also have that the operator $2^{A-t}V^t_{*,A}f$ is of strong type (p,p) for all $1 \leq p \leq \infty$.

As we remember $A = |n|$ gives $V^t_n f = V^t_n(E_{A+1}f)$, and also that $V^t_n(E_Af) = 0$ (recall that $A > t, n_A \neq 0$). Thus, $V^t_{*,A}f = V^t_{*,A}(E_{A+1}f) = V^t_{*,A}(E_{A+1}f - E_Af)$. Thus, $V^k_* f = \sup_A V^{A-k}_{*,A}(E_{A+1}f - E_Af)$. It is also simple to have from the definition of V^t_n that for any integrable function f and $\mathcal{A}_t$ measurable function g the equality $V^t_{*,A}(gf) = gV^t_{*,A}(f)$. This will be applied below for the function $1_{\nu<A-k} = g$.

The rest of the proof of Lemma 10 is quite similar to the proof of Lemma 8, 9. Define the stopping time ν as in the proof of Lemma 8, that is,

$$\nu(x) := \inf\{k \in \mathbb{N} : E_k(|f|)(x) > \lambda\} \quad (\inf\emptyset = +\infty).$$

It is known that $\mu(\nu<\infty) \leq \|f\|_1/\lambda$.

$$\mu\left(2^k\mathbf{V}^k_* f > \lambda\right) \leq \mu\left(1_{\{\nu<\infty\}}2^k\mathbf{V}^k_* f > \frac{\lambda}{2}\right)$$
$$+\mu\left(1_{\{\nu=\infty\}}2^k \sup_{A\geq k} V^k_{*,A}(1_{\{\nu>A+1\}}(E_{A+1}f - E_Af)) > \frac{\lambda}{6}\right)$$
$$+\mu\left(1_{\{\nu=\infty\}}2^k \sup_{A\geq k} V^k_{*,A}(1_{\{\nu<A-k\}}(E_{A+1}f - E_Af)) > \frac{\lambda}{6}\right)$$
$$+\mu\left(1_{\{\nu=\infty\}}2^k \sup_{A\geq k} V^k_{*,A}(1_{\{A-k\leq\nu\leq A+1\}}(E_{A+1}f - E_Af)) > \frac{\lambda}{6}\right)$$
$$=: I + II + III + IV.$$

We already have that $I \leq C\|f\|_1/\lambda$. The equality $III = 0$ is given by the equality that

$$V^{A-k}_{*,A}(1_{\{\nu<A-k\}}f) = 1_{\{\nu<A-k\}}V^{A-k}_{*,A}f$$

for any integrable function f and by the fact that the product on the right hand side is zero on the set $\{\nu=\infty\}$. In order to have an appropriate upper bound for II we use the fact that the operator $2^{A-t}V^t_{*,A}f$ is of type $(2,2)$.

$$II \leq \frac{C}{\lambda^2}\left\|2^k \sup_{A\geq k} V^k_{*,A}(1_{\{\nu>A+1\}}(E_{A+1}f - E_Af))\right\|_2^2$$
$$\leq \frac{C}{\lambda^2}\sum_{A=k}^\infty \left\|2^k V^k_{*,A}(1_{\{\nu>A+1\}}(E_{A+1}f - E_Af))\right\|_2^2$$
$$\leq \frac{C}{\lambda^2}\sum_{A=k}^\infty \left\|1_{\{\nu>A+1\}}(E_{A+1}f - E_Af)\right\|_2^2$$

By the lemma of Burkholder [19, 22] we get

$$II \leq \frac{C}{\lambda}\|f\|_1.$$

So, we have to investigate IV.

$$\begin{aligned}
IV &\leq \frac{C}{\lambda}\sum_{A=k}^{\infty}\sum_{l=A-k}^{A+1}\left\|2^k V_{*,A}^k(1_{\{\nu=l\}}(E_{A+1}f - E_A f))\right\|_1 \\
&\leq \frac{C}{\lambda}\sum_{A=k}^{\infty}\sum_{l=A-k}^{A+1}\left(\left\|1_{\{\nu=l\}}E_{A+1}f\right\|_1 + \left\|1_{\{\nu=l\}}E_(A)f\right\|_1\right) \\
&= \frac{C}{\lambda}\sum_{l=0}^{\infty}\sum_{A=l-1}^{k+l}\left\|1_{\{\nu=l\}}f\right\|_1 \\
&= (k+2)\frac{C}{\lambda}\sum_{l=0}^{\infty}\left\|1_{\{\nu=l\}}f\right\|_1 = (k+2)\frac{C}{\lambda}\|f\|_1.
\end{aligned}$$

