

# ON THE DIVERGENCE OF NÖRLUND LOGARITHMIC MEANS OF WALSH-FOURIER SERIES

GYÖRGY GÁT AND USHANGI GOGINAVA

ABSTRACT. It is well known in the literature that the logarithmic means

$$\frac{1}{\log n} \sum_{k=1}^{n-1} \frac{S_k(f)}{k}$$

of Walsh or trigonometric Fourier series converge a.e. to the function for each integrable function on the unit interval. This is not the case if we take the partial sums. In this paper we prove that the behavior of the so-called Nörlund logarithmic means

$$\frac{1}{\log n} \sum_{k=1}^{n-1} \frac{S_k(f)}{n-k}$$

is closer to the properties of partial sums in this point of view.

## 1. INTRODUCTION

The question of almost everywhere convergence is highly celebrated in the theory of Fourier series. It is quite well known for both Walsh and trigonometric Fourier series, that the behavior of the logarithmic means

$$\frac{1}{\log n} \sum_{k=1}^{n-1} \frac{S_k(f)}{k}$$

is very nice. That is, for each function  $f$  which is integrable on the unit interval, the logarithmic means converge to  $f$  almost everywhere. That is, to take these means is a good idea, since for the partial sums there are divergence results. For instance for the Walsh system it is known [1] that for each measurable function  $\phi(u) = o(u\sqrt{\log u})$  there exists an integrable function  $f$  such that

$$\int_I \phi(|f(x)|) d\mu(x) < \infty,$$

and the Walsh-Fourier series of  $f$  diverges everywhere. The notion of the Nörlund logarithmic means is similar to the definition of the logarithmic means, the difference is that the denominators are taken in the reversed order. More precisely,

$$\frac{1}{\log n} \sum_{k=1}^{n-1} \frac{S_k(f)}{n-k}.$$

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The aim of this paper is to prove that these means are much closer to the partial sums than to the logarithmic means. In other words, we prove that in the function class above (see the result of Bochkarev) there exists a function and a set with positive measure, such that the Walsh-Nörlund logarithmic means of the function diverge on this set. This also says that not all classical summation methods improve the convergence properties of the partial sums - in this point of view. During the proof we apply some Bochkarev's idea from [1]. We also remark that the issue of divergence of the partial sums of Walsh-Fourier and Vilenkin-Fourier series of functions are discussed by several authors. See for instance [6, 7, 8, 9, 10].

## 2. DEFINITIONS AND NOTATIONS

Let  $\mathbf{P}$  denote the set of positive integers,  $\mathbf{N} := \mathbf{P} \cup \{0\}$ . Denote by  $Z_2$  the discrete cyclic group of order 2, that is  $Z_2 = \{0, 1\}$ , where the group operation is the addition modulo 2 and every subset is open. The Haar measure on  $Z_2$  is given such that the measure of a singleton is  $1/2$ . Let  $I$  be the complete direct product of the countable infinite copies of the compact groups  $Z_2$ . The elements of  $I$  are of the form  $x = (x_0, x_1, \dots, x_k, \dots)$  with  $x_k \in \{0, 1\}$  ( $k \in \mathbf{N}$ ). The group operation on  $I$  is the coordinate-wise addition, the measure (denote by  $\mu$ ) and the topology are the product measure and topology. The compact Abelian group  $I$  is called the Walsh group. A base for the neighborhoods of  $I$  can be given in the following way:

$$I_0(x) := I, \quad I_n(x) := I_n(x_0, \dots, x_{n-1}) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\},$$

$$(x \in I, n \in \mathbf{N}).$$

These sets are called the dyadic intervals. Let  $0 = (0 : i \in \mathbf{N}) \in I$  denote the null element of  $I$ ,  $I_n := I_n(0)$  ( $n \in \mathbf{N}$ ). Set  $e_n := (0, \dots, 0, 1, 0, \dots) \in I$  the  $n$ th coordinate of which is 1 and the rest are zeros ( $n \in \mathbf{N}$ ). Denote  $\bar{I}_n := I \setminus I_n$ .

For  $k \in \mathbf{N}$  and  $x \in I$  denote

$$r_k(x) := (-1)^{x_k}$$

the  $k$ -th Rademacher function. If  $n \in \mathbf{N}$ , then  $n = \sum_{i=0}^{\infty} n_i 2^i$ , where  $n_i \in \{0, 1\}$  ( $i \in \mathbf{N}$ ), i. e.  $n$  is expressed in the number system of base 2. Denote  $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$ , that is,  $2^{|n|} \leq n < 2^{|n|+1}$ .

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in I, n \in \mathbf{N}).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$(1) \quad D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \in \bar{I}_n. \end{cases}$$

Suppose that  $f$  is a Lebesgue integrable ( $f \in L(I)$ ) function on  $I$  and 1-periodic. Then its Walsh-Fourier series is defined by

$$\sum_{k=0}^{\infty} \hat{f}(k) w_k(x),$$

where

$$\hat{f}(k) = \int_I f(t) w_k(t) d\mu(t)$$

is called the  $k$ -th Walsh-Fourier coefficient of function  $f$ .

Denote the  $n$ -th partial sum of the Walsh-Fourier series of the function  $f$  by  $S_n(f, x)$ . Namely ,

$$S_n(f, x) = \sum_{k=0}^{n-1} \hat{f}(k) w_k(x).$$

The (Nörlund) logarithmic means (see for instance [3]) of the Walsh-Fourier series is defined as follows

$$t_n(f, x) = \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k(f, x)}{n-k},$$

where

$$l_n = \sum_{k=1}^{n-1} \frac{1}{k}.$$

It is evident that

$$t_n(f, x) - f(x) = \int_I [f(x+t) - f(x)] F_n(t) \mu(t),$$

where

$$F_n(t) = \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{D_k(t)}{n-k}.$$

For  $n \in \mathbf{P}$  and a function  $f$  the Fejér mean of order  $n$  of the Walsh-Fourier series of the function  $f$  is given by

$$\sigma_n(f, x) = \frac{1}{n} \sum_{i=1}^n S_i(f, x).$$

The Fejér kernel of order  $n$  of the Walsh-Fourier series defined by

$$K_n(x) := \frac{1}{n} \sum_{k=1}^n D_k(x).$$

The space weak- $L(I)$  consists of all measurable functions  $f$  for which

$$\|f\|_{\text{weak-}L(I)} := \sup_{\lambda > 0} \lambda \mu(|f| > \lambda) < +\infty.$$

## 3. MAIN RESULT

**Theorem 1.** *Let  $\varphi : [0, \infty) \rightarrow [0, \infty)$  be a function such that  $\varphi(u)/u$  is nondecreasing and  $\varphi(u) = o(u\sqrt{\log u})$ . Then there exists a function  $f \in L(I)$  and a measurable set  $E \subset I$  with positive measure for which*

$$\int_I \varphi(|f(x)|) d\mu(x) < \infty$$

and

$$\overline{\lim}_{n \rightarrow \infty} t_n(f, x) = +\infty, \quad x \in E.$$

## 4. AUXILIARY RESULTS

**Lemma 1.** [2] *Let  $n = 2^{n_1} + 2^{n_2} + \dots + 2^{n_r}$  with  $n_1 > n_2 > \dots > n_r \geq 0$ , let  $n^{(0)} = n$  and  $n^{(i)} = n^{(i-1)} - 2^{n_i}$ ,  $i = 1, \dots, r-1$ . Then*

$$\begin{aligned} l_n F_n^*(x) & : = w_n(x) l_n F_n(x) \\ & = \sum_{j=1}^{r-1} w_{2^{n_j}}(x) D_{2^{n_j}}(x) l_{n^{(j-1)}} \\ & \quad + \sum_{j=1}^{r-1} w_{n^{(j-1)}}(x) w_{2^{n_j-1}} \sum_{k=0}^{2^{n_j}-1} \frac{D_k(x)}{k+n^{(j)}} \\ & \quad + l_{2^{n_r}} F_{2^{n_r}}^*(x). \end{aligned}$$

**Lemma 2.** [4] *Let  $N \in \mathbb{N}$ . Then*

$$\int_{\bar{I}_N} \sup_{n \geq 2^N} |K_n(x)| d\mu(x) < \infty.$$

Denote

$$G_n(x) := \frac{w_n(x)}{l_n} \sum_{j=1}^{r-1} w_{n-n^{(j-1)}}(x) w_{2^{n_j-1}}(x) \sum_{k=1}^{2^{n_j}-1} \frac{D_k(x)}{k+n^{(j)}}.$$

**Lemma 3.** *Let  $N \in \mathbb{N}$ . Then*

$$\int_{\bar{I}_N} \sup_{n \geq 2^N} |G_n(x)| d\mu(x) < \infty.$$

**Proof.** Applying Abel's transform we have

$$\begin{aligned} & (2) \quad G_n(x) \\ & = \frac{w_n(x)}{l_n} \sum_{j=1}^{r-1} w_{n-n^{(j-1)}}(x) w_{2^{n_j-1}}(x) \left[ \sum_{k=1}^{2^{n_j}-2} \frac{k K_k(x)}{(k+n^{(j)})(k+n^{(j)}+1)} + \frac{(2^{n_j}-1) K_{2^{n_j}-1}(x)}{n^{(j-1)}-1} \right] \\ & = G_{n,1}(x) + G_{n,2}(x). \end{aligned}$$

First we discuss  $G_{n,1}(x)$ . Set  $K_T^* := \sup_{n \geq T} |K_n|$  and  $\tau := \max \{j : n^{(j)} > 2^N\}$ . We write

$$\begin{aligned}
(3) \quad G_{n,1}(x) &= \frac{w_n(x)}{l_n} \sum_{j=1}^{\tau} w_{n-n^{(j-1)}}(x) w_{2^{n_j-1}}(x) \sum_{k=1}^{2^N-1} \frac{k K_k(x)}{(k+n^{(j)})(k+n^{(j)}+1)} \\
&\quad + \frac{1}{l_n} \sum_{j=1}^{\tau} w_{n-n^{(j-1)}}(x) w_{2^{n_j-1}}(x) \sum_{k=2^N}^{2^{n_j}-2} \frac{k K_k(x)}{(k+n^{(j)})(k+n^{(j)}+1)} \\
&\quad + \frac{1}{l_n} \sum_{j=\tau+1}^{r-1} w_{n-n^{(j-1)}}(x) w_{2^{n_j-1}}(x) \sum_{k=1}^{2^{n_j}-2} \frac{k K_k(x)}{(k+n^{(j)})(k+n^{(j)}+1)} \\
&= G_{n,1,1}(x) + G_{n,1,2}(x) + G_{n,1,3}(x).
\end{aligned}$$