This completes the proof of Lemma 10. $\square$

Define the operator $U_n^t : L^1(G_m) \to L^0(G_m)$ (and some others), where $n \in \mathbb{P}$ and $A = |n|$ as

$$U_n f(y) := \frac{1}{m_A}\left|\sum_{x_A=0, x_A\neq y_A}^{m_A-1} E_{A+1}f(y + e_A(x_A - y_A))\frac{r_A^{n_A}(y-x) - 1}{n_A(r_A(y-x)-1)^2}\right|,$$

and

$$U_{*,A}f := \sup\left\{U_n f : n \in \mathbb{P}, |n| = A\right\}, \quad \mathbf{U}_* f := \sup\left\{U_{*,A}f : A \in \mathbb{P}\right\}.$$

Lemma 11. *The operator $\mathbf{U}_*$ is of weak type $(1,1)$, that is, we have a constant $C > 0$ such that for all $\lambda > 0$ and $f \in L^1(G_m)$ the inequality*

$$\mu\left(\mathbf{U}_* f > \lambda\right) \leq C\frac{1}{\lambda}\|f\|_1$$

holds.

Proof. The very base of the proof of this lemma is the theorem [9, Theorem 2.8] which reads as follows:

$$\frac{1}{m_A}\sum_{y_A=0}^{m_A-1}(U_{*,A}f(y))^p \leq C\frac{1}{m_A}\sum_{y_A=0}^{m_A-1}|E_{A+1}f(y)|^p$$

for all $1 < p < \infty$, and $y_i \in Z_{m_i} (i < A)$ (C depends on p). Besides, the number of the set $\{y_A : U_{*,A}f(y) > \lambda\}$ divided by m_A is bounded by $C\frac{1}{m_A}\sum_{y_A=0}^{m_A-1}|E_{A+1}f(y)|$. In other words, the operator $U_{*,A}$ is of type (p,p) for $1 < p < \infty$ and of weak type $(1,1)$ on the discrete group Z_{m_A}. This is what one can find in [9, Theorem 2.8]. It is not so difficult to have

$$\frac{1}{m_A}\left|\sum_{x_A=0, x_A\neq y_A}^{m_A-1}\frac{r_A^{n_A}(y-x) - 1}{n_A(r_A(y-x)-1)^2}\right| \leq C.$$

Basically, behind of this inequality is the fact that

$$\frac{1}{m_A}\sum_{z_A=0}^{m_A-1} r_A^k(z) = \begin{cases} 1 & \text{, if } k = 0, \\ 0 & \text{, if } k \neq 0 \end{cases}$$

for $k \in \{0, 1, \ldots, m_A - 1\}$. This provide that

$$U_n(E_A f)(y) := \frac{1}{m_A} \left| E_A f(y) \sum_{x_A=0, x_A \neq y_A}^{m_A-1} \frac{r_A^{n_A}(y-x) - 1}{n_A(r_A(y-x)-1)^2} \right| \leq C\,|E_A f(y)| \leq Cf^*(y).$$

Consequently, we get

$$\mathbf{U}_* f \leq \sup\{U_{*,A}(E_{A+1}f - E_A f) : A \in \mathbb{P}\} + Cf^* =: \mathbf{U}_*^1 f + Cf^*. \tag{23}$$

So, by the help of (23) we have that it is enough to prove that operator $\mathbf{U}_*^1 f$ is of weak type $(1,1)$. Define the stopping time ν as we have done it earlier, e.g. in the proof of Lemma 8, that is,

$$\nu(x) := \inf\{k \in \mathbb{N} : E_k(|f|)(x) > \lambda\} \quad (\inf \emptyset = +\infty).$$

It is known that $\mu(\nu < \infty) \leq \|f\|_1/\lambda$.

$$\begin{aligned}
&\mu\left(\mathbf{U}_*^1 f > \lambda\right) \leq \mu\left(1_{\{\nu<\infty\}} \mathbf{U}_*^1 f > \frac{\lambda}{2}\right) \\
&+ \mu\left(1_{\{\nu=\infty\}} \sup_{A\in\mathbb{P}} U_{*,A}(1_{\{\nu>A+1\}}(E_{A+1}f - E_A f)) > \frac{\lambda}{6}\right) \\
&+ \mu\left(1_{\{\nu=\infty\}} \sup_{A\in\mathbb{P}} U_{*,A}(1_{\{\nu<A\}}(E_{A+1}f - E_A f)) > \frac{\lambda}{6}\right) \\
&+ \mu\left(1_{\{\nu=\infty\}} \sup_{A\in\mathbb{P}} U_{*,A}(1_{\{A\leq\nu\leq A+1\}}(E_{A+1}f - E_A f)) > \frac{\lambda}{6}\right) \\
&=: I + II + III + IV.
\end{aligned}$$