Since  $n^{(j)} > 2^N$  for  $j = 1, \dots, \tau$ , we have

$$\begin{aligned}
|G_{n,1,1}(x)| &\leq \frac{1}{l_n} \sum_{j=1}^{\tau} \sum_{k=1}^{2^N-1} \frac{k |K_k(x)|}{(k+n^{(j)})(k+n^{(j)}+1)} \\
&\leq \frac{1}{l_n} \sum_{j=1}^{\tau} \frac{1}{n^{(j)}} \sum_{k=1}^{2^N-1} |K_k(x)| \\
&\leq \frac{1}{2^N} \sum_{k=1}^{2^N-1} |K_k(x)|,
\end{aligned}$$

and consequently,

$$(4) \quad \int_{\bar{I}_N} \sup_{n \geq 2^N} |G_{n,1,1}(x)| d\mu(x) \leq \frac{c}{2^N} \sum_{k=1}^{2^N-1} \|K_k\|_1 \leq c < \infty.$$

It is evident that

$$\begin{aligned}
|G_{n,1,2}(x)| &\leq \frac{1}{l_n} \sum_{j=1}^{\tau} \sum_{k=2^N}^{2^{n_j}-1} \frac{k |K_k(x)|}{(k+n^{(j)})(k+n^{(j)}+1)} \\
&\leq \frac{K_{2^N}^*(x)}{l_n} \sum_{j=1}^{\tau} \sum_{k=2^N}^{2^{n_j}-1} \frac{1}{k+n^{(j)}} \\
&\leq \frac{K_{2^N}^*(x)}{l_n} \sum_{j=1}^{\tau} \log \frac{n^{(j-1)}}{n^{(j)}} \\
&\leq \frac{K_{2^N}^*(x)}{l_n} \log \frac{n}{n^{(\tau)}} \\
&\leq c K_{2^N}^*(x).
\end{aligned}$$

Hence from Lemma 2 we get

$$(5) \quad \int_{\bar{I}_N} \sup_{n \geq 2^N} |G_{n,1,2}(x)| d\mu(x) \leq \int_{\bar{I}_N} K_{2^N}^*(x) d\mu(x) \leq c < \infty.$$

For  $G_{n,1,3}(x)$  we write

$$\begin{aligned}
(6) \quad |G_{n,1,3}(x)| &\leq \frac{1}{l_n} \sum_{j=\tau+2}^{r-1} \sum_{k=1}^{2^{n_j}-1} \frac{k |K_k(x)|}{(k+n^{(j)})(k+n^{(j)}+1)} \\
&\quad + \frac{1}{l_n} \sum_{k=1}^{2^{n_{\tau+1}}-1} \frac{k |K_k(x)|}{(k+n^{(\tau+1)})(k+n^{(\tau+1)}+1)} \\
&= G_{n,1,3,1}(x) + G_{n,1,3,2}(x).
\end{aligned}$$

It is clear that

$$G_{n,1,3,2}(x) \leq \frac{1}{l_n} \sum_{k=1}^{2^{2^N}-1} \frac{|K_k(x)|}{k+1} + K_{2^N}^*(x).$$

Since  $\|K_j\| \leq c < \infty$ , from Lemma 2 we obtain

$$(7) \quad \int_{\bar{I}_N} \sup_{n \geq 2^N} |G_{n,1,3,2}(x)| d\mu(x) \leq c < \infty.$$

We write

$$\begin{aligned}
G_{n,1,3,1}(x) &\leq \frac{1}{l_n} \sum_{j=\tau+2}^{r-1} \sum_{k=1}^{2^{n_{j+1}}-1} \frac{k |K_k(x)|}{(k+2^{n_{j+1}})^2} + \frac{1}{l_n} \sum_{j=\tau+2}^{r-1} \sum_{k=2^{n_{j+1}}}^{2^{n_j}-1} \frac{k |K_k(x)|}{(k+2^{n_{j+1}})^2} \\
&\leq \frac{1}{l_n} \sum_{j=\tau+2}^{r-1} \frac{1}{2^{n_{j+1}}} \sum_{k=1}^{2^{n_{j+1}}-1} |K_k(x)| + \frac{1}{l_n} \sum_{j=\tau+2}^{r-1} \sum_{k=2^{n_{j+1}}}^{2^{n_j}-1} \frac{|K_k(x)|}{k} \\
&\leq \frac{1}{N} \sum_{j=0}^N \frac{1}{2^j} \sum_{k=1}^{2^j} |K_k(x)| + \frac{1}{N} \sum_{k=1}^{2^N} \frac{|K_k(x)|}{k}.
\end{aligned}$$

Consequently,

$$(8) \quad \int_{\bar{I}_N} \sup_{n \geq 2^N} |G_{n,1,3,1}(x)| d\mu(x) \leq \frac{c}{N} \sum_{j=0}^N \frac{1}{2^j} \sum_{k=1}^{2^j} \|K_k\|_1 + \frac{c}{N} \sum_{k=1}^{2^N} \frac{\|K_k\|_1}{k} \leq c < \infty.$$

>From (3)-(8) we get

$$(9) \quad \int_{\bar{I}_N} \sup_{n \geq 2^N} |G_{n,1}(x)| d\mu(x) \leq c < \infty.$$

The estimation of  $G_{n,2}(x)$  is analogous to the estimation of  $G_{n,1}(x)$  and we have

$$(10) \quad \int_{\bar{I}_N} \sup_{n \geq 2^N} |G_{n,2}(x)| d\mu(x) \leq c < \infty.$$

Combining (2), (9) and (10) we complete the proof of Lemma 3.