We already have that $I \leq C\|f\|_1/\lambda$. The equality $III = 0$ is given by the equality that

$$U_{*,A}(1_{\{\nu<A\}} f) = 1_{\{\nu<A\}} U_{*,A} f$$

for any integrable function f and by the fact that the product on the right hand side is zero on the set $\{\nu = \infty\}$. In order to have an appropriate upper bound for II we use the fact that the operator $U_{*,A} f$ is of type $(2,2)$ (see the beginning of the proof of this lemma). Thus,

$$\begin{aligned}
II &\leq C\frac{1}{\lambda^2} \sum_{A=1}^{\infty} \|U_{*,A}(1_{\{\nu>A+1\}}(E_{A+1}f - E_A f))\|_2^2 \\
&\leq C\frac{1}{\lambda^2} \sum_{A=1}^{\infty} \|1_{\{\nu>A+1\}}(E_{A+1}f - E_A f)\|_2^2 \leq C\frac{1}{\lambda^2}\|f\|_1\lambda \\
&= C\frac{\|f\|_1}{\lambda}.
\end{aligned}$$

The last inequality is given by the lemma of Burkholder [19, 22] again. So, we have to investigate IV.

$$\begin{aligned}
IV &\leq \sum_{A=1}^{\infty} \mu\left(U_{*,A}(1_{\{\nu=A\}}(E_{A+1}f - E_A f)) > \frac{\lambda}{12}\right) \\
&+ \sum_{A=1}^{\infty} \mu\left(U_{*,A}(1_{\{\nu=A+1\}}(E_{A+1}f - E_A f)) > \frac{\lambda}{12}\right) \\
&\leq C\frac{1}{\lambda}\sum_{A=1}^{\infty}\sum_{l=A}^{A+1}\|1_{\{\nu=l\}}(E_{A+1}f - E_A f)\|_1 \\
&\leq C\frac{1}{\lambda}\sum_{A=1}^{\infty}\sum_{l=A-1}^{A+1}\|1_{\{\nu=l\}}(E_A f)\|_1 \\
&\leq C\frac{1}{\lambda}\|f\|_1.
\end{aligned}$$

This completes the proof of this lemma. □

Now, we are ready to prove the main theorem of this paper.

Proof of Theorem 1.

For $l, L \in \mathbb{N}$ define the following sum of Dirichlet kernel functions

$$K_{l,L} := \sum_{i=l}^{l+L-1} D_i.$$

Let $n^{(j)} := \sum_{i=j}^{\infty} n_i M_i$ $(n, j \in \mathbb{N})$ and for $n \in \mathbb{N}$ set $|n| := \max(i \in \mathbb{N} : n_i \neq 0)$. That is, $|n| = A$ if and only if $M_A \leq n < M_{A+1}$. Then by elementary calculations we have

$$nK_n = \sum_{s=0}^{|n|}\sum_{j=0}^{n_s-1} K_{n^{(s+1)}+jM_s, M_s}. \tag{24}$$

By (24) we have

$$\begin{aligned}
&\mu(\sigma^* f > \lambda) = \mu(y : \sup_{n\in\mathbb{P}}\left|\int_{G_m} f(x)K_n(y-x)d\mu(x)\right| > \lambda) \\
&\leq \mu(y : \sup_{n\in\mathbb{P}}\left|\int_{I_{|n|+1}(y)} f(x)K_n(y-x)d\mu(x)\right| > \lambda/2) \\
&+ \mu(y : \sup_{n\in\mathbb{P}}\frac{1}{n}\left|\sum_{t=0}^{|n|}\int_{I_t(y)\setminus I_{t+1}(y)} f(x)\sum_{s=0}^{|n|}\sum_{j=0}^{n_s-1} K_{n^{s+1}+jM_s, M_s}(y-x)d\mu(x)\right| > \lambda/2) \\
&=: \mathbf{J}_1 + \mathbf{J}_2.
\end{aligned}$$

Discuss $\mathbf{J}_2$.

$$
\begin{aligned}
\mathbf{J}_2 \leq\ & \mu(y:\ \sup_{n\in\mathbb{P}} \frac{1}{n}\left|\sum_{t=1}^{|n|}\sum_{s=0}^{t-1}\int_{I_t(y)\setminus I_{t+1}(y)} f(x)\sum_{j=0}^{n_s-1} K_{n^{s+1}+jM_s,M_s}(y-x)d\mu(x)\right| > \lambda/10)\\
&+\mu(y:\ \sup_{n\in\mathbb{P}} \frac{1}{n}\left|\sum_{s=1}^{|n|-1}\sum_{t=0}^{s-1}\int_{I_t(y)\setminus I_{t+1}(y)} f(x)\sum_{j=0}^{n_s-1} K_{n^{s+1}+jM_s,M_s}(y-x)d\mu(x)\right| > \lambda/10)\\
&+\mu(y:\ \sup_{n\in\mathbb{P}} \frac{1}{n}\left|\sum_{t=0}^{|n|-1}\int_{I_t(y)\setminus I_{t+1}(y)} f(x)\sum_{j=0}^{n_{|n|}-1} K_{jM_{|n|},M_{|n|}}(y-x)d\mu(x)\right| > \lambda/10)\\
&+\mu(y:\ \sup_{n\in\mathbb{P}} \frac{1}{n}\left|\sum_{t=0}^{|n|-1}\int_{I_t(y)\setminus I_{t+1}(y)} f(x)\sum_{j=0}^{n_t-1} K_{n^{(t+1)}+jM_t,M_t}(y-x)d\mu(x)\right| > \lambda/10)\\
&+\mu(y:\ \sup_{n\in\mathbb{P}} \frac{1}{n}\left|\int_{I_{|n|}(y)\setminus I_{|n|+1}(y)} f(x)\sum_{j=0}^{n_{|n|}-1} K_{jM_{|n|},M_{|n|}}(y-x)d\mu(x)\right| > \lambda/10)\\
&=: \mathbf{J}_{2,1}+\mathbf{J}_{2,2}+\mathbf{J}_{2,3}+\mathbf{J}_{2,4}+\mathbf{J}_{2,5}.
\end{aligned}
$$

Basically, $\mathbf{J}_{2,1}, \mathbf{J}_{2,2}, \mathbf{J}_{2,3}, \mathbf{J}_{2,4}, \mathbf{J}_{2,5}$ means the cases $s<t$, $|n|>s>t$, $|n|=s>t$, $|n|>s=t$, $|n|=s=t$.

Next, we give an upper bound for $\mathbf{J}_{2,1}$ and for $\mathbf{J}_1$. Let $t\in\mathbb{N}$, $z\in I_t\setminus I_{t+1}$. Then for $s<t$ by (2) we have

$$
|K_{n^{(s+1)}+jM_s,M_s}(z)| \leq C n_t M_t M_s.
$$

Thus,

$$
\begin{aligned}
&\mu\left(y\in G_m : \sup_{n\in\mathbb{P}}\frac{1}{n}\sum_{t=1}^{|n|}\sum_{s=0}^{t-1}\left|\int_{I_t(y)\setminus I_{t+1}(y)} f(x)\sum_{j=0}^{n_s-1}K_{n^{(s+1)}+jM_s,M_s}(y-x)d\mu(x)\right| > \lambda\right)\\
&\leq \mu\left(y\in G_m : C\sup_{n\in\mathbb{P}}\frac{1}{n}\sum_{t=1}^{|n|}\sum_{s=0}^{t-1}\int_{I_t(y)\setminus I_{t+1}(y)}|f(x)|n_sM_sn_tM_t d\mu(x) > \lambda\right)\\
&\leq \mu\left(y\in G_m : C\int_{G_m}\sup_{n\in\mathbb{P}}\frac{1}{n}\sum_{t=1}^{|n|}n_tM_tM_t\int_{I_t(y)}|f(x)|d\mu(x) > \lambda\right)\\
&\leq \mu\left(y\in G_m : C\sup_{n\in\mathbb{P}}\frac{1}{n}\sum_{t=1}^{|n|}n_tM_t|f|^*(y) > \lambda\right)\\
&\leq \mu\left(y\in G_m : C|f|^*(y) > \lambda\right)\\
&\leq C\||f|\|_1/\lambda = C\|f\|_1/\lambda.
\end{aligned}
$$