**Lemma 4.** *Let  $N \in \mathbf{N}$ . Then*

$$\int_{\bar{I}_N} \sup_{n \geq N} \frac{1}{l_n} \sup_{1 \leq s \leq n} \sum_{j=1}^{2^s-1} \frac{|K_j|}{j} \leq c < \infty.$$

**Proof.** It is evident

$$(11) \quad \begin{aligned} & \frac{1}{l_n} \sup_{1 \leq s \leq n} \sum_{j=1}^{2^s-1} \frac{|K_j|}{j} \leq \frac{1}{l_n} \sup_{1 \leq s \leq N} \sum_{j=1}^{2^s-1} \frac{|K_j|}{j} \\ & + \frac{1}{l_n} \sup_{N < s \leq n} \sum_{j=1}^{2^N-1} \frac{|K_j|}{j} + \frac{1}{l_n} \sup_{N < s \leq n} \sum_{j=2^N}^{2^s-1} \frac{|K_j|}{j} = I + II + III. \end{aligned}$$

Consequently,

$$(12) \quad I, II \leq \frac{c}{N} \sum_{j=1}^{2^N-1} \frac{|K_j|}{j},$$

$$(13) \quad III \leq \frac{1}{l_n} \left( \sup_{N < s \leq n} \sum_{j=2^N}^{2^s-1} \frac{1}{j} \right) K_{2^N}^* \leq cK_{2^N}^*.$$

Combining (11)-(13) from Lemma 2 we obtain the proof of Lemma 4.

Analogously, we can prove

**Lemma 5.** *Let  $N \in \mathbf{N}$ . Then*

$$\int_{\bar{I}_N} \sup_{n \geq N} \frac{1}{l_n} \sup_{1 \leq s \leq n} |K_{2^s}| \leq c < \infty.$$

**Lemma 6.** *Let  $f \in L_\infty(I)$ . Then*

$$\left\| \sup_n |f * G_n| \right\|_\infty \leq C \|f\|_\infty.$$

**Proof.** Since  $\|K_j\|_1 \leq C < \infty$  and

$$|G_n(x)| \leq \frac{C}{\log n} \sum_{j=1}^{r-1} \sum_{k=1}^{2^{n_j}-2} \frac{k |K_k(x)|}{(k + n^{(j)})^2} + \frac{C}{\log n} \sum_{j=1}^{r-1} |K_{2^{n_j-1}}(x)|$$

we have

$$\begin{aligned} \|G_n\|_1 & \leq \frac{C}{\log n} \sum_{j=1}^{r-1} \sum_{k=1}^{2^{n_j}-2} \frac{k}{(k + n^{(j)})^2} \|K_k\|_1 + \frac{C}{\log n} \sum_{j=1}^{r-1} \|K_{2^{n_j-1}}\|_1 \\ & \leq \frac{C}{\log n} \sum_{j=1}^{r-1} \sum_{k=1}^{2^{n_j}-2} \left( \frac{k}{k + n^{(j)}} - \frac{n^{(j)}}{(k + n^{(j)})^2} \right) + C \leq C < \infty, \end{aligned}$$

which means that the maximal operator  $\sup_n |f * G_n|$  is of type  $(\infty, \infty)$ .

Lemma 6 is proved.

**Lemma 7.** ([5]) Let  $f \in L_\infty(I)$ . Then

$$\left\| \sup_n |f * F_{2^n}| \right\|_\infty \leq C \|f\|_\infty.$$

**Lemma 8.** ([4]) Let  $f, Q_n \in L(I)$ ,  $n \in \mathbf{P}$  for which

- 1)  $\int_{I_N} f = 0$ ,  $\text{supp}(f) \subset I_N$ ;
- 2)  $f * Q_n = 0$ ,  $n < 2^N$ ;
- 3)  $\int_{\bar{I}_N} \sup_{n \geq 2^N} |Q_n| \leq c < \infty$ .

If  $T := \sup_{n \in \mathbf{P}} |f * Q_n|$  is bounded from  $L_\infty(I)$  to  $L_\infty(I)$ , then

$$\|Tf\|_{\text{weak-L}} \leq C \|f\|_1, \quad (f \in L(I)).$$

**Lemma 9.** Let the operator  $T$  be of weak type  $(1,1)$ . Then  $|Tf(x)| < \infty$  a. e. ( $f \in L(I)$ ).

**Proof.** Denote

$$\Omega_n := \{x \in I : |Tf(x)| > n\}$$

and

$$\Omega := \bigcap_{n=1}^{\infty} \Omega_n.$$

Since

$$\Omega_{n+1} \subset \Omega_n$$

and

$$\mu(\Omega_n) \leq \frac{C}{n} \|f\|_1 = O\left(\frac{1}{n}\right),$$

we can write

$$\mu(\Omega) = \lim_{n \rightarrow \infty} \mu(\Omega_n) = 0,$$

which means that  $|Tf(x)| < \infty$  a. e.

Lemma 9 is proved.

## 5. PROOF OF THE MAIN RESULT

**Proof of Theorem 1.** Denote

$$L_n(x) := \frac{w_n(x)}{l_n} \sum_{j=1}^{r-1} w_{2^{n_j}}(x) D_{2^{n_j}}(x) l_{n^{(j-1)}},$$

then from Lemma 1 we have

$$(14) \quad t_n(f) := f * F_n = f * L_n + f * G_n + f * \left( \frac{l_{2^{n_r}}}{l_n} w_n w_{2^{n_r}} F_{2^{n_r}} \right).$$