For the inequality $\mu\left(y \in G_m : f^*(y) > \lambda\right) \leq C\|f\|_1/\lambda$ see the paper of Burkholder [19]. That is, $\mathbf{J}_{2,1} \leq C\|f\|_1/\lambda$. Moreover, by $|K_n| \leq (n+1)/2$ we have

$$\begin{aligned}
&\mu\left(y \in G_m : \sup_{n\in\mathbb{P}}\left|\int_{I_{|n|+1}(y)} f(x)K_n(y-x)d\mu(x)\right| > \lambda/10\right)\\
&\leq \mu\left(y \in G_m : \sup_{n\in\mathbb{P}}\int_{I_{|n|+1}(y)} |f(x)|\frac{n+1}{2}d\mu(x) > \lambda/10\right)\\
&\leq \mu\left(y \in G_m : C\sup_{n\in\mathbb{P}}\frac{n+1}{2M_{|n|+1}}|f|^*(y) > \lambda/10\right)\\
&\leq C\|f\|_1/\lambda.
\end{aligned}$$

That is, $\mathbf{J}_1 \leq C\|f\|_1/\lambda$.

In [9, Lemma 4.3] one can find the following: Let $|n| \geq s > t$, $n, s, t \in \mathbb{N}$, $z \in I_t \setminus I_{t+1}$. Then,

$$K_{n^{(s)},M_s}(z) = \begin{cases} 0, & \textit{if} \;\; z - e_t z_t \notin I_s, \\ M_t M_s \psi_{n^{(s)}}(z)\frac{1}{1-r_t(z)}, & \textit{if} \;\; z - e_t z_t \in I_s. \end{cases} \tag{25}$$

Next, we discuss the form $\mathbf{J}_{2,2}$ by the help this equality and by the help of Lemma 8.

$$\mathbf{J}_{2,2} = \mu(y : \sup_{n\in\mathbb{P}}\frac{1}{n}\left|\sum_{s=1}^{|n|-1}\sum_{t=0}^{s-1}\int_{I_t(y)\setminus I_{t+1}(y)} f(x)\sum_{j=0}^{n_s-1}K_{n^{s+1}+jM_s,M_s}(y-x)d\mu(x)\right| > \lambda/10)$$

Recall the definition of the operators $T_n^{s,t}f$.

$$T_n^{s,t}f(y) := \frac{1}{m_t}\left|\sum_{x_t=0,x_t\neq y_t}^{m_t-1} E_s(f\bar{\psi}_{n^{(s)}})(y + e_t(x_t - y_t))\frac{1}{1-r_t(y-x)}\right|.$$

So, we have

$$\begin{aligned}
\mathbf{J}_{2,2} &\leq \mu\left(\sup_{n\in\mathbb{P}}\frac{1}{n}\sum_{s=1}^{|n|-1}\sum_{t=0}^{s-1}M_{t+1}n_s T_n^{s,t}f > \lambda/10\right)\\
&\leq \mu\left(\sup_{A\in\mathbb{P}}\sum_{s=1}^{A-1}\sum_{t=0}^{s-1}\left(\frac{1}{2^{A-t-2}}\right)^{1-\alpha}\left(\frac{M_{t+1}m_s}{M_A}\right)^{\alpha}T_{*,A}^{s,t}f > \lambda/10\right)
\end{aligned}$$

for any fixed $0 < \alpha < 1$. Since $\left(\frac{1}{2^{A-t-2}}\right)^{1-\alpha} \leq 1/\delta^{A-t}$ for some $\delta > 1$, then we have

$$\begin{aligned}
\mathbf{J}_{2,2} &\leq \mu\left(\sup_{A\in\mathbb{P}} \sum_{j=1}^{A-1}\sum_{k=1}^{A-j} \frac{1}{\delta^{j+k}} \left(\frac{M_{A-j-k+1}m_{A-j}}{M_A}\right)^{\alpha} T_{*,A}^{A-j,A-j-k} f > \lambda/10\right) \\
&\leq \mu\left(\sup_{A\in\mathbb{P}} \sum_{j=1}^{A-1}\sum_{k=1}^{A-j} \frac{1}{\delta^{j+k}} \mathbf{T}_*^{j,k} f > \lambda/10\right) \\
&\leq \mu\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{\delta^{j+k}} \mathbf{T}_*^{j,k} f > \lambda/10\right) \\
&\leq \mu\left(\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} \left\{\frac{C}{\delta^{j/2+k/2}} \mathbf{T}_*^{j,k} f > \lambda/10\right\}\right).
\end{aligned}$$