First we prove that ( $f \in L(I)$ )

$$(15) \quad \sup_n \left| f * \left( \frac{l_{2^{n_r}}}{l_n} w_n w_{2^{n_r}} F_{2^{n_r}} \right) \right| < \infty \quad \text{a. e.}$$

and

$$(16) \quad \sup_n |f * G_n| < \infty \quad \text{a. e.}$$

Since

$$\begin{aligned} l_{2^{nr}} F_{2^{nr}} &= \sum_{j=1}^{2^{nr}-1} \frac{D_{2^{nr}-j}}{j} = l_{2^{nr}} D_{2^{nr}} - w_{2^{nr}-1} \sum_{j=1}^{2^{nr}-1} \frac{D_j}{j} \\ &= l_{2^{nr}} D_{2^{nr}} - w_{2^{nr}-1} \sum_{j=1}^{2^{nr}-2} \frac{K_j}{j} - w_{2^{nr}-1} K_{2^{nr}-1}, \end{aligned}$$

we can write

$$\begin{aligned} & f * \left( \frac{l_{2^{nr}}}{l_n} w_n w_{2^{nr}} F_{2^{nr}} \right) \\ &= f * \left( \frac{l_{2^{nr}}}{l_n} w_n w_{2^{nr}} D_{2^{nr}} \right) \\ &- f * \left( \frac{l_{2^{nr}}}{l_n} w_n w_{2^{nr}} w_{2^{nr}-1} \sum_{j=1}^{2^{nr}-2} \frac{K_j}{j} \right) \\ &- f * \left( \frac{l_{2^{nr}}}{l_n} w_n w_{2^{nr}} w_{2^{nr}-1} K_{2^{nr}-1} \right) \\ (17) \quad &= A_n - B_n - C_n. \end{aligned}$$

>From (1) we obtain

$$\sup_n |A_n| \leq \sup_n 2^n \int_{I_n(x)} |f(t)| d\mu(t).$$

Hence

$$(18) \quad \left\| \sup_n |A_n| \right\|_{weak-L} \leq C \|f\|_1.$$

>From Lemmas 3-8 we get that

$$(19) \quad \left\| \sup_n |B_n| \right\|_{weak-L} \leq C \|f\|_1.$$

$$(20) \quad \left\| \sup_n |C_n| \right\|_{weak-L} \leq C \|f\|_1.$$

Combining (17)-(20) by Lemma 9 we obtain (15) and (16).

Denote

$$E_N := \bigcup_{x_0=0}^1 \cdots \bigcup_{x_{2N-1}=0}^1 \bigcup_{\tilde{C}N \leq x_{2N} + \cdots + x_{3N-1} \leq N/2} I_{3N+1}(x_0, \dots, x_{3N-1}, 1 - x_{2N}),$$

where  $\tilde{C} < 1/2$  is some positive constant discussed later.

It is evident that

$$(21) \quad I_{3N+1}(x_0, \dots, x_{3N-1}, 1 - x_{2N}) \cap I_{4N}(x_0, \dots, x_{2N-1}, x_{2N}, \dots, x_{3N-1}, x_{2N}, \dots, x_{3N-1}) = \emptyset$$

and

$$\mu(E_N) \geq C > 0.$$

Denote

$$\tilde{E} := \bigcap_{k=1}^{\infty} \bigcup_{N=k}^{\infty} E_N.$$

Then it is easy to show that  $\mu(\tilde{E}) \geq C > 0$ .

Let  $x \in \tilde{E}$  and  $\{N_v : v \geq 1\}$  is such that  $x \in E_{N_v}$ ,  $v = 1, 2, \dots$  and  $\{M_a : a \geq 1\}$  is subsequence of  $\{N_v : v \geq 1\}$  for which

$$(22) \quad \begin{aligned} \varphi(2^{4M_1}) &> 1, \\ M_a &> 2M_{a-1}, \end{aligned}$$

$$(23) \quad \frac{\varphi(2^{4M_a})}{2^{4M_a} \sqrt{M_a}} \leq \frac{1}{8^a 2^{4M_{a-1}}},$$

$$(24) \quad \sum_{k=1}^{a-1} \frac{2^{8M_k}}{4^k 2^{4M_{k-1}} \varphi(2^{4M_k})} < \frac{2^{4M_a} \sqrt{M_a}}{8^a 2^{4M_{a-1}} \varphi(2^{4M_a})}.$$

Consider the function  $p_{M_a}$ , defined by

$$\begin{aligned} &p_{M_a}(t) \\ : &= \begin{cases} \frac{2^{M_a}}{\sqrt{M_a}} \sum_{j=2M_a}^{3M_a-1} w_{2^j}(t), & t \in \bigcup_{x_0=0}^1 \cdots \bigcup_{x_{3M_a-1}=0}^1 I_{4M_a}(x_0, \dots, x_{3M_a-1}, x_{2M_a}, \dots, x_{3M_a-1}) \\ 0, & \text{otherwise,} \end{cases} \\ &p_{M_a}(t+l) = p_{M_a}(t), \quad l = 0, \mp 1, \mp 2, \dots \end{aligned}$$

It is easy to show that

$$\begin{aligned} \|p_{M_a}\|_1 &= \frac{2^{M_a}}{\sqrt{M_a}} \bigcup_{x_0=0}^1 \cdots \bigcup_{x_{3M_a-1}=0}^1 \int_{I_{4M_a}(x_0, \dots, x_{3M_a-1}, x_{2M_a}, \dots, x_{3M_a-1})} \left| \sum_{j=2M_a}^{3M_a-1} w_{2^j}(t) \right| d\mu(t) \\ &= \frac{1}{\sqrt{M_a}} \bigcup_{x_0=0}^1 \cdots \bigcup_{x_{3M_a-1}=0}^1 \int_{I_{3M_a}(x_0, \dots, x_{2M_a-1}, x_{2M_a}, \dots, x_{3M_a-1})} \left| \sum_{j=2M_a}^{3M_a-1} w_{2^j}(t) \right| d\mu(t) \\ &= \frac{1}{\sqrt{M_a}} \int_I \left| \sum_{j=2M_a}^{3M_a-1} w_{2^j}(t) \right| d\mu(t) \leq \frac{1}{\sqrt{M_a}} \left( \int_I \left| \sum_{j=2M_a}^{3M_a-1} w_{2^j}(t) \right|^2 d\mu(t) \right)^{1/2} = 1. \end{aligned}$$

Let

$$f_0(x) := \sum_{a=1}^{\infty} c_a p_{M_a}(x),$$

where

$$c_a := \frac{2^{4M_a}}{4^a 2^{4M_{a-1}} \varphi(2^{4M_a})}.$$

Denote

$$n_{a(x)} := s_{a(x)} + q_{a(x)},$$

where

$$s_{a(x)} := \sum_{j=2M_a}^{3M_a-1} (1-x_j) 2^j, \quad q_{a(x)} := \sum_{j=3M_a}^{4M_a-1} (1-x_{j-M_a}) 2^j.$$

Then it is easy to show that

$$(25) \quad w_{n_{a(x)}}(t) = w_{s_{a(x)}}(t) w_{q_{a(x)}}(t) = 1 \quad \text{for } t \in \text{supp } p_{M_a}.$$

We write

$$(26) \quad f_0 * L_{n_{a(x)}} = c_a p_{M_a} * L_{n_{a(x)}} + \sum_{k=1}^{a-1} c_k p_{M_k} * L_{n_{a(x)}} + \sum_{k=a+1}^{\infty} c_k p_{M_k} * L_{n_{a(x)}}.$$

Let  $n := \sum_{i=0}^{\infty} n_i 2^i$  and denote  $n(j) := \sum_{i=0}^j n_i 2^i$ . Then for  $k < a$ , we have

$$(27) \quad \begin{aligned} & p_{M_k} * l_{n_{a(x)}} L_{n_{a(x)}} \\ &= \int_I p_{M_k}(t) l_{n_{a(x)}} L_{n_{a(x)}}(x+t) d\mu(t) \\ &= \int_I p_{M_k}(t) w_{n_{a(x)}}(x+t) l_{n_{a(x)}} L_{n_{a(x)}}^*(x+t) d\mu(t) \\ &= \sum_{j=2M_a}^{3M_a-1} (1-x_j) l_{n_{a(x)}(j)} \int_I p_{M_k}(t) w_{n_{a(x)}}(x+t) w_{2^j}(x+t) D_{2^j}(x+t) d\mu(t) \\ &\quad + \sum_{j=3M_a}^{4M_a-1} (1-x_{j-M_a}) l_{n_{a(x)}(j)} \int_I p_{M_k}(t) w_{n_{a(x)}}(x+t) w_{2^j}(x+t) D_{2^j}(x+t) d\mu(t) \\ &= (1-x_{\widehat{M}_a-M_a}) l_{n_{a(x)}(\widehat{M}_a)} \int_I p_{M_k}(t) w_{n_{a(x)}}(x+t) w_{2^{\widehat{M}_a}}(x+t) D_{2^{\widehat{M}_a}}(x+t) d\mu(t), \end{aligned}$$

where

$$\widehat{M}_a := \max \{j : 2M_a \leq j < 3M_a, x_j = 0\}.$$

It is evident that

$$\int_I p_{M_k}(t) w_{n_{a(x)}}(t) w_{2^{\widehat{M}_a}}(t) w_l(t) dt \neq 0,$$

implies that

$$l = n_{a(x)} - 2^{\widehat{M}_a} + s, \quad 0 \leq s < 2^{4M_k}.$$

Consequently from (27), we can write

$$(28) \quad \left| p_{M_k} * L_{n_{a(x)}} \right| \leq \frac{CM_a 2^{4M_k}}{M_a} \|p_{M_k}\|_1 \leq C 2^{4M_k}.$$

For  $k > a$ , we have

$$(29) \quad \left| p_{M_k} * L_{n_{a(x)}} \right| \leq C n_{a(x)} \|p_{M_k}\|_1 \leq C 2^{4M_a}.$$

Since (see(21))

$$(30) \quad x + t \notin I_{3N+1} \quad \text{for} \quad x \in E_N \quad \text{and} \quad t \in \text{supp}(p_N),$$

from (1) and (25) we have

$$\begin{aligned} p_{M_a} * l_{n_{a(x)}} L_{n_{a(x)}} &= \int_I p_{M_a}(t) l_{n_{a(x)}} L_{n_{a(x)}}(x+t) d\mu(t) \\ &= \int_I p_{M_a}(t) w_{n_{a(x)}}(x+t) l_{n_{a(x)}} L_{n_{a(x)}}^*(x+t) d\mu(t) \\ &= w_{n_{a(x)}}(x) \int_I p_{M_a}(t) l_{n_{a(x)}} L_{n_{a(x)}}^*(x+t) d\mu(t) \\ &= w_{n_{a(x)}}(x) \sum_{j=2M_a}^{3M_a-1} (1-x_j) l_{n_{a(x)}(j)} \int_I p_{M_a}(t) D_{2^j}^*(x+t) d\mu(t) \\ &+ w_{n_{a(x)}}(x) \sum_{j=3M_a}^{4M_a-1} (1-x_{j-M_a}) l_{n_{a(x)}(j)} \int_I p_{M_a}(t) D_{2^j}^*(x+t) d\mu(t) \\ &= w_{n_{a(x)}}(x) \sum_{j=2M_a}^{3M_a-1} (1-x_j) l_{n_{a(x)}(j)} \int_I p_{M_a}(t) D_{2^j}^*(x+t) d\mu(t) \\ &+ w_{n_{a(x)}}(x) (1-x_{M_a}) l_{n_{a(x)}(3M_a)} \int_I p_{M_a}(t) D_{2^{3M_a}}^*(x+t) d\mu(t) \\ &= \frac{2^{M_a}}{\sqrt{M_a}} w_{n_{a(x)}}(x) \sum_{j=2M_a}^{3M_a-1} (1-x_j) l_{n_{a(x)}(j)} \bigcup_{s_0=0}^1 \cdots \bigcup_{s_{3M_a-1}=0}^1 \\ &\quad \int_{I_{4M_a}(s_0, \dots, s_{2M_a-1}, s_{2M_a}, \dots, s_{3M_a-1}, s_{2M_a}, \dots, s_{3M_a-1})} \left( \sum_{k=2M_a}^{3M_a-1} w_{2^k}(t) \right) D_{2^j}^*(x+t) d\mu(t) \\ &+ w_{n_{a(x)}}(x) (1-x_{M_a}) l_{n_{a(x)}(3M_a)} \int_I p_{M_a}(t) w_{2^{3M_a}}(x+t) D_{2^{3M_a}}^*(x+t) d\mu(t) =: B \end{aligned}$$