The last inequality is implied by the fact that if inequality $\frac{C}{\delta^{j/2+k/2}}\mathbf{T}_*^{j,k} f(y) \leq \lambda/10$ holds for all j,k, then we have

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{\delta^{j+k}} \mathbf{T}_*^{j,k} f(y) \leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{\lambda}{10}\frac{C}{\delta^{j/2+k/2}} < \frac{\lambda}{10}.$$

Consequently,

$$\begin{aligned}
\mathbf{J}_{2,2} &\leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mu\left(\mathbf{T}_*^{j,k} f > C\lambda\delta^{j/2+k/2}\right) \\
&\leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} C\|f\|_1 \frac{1}{\lambda\delta^{j/2+k/2}}(j+k+2)^2 \\
&\leq C\|f\|_1/\lambda.
\end{aligned}$$

Next, we discuss the form $\mathbf{J}_{2,3}$ by the help of equality (25) and by the help of Lemma 9. This is the case $|n| = A = s > t$.

$$\begin{aligned}
\mathbf{J}_{2,3} &= \mu\left(\sup_{A,n_A\neq 0} \frac{1}{n_A M_A}\left|\sum_{t=0}^{A-1}\int_{I_t(y)\setminus I_{t+1}(y)} f(x)\sum_{j=0}^{n_A-1} K_{jM_A,M_A}(y-x)d\mu(x)\right| > \lambda/10\right) \\
&\leq \mu\left(\sup_{A,n_A\neq 0} \sum_{t=0}^{A-1} \frac{M_{t+1}}{n_A M_A}\sum_{j=0}^{n_A-1} R_{jM_A}^t f > \lambda/10\right) \\
&\leq \mu\left(\sup_{A,n_A\neq 0} \sum_{t=0}^{A-1} \frac{M_{t+1}}{M_A} R_{n_A M_A}^t f > \lambda/10\right) \\
&\leq \mu\left(\sup_{A} \sum_{t=0}^{A-1} \frac{M_{t+1}}{M_A} R_{*,A}^t f > \lambda/10\right) \\
&\leq \mu\left(\sup_{A} \sum_{k=1}^{A} \left(\frac{1}{2^{k-1}}\right)^{1-\alpha}\left(\frac{M_{A-k+1}}{M_A}\right)^{\alpha} R_{*,A}^{A-k} f > \lambda/10\right).
\end{aligned}$$

Then, using again here $(2^{k-1})^{1-\alpha} \geq \delta^k$ for some $\delta > 1$ we have

$$\begin{aligned}
\mathbf{J}_{2,3} &\leq \mu\left(\sup_A \sum_{k=1}^{A} \frac{1}{\delta^k}\mathbf{R}_*^k f > \lambda/10\right) \\
&\leq \mu\left(\sum_{k=1}^{\infty} \frac{1}{\delta^k}\mathbf{R}_*^k f > \lambda/10\right) \\
&\leq \mu\left(\bigcup_{k=1}^{\infty} \left\{\frac{C}{\delta^{k/2}}\mathbf{R}_*^k f > \lambda/10\right\}\right).
\end{aligned}$$

The last inequality is implied by the fact that if inequality $\frac{C}{\delta^{k/2}}\mathbf{R}_*^k f(y) \leq \lambda/10$ holds for all k, then we have

$$\sum_{k=1}^{\infty} \frac{1}{\delta^k}\mathbf{R}_*^k f(y) \leq \sum_{k=1}^{\infty} \frac{\lambda}{10}\frac{C}{\delta^{k/2}} < \frac{\lambda}{10}.$$

Consequently, by Lemma 9 we have

$$\begin{aligned}
\mathbf{J}_{2,3} &\leq \sum_{k=1}^{\infty} \mu\left(\mathbf{R}_*^k f > C\lambda\delta^{k/2}\right) \\
&\leq \sum_{k=1}^{\infty} C\|f\|_1 \frac{1}{\lambda\delta^{k/2}}(k+2)^2 \\
&\leq C\|f\|_1/\lambda.
\end{aligned}$$