We have  $x \in E_{M_a}$ . It means that

$$x_{3M_a} = 1 - x_{2M_a}.$$

$D_{2^{3M_a}}^*(x+t) \neq 0$  implies that  $t_0 = x_0, \dots, t_{3M_a-1} = x_{3M_a-1}$ . On the other hand,  $t \in \text{supp} p_{M_a}(t)$  and

$$t_{3M_a} = x_{2M_a}.$$

Consequently,

$$w_{2^{3M_a}}(x+t) = (-1)^{x_{3M_a} + t_{3M_a}} = -1.$$

By the help of this we get for  $B$

$$\begin{aligned}
B &= \frac{1}{\sqrt{M_a}} w_{n_a(x)}(x) \sum_{j=2M_a}^{3M_a-1} (1-x_j) l_{n_a(x)(j)} \bigcup_{s_0=0}^1 \cdots \bigcup_{s_{3M_a-1}=0}^1 \\
&\quad \int_{I_{3M_a}(s_0, \dots, s_{3M_a-1})} \left( \sum_{k=2M_a}^{3M_a-1} w_{2^k}(t) \right) D_{2^j}^*(x+t) d\mu(t) \\
&\quad - w_{n_a(x)}(x) (1-x_{M_a}) l_{n_a(x)(3M_a)} \int_I p_{M_a}(t) D_{2^{3M_a}}(x+t) d\mu(t) \\
&= \frac{1}{\sqrt{M_a}} w_{n_a(x)}(x) \sum_{j=2M_a}^{3M_a-1} (1-x_j) l_{n_a(x)(j)} \int_I \left( \sum_{k=2M_a}^{3M_a-1} w_{2^k}(t) \right) D_{2^j}^*(x+t) d\mu(t) \\
&\quad - w_{n_a(x)}(x) (1-x_{M_a}) l_{n_a(x)(3M_a)} \frac{1}{\sqrt{M_a}} \sum_{j=2M_a}^{3M_a-1} (-1)^{x_j}.
\end{aligned}$$

Since

$$\sum_{j=2M_a}^{3M_a-1} (-1)^{x_j} \leq M_a (1 - 2\tilde{C}), x \in E_{M_a}$$

and

$$l_{n_a(x)(j)} \geq Cj(1-x_j)$$

we have

$$\begin{aligned}
(31) \quad & \left| p_{M_a} * L_{n_a(x)} \right| \\
& \geq \frac{1}{l_{n_a(x)} \sqrt{M_a}} \sum_{j=2M_a}^{3M_a-1} (1-x_j) l_{n_a(x)(j)} (-1)^{x_j} - \frac{l_{n_a(x)(3M_a)}}{l_{n_a(x)}} \frac{1}{\sqrt{M_a}} \sum_{j=2M_a}^{3M_a-1} (-1)^{x_j} \\
& \geq \frac{1}{l_{n_a(x)} \sqrt{M_a}} \sum_{j=2M_a}^{3M_a-1} (1-x_j) j (-1)^{x_j} - \frac{(1-2\tilde{C}) M_a}{\sqrt{M_a}} \\
& \geq \left( C - (1-2\tilde{C}) \right) \sqrt{M_a}.
\end{aligned}$$

Let  $1 - 2\tilde{C} < C/2$ . Then from (31) we obtain

$$(32) \quad \left| p_{M_a} * L_{n_a(x)} \right| \geq C/2 \sqrt{M_a}.$$

Combining (24), (26), (28), (29), (32) and using the fact that  $\varphi(u)/u$  is nondecreasing for  $x \in E_{n_a}$  we obtain that

$$\begin{aligned}
(33) \quad & \left| f_0 * L_{n_a(x)} \right| \\
& \geq c_a \left| p_{M_a} * L_{n_a(x)} \right| - \sum_{k=1}^{a-1} c_k \left| p_{M_k} * L_{n_a(x)} \right| - \sum_{k=a+1}^{\infty} c_k \left| p_{M_k} * L_{n_a(x)} \right| \\
& \geq \frac{2^{4M_a} \sqrt{M_a}}{4^a 2^{4M_{a-1}} \varphi(2^{4M_a})} - C \sum_{k=1}^{a-1} \frac{2^{4M_k}}{4^k 2^{4M_{k-1}} \varphi(2^{4M_k})} 2^{4M_k} - C \sum_{k=a+1}^{\infty} \frac{2^{4M_k}}{4^k 2^{4M_{k-1}} \varphi(2^{4M_k})} 2^{4M_a} \\
& \geq \frac{C 2^{4M_a} \sqrt{M_a}}{4^a 2^{4M_{a-1}} \varphi(2^{4M_a})} - C \sum_{k=a+1}^{\infty} \frac{2^{8M_k}}{4^k 2^{4M_{k-1}} \varphi(2^{4M_k})} \\
& \geq \frac{C 2^{4M_a} \sqrt{M_a}}{4^a 2^{4M_{a-1}} \varphi(2^{4M_a})} - C \sum_{k=a+1}^{\infty} \frac{2^{4M_{k-1}}}{4^k 2^{4M_{k-1}} \varphi(2^{4M_{k-1}})} 2^{4M_a} \\
& \geq \frac{C 2^{4M_a} \sqrt{M_a}}{4^a 2^{4M_{a-1}} \varphi(2^{4M_a})} - C \\
& \geq C 4^a.
\end{aligned}$$