Next, we turn our attention to the case $|n| > s = t$, that is, to $\mathbf{J}_{2,4}$. In [9, page 26] one can find: If $z \in I_t \setminus I_{t+1}$, then

$$\begin{aligned}
&\sum_{j=0}^{n_t-1} K_{n^{(t+1)}+jM_t,M_t}(z) \\
&= \sum_{j=0}^{n_t-1} \psi_{n^{(t+1)}+jM_t}(z)\frac{M_t(M_t-1)}{2} + \psi_{n^{(t+1)}}M_t^2\frac{r_t^{n_t}(z)-1}{(r_t(z)-1)^2} - \psi_{n^{(t+1)}}M_t^2 n_t\frac{1}{r_t(z)-1} \\
&=: A_1(z) + A_2(z) + A_3(z).
\end{aligned}$$

It is easy to investigate A_1.

$$\begin{aligned}
&\mu\left(y : \sup_{n\in\mathbb{P}} \frac{1}{n}\left|\sum_{t=0}^{|n|-1} \int_{I_t(y)\setminus I_{t+1}(y)} f(x)A_1(y-x)d\mu(x)\right| > \lambda/30\right) \\
&\leq \mu\left(y : \sup_{n\in\mathbb{P}} \sum_{t=0}^{|n|-1} \frac{M_{t+1}}{M_{|n|}}M_t \int_{I_t(y)} |f| > \lambda/30\right) \\
&\leq \mu\left(y : 2|f|^*(y) > \lambda/30\right) \\
&\leq C\|f\|_1/\lambda.
\end{aligned}$$

Next, discuss A_3.

$$\begin{aligned}
&\frac{1}{n}\left|\sum_{t=0}^{|n|-1} M_t^2 n_t \int_{I_t(y)\setminus I_{t+1}(y)} f(x)\psi_{n^{(t+1)}}(y-x)\frac{1}{r_t(y-x)-1}d\mu(x)\right| \\
&= \sum_{t=0}^{|n|-1} \frac{M_t n_t}{n} T_n^{t+1,t} f(y) \\
&\leq \sum_{t=0}^{|n|-1} \frac{1}{2^{|n|-t-1}} T_n^{t+1,t} f(y) \\
&= \sum_{j=0}^{|n|-1} \frac{1}{2^j} T_n^{|n|-j,|n|-j-1} f(y) \\
&\leq \sum_{j=0}^{\infty} \frac{1}{2^j} \mathbf{T}_*^{j,1} f(y).
\end{aligned}$$

Since, by the method we used earlier it is simple to prove that

$$\mu\left(\sum_{j=0}^{\infty}\frac{1}{2^j}\mathbf{T}_*^{j,1} f > \lambda/30\right) \leq C\|f\|_1/\lambda.$$

So, in order to have a proper bound for $\mathbf{J}_{2,4}$ we have to discuss only A_2.

$$\begin{aligned}
&\frac{1}{n}\left|\sum_{t=0}^{|n|-1} M_t^2 \int_{I_t(y)\setminus I_{t+1}(y)} f(x)\psi_{n^{(t+1)}}(y-x)\frac{r_t^{n_t}(y-x)-1}{(r_t(y-x)-1)^2}d\mu(x)\right| \\
&\leq \sum_{t=0}^{|n|-1} V_n^t f(y) \leq \sum_{t=0}^{A-1} V_{*,A}^t f(y) = \sum_{k=1}^{A} V_{*,A}^{A-k} f(y) \leq \sum_{k=1}^{\infty} \mathbf{V}_*^k f(y).
\end{aligned}$$

On the other hand,

$$\mu\left(\sum_{k=1}^{\infty}\mathbf{V}_*^k f > \lambda/30\right) \leq \mu\left(\bigcup_{k=1}^{\infty}\left\{C2^{k/2}\mathbf{V}_*^k f > \lambda/30\right\}\right).$$

The last inequality is implied by the fact that if inequality $C2^{k/2}\mathbf{V}_*^k f(y) \leq \lambda/30$ holds for all k, then we have $\sum_{k=1}^{\infty}\mathbf{V}_*^k f(y) \leq \sum_{k=1}^{\infty}\frac{C}{2^{k/2}}\frac{\lambda}{30} < \frac{\lambda}{30}$. Consequently, by Lemma 10 we get

$$\mu\left(\sum_{k=1}^{\infty}\mathbf{V}_*^k f > \lambda/30\right) \leq \sum_{k=1}^{\infty}\mu\left(\mathbf{V}_*^k f > \frac{C\lambda}{2^{k/2}}\right) \leq \sum_{k=1}^{\infty} C\|f\|_1\frac{k+2}{2^k\lambda 2^{-k/2}} \leq C\|f\|_1/\lambda.$$

So, cases A_1, A_3 and A_2 are discussed. That is, we have $\mathbf{J}_{2,4} \leq C\|f\|_1/\lambda$.

The last thing in order to have that the maximal operator $\sigma^* f = \sup_{n\in\mathbb{P}}|\sigma_n f|$ is of weak type $(1,1)$ is to prove the inequality $\mathbf{J}_{2,5} \leq C\|f\|_1/\lambda$.

In [9, page 31, 32] one can find for the case $|n| = s = t$ and $y \in I_{|n|} \setminus I_{|n|+1}$, $j < n_{|n|}$ that

$$\begin{aligned}
&K_{n^{(t+1)}+jM_t,M_t} = K_{jM_{|n|},M_{|n|}}(z)\\
&= \sum_{k=jM_{|n|}}^{(j+1)M_{|n|}-1} r_{|n|}^j(z)\left(\sum_{l=0}^{|n|-1} k_l M_l + M_{|n|}\sum_{l=m_{|n|}-j}^{m_{|n|}-1} r_{|n|}^l(z)\right)\\
&=: A_4(z) + A_5(z).
\end{aligned}$$

It is very simple to have that $|A_4(z)| \le M_{|n|}^2$ and consequently

$$\begin{aligned}
&\sup_{n\in\mathbb{P}}\left|\frac{1}{n}\sum_{j=0}^{n_{|n|}-1}\int_{I_{|n|}(y)\setminus I_{|n|+1}(y)} f(x)A_4(y-x)d\mu(x)\right|\\
&\le \sup_{n\in\mathbb{P}}\frac{n_{|n|}M_{|n|}^2}{n_{|n|}M_{|n|}}\int_{I_{|n|}(y)\setminus I_{|n|+1}(y)}|f(x)|d\mu(x)\\
&\le |f|^*(y).
\end{aligned}$$

On the other hand, $A_5(z) = M_{|n|}^2 \frac{r_{|n|}^j(z)-1}{r_{|n|}(z)-1}$ gives

$$\sum_{j=0}^{n_{|n|}-1} A_5(z) = M_{|n|}^2\frac{r_{|n|}^{n_{|n|}}(z)-1}{(r_{|n|}(z)-1)^2} + M_{|n|}^2 n_{|n|}\frac{1}{1-r_{|n|}(z)} =: A_{5,1}(z) + A_{5,2}(z).$$

By Lemma 11 we get for the kernel $A_{5,1}(z)$

$$\begin{aligned}
&\mu\left(y : \sup_{n\in\mathbb{P}}\left|\frac{n_{|n|}M_{|n|}}{n}M_{|n|}\int_{I_{|n|}(y)\setminus I_{|n|+1}(y)} f(x)\frac{r_{|n|}^{n_{|n|}}(y-x)-1}{n_{|n|}(r_{|n|}(y-x)-1)^2}d\mu(x)\right| > \lambda/30\right)\\
&\le \mu\left(y : \mathbf{U}_* f(y) > \lambda/30\right) \le C\|f\|_1/\lambda.
\end{aligned}$$

To discuss the maximal operator coming from kernel $A_{5,2}(z)$ we need a lemma [10, Lemma 2.4]. For an integrable function f we define the following operator:

$$H_1 f(y) := \sup_{A\in\mathbb{P}}\left|M_A\int_{I_A(y)\setminus I_{A+1}(y)} f(x)\frac{1}{1-r_A(y-x)}dx\right|.$$

That lemma [10, Lemma 2.4] states that the operator H_1 is of weak type $(1,1)$. This immediately gives

$$\begin{aligned}
&\mu\left(y : \sup_{n\in\mathbb{P}}\left|\frac{n_{|n|}M_{|n|}}{n}M_{|n|}\int_{I_{|n|}(y)\setminus I_{|n|+1}(y)} f(x)\frac{1}{r_{|n|}(y-x)-1}d\mu(x)\right| > \lambda/30\right)\\
&\le \mu\left(y : H_1 f(y) > \lambda/30\right) \le C\|f\|_1/\lambda.
\end{aligned}$$

The results for the kernels $A_4(z)$, $A_{5,1}(z)$ and $A_{5,2}(z)$ give the inequality $\mathbf{J}_{2,5} \le C\|f\|_1/\lambda$. That is, we proved that the maximal operator σ^* is of weak type $(1,1)$. □

Proof of Theorem 2. Theorem 1 and the standard density argument give that for all integrable function f on any rarely unbounded Vilenkin group the a.e. relation $\sigma_n f \to f$ $(n \to \infty)$ holds. □

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