Let  $E_0$  be the set for which (15) and (16) does not hold. Denote  $E := \tilde{E} \setminus E_0$ . Then it is evident that  $\mu(E) > 0$ . Let  $x \in E$ . Then combining (14)-(16) and (33) we conclude that

$$\overline{\lim}_{n \rightarrow \infty} |t_n(f_0, x)| = +\infty \quad (x \in E).$$

Finally, we prove that

$$(34) \quad \int_I \varphi(|f(x)|) d\mu(x) < \infty.$$

Since

$$\sum_k u_k \leq \sup_k 2^k u_k,$$

we write

$$\varphi\left(\sum_k u_k\right) \leq \varphi\left(\sup_k 2^k u_k\right) \leq \sum_k \varphi(2^k u_k),$$

consequently,

$$\int_I \varphi(|f(x)|) dx \leq \sum_{a=1}^{\infty} \int_I \varphi(2^a c_a |P_{M_a}(x)|) dx.$$

Since  $\varphi(u)/u$  is nondecreasing, we obtain that

$$\begin{aligned}
& \int_I \varphi(2^a c_a |P_{M_a}(x)|) d\mu(x) \\
& \leq 2^a c_a \int_I \varphi(|P_{M_a}(x)|) d\mu(x) \\
& \leq 2^a c_a \int_I \frac{\varphi(2^{4M_a})}{2^{4M_a}} |P_{M_a}(x)| d\mu(x) \\
& \leq 2^a c_a \frac{\varphi(2^{4M_a})}{2^{4M_a}} \\
& \leq 2^a \frac{2^{4M_a}}{4^a 2^{4M_{a-1}} \varphi(2^{4M_a})} \frac{\varphi(2^{4M_a})}{2^{4M_a}} \\
& \leq C \frac{1}{2^a}.
\end{aligned}$$

Consequently we can write

$$\int_I \varphi(|f(x)|) d\mu(x) \leq C \sum_{a=1}^{\infty} \frac{1}{2^a} < \infty.$$

Theorem 1 is proved.

**Remark 1.** We note that Theorem 1 is true in the case of  $\varphi(u)/u$  is not nondecreasing.

Indeed, let  $\varphi : [0, \infty) \rightarrow [0, \infty)$  is arbitrary function. Define

$$\tilde{\varphi}(u) := \begin{cases} \varphi(1)u, & 0 \leq u \leq 1 \\ \sup_{1 \leq v \leq u} \frac{\varphi(v)}{v}u, & u > 1. \end{cases}$$

Then it is evident that

$$(35) \quad \tilde{\varphi}(u) \geq \varphi(u), \quad u > 1.$$

On the other hand ( $u > 1$ ),

$$(36) \quad \frac{\tilde{\varphi}(u)}{u\sqrt{\log u}} = \frac{\sup_{1 \leq v \leq u} \frac{\varphi(v)}{v}u}{u\sqrt{\log u}} = \frac{\sup_{1 \leq v \leq u} \frac{\varphi(v)}{v}}{\sqrt{\log u}}.$$

Let  $\varepsilon > 0$ . Then from the condition of Theorem 1, there exists an  $u(\varepsilon)$  such that for every  $u > u(\varepsilon)$  we have

$$\varphi(u) \leq \varepsilon u \sqrt{\log u}.$$

Then from (36) we can write

$$\frac{\tilde{\varphi}(u)}{u\sqrt{\log u}} \leq \frac{\sup_{1 \leq v \leq u(\varepsilon)} \frac{\varphi(v)}{v}}{\sqrt{\log u}} + \frac{\varepsilon \sup_{u(\varepsilon) < v \leq u} \sqrt{\log v}}{\sqrt{\log u}}.$$

Consequently,

$$\tilde{\varphi}(u) = o\left(u\sqrt{\log u}\right).$$

Since  $\tilde{\varphi}(u)/u$  is nondecreasing, from Theorem 1 there exists a function  $f_0$  and set  $E \subset I$  with positive measure, for which

$$\int_I \tilde{\varphi}(|f(x)|) d\mu(x) < \infty$$

and

$$\overline{\lim}_{n \rightarrow \infty} |t_n(f_0, x)| = +\infty \quad (x \in E) .$$

On the other hand from (35), we write

$$\int_I \varphi(|f(x)|) d\mu(x) \leq \int_I \tilde{\varphi}(|f(x)|) d\mu(x) < \infty.$$

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G. GÁT, INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE, COLLEGE OF NYÍREGYHÁZA, P.O. BOX 166, NYIREGYHÁZA, H-4400 HUNGARY

*E-mail address:* gatgy@zeus.nyf.hu

U. GOGINAVA, DEPARTMENT OF MECHANICS AND MATHEMATICS, TBILISI STATE UNIVERSITY, CHAVCHAVADZE STR. 1, TBILISI 0128, GEORGIA

*E-mail address:* z\_goginava@hotmail.com