

NORM SUMMABILITY OF NÖRLUND LOGARITHMIC MEANS ON UNBOUNDED VILENKIN GROUPS

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ABSTRACT. The (Nörlund) logarithmic means of the Fourier series is:

$$t_n f = \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k f}{n-k}, \text{ where } l_n = \sum_{k=1}^{n-1} \frac{1}{k}.$$

In general, the Fejér $(C, 1)$ means have better properties, than the logarithmic ones. We compare them, and we show that in the case of some unbounded Vilenkin systems the situation changes.

1. Introduction

Introduce the necessary notations and definitions.

Let $\mathbb{P} = \mathbb{N} \setminus \{0\}$, and let $m := (m_0, m_1, \dots)$ denote a sequence of positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, \dots, m_k - 1\}$ the additive group of integers modulo m_k .

Define the set G_m as the complete direct product of the sets Z_{m_j} , with the product of the discrete topologies of Z_{m_j} 's.

The direct product μ of the measures

$$\mu_k(\{j\}) := \frac{1}{m_k} \quad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

If the sequence m is bounded, then G_m is called a bounded Vilenkin group, else its name is an unbounded one. The elements of G_m can be represented by sequences $x := (x_0, x_1, \dots, x_j, \dots)$ ($x_j \in Z_{m_j}$). It is easy to give a base for the neighborhoods of G_m :

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$$

for $x \in G_m$, $n \in \mathbb{N}$. Define $I_n := I_n(0)$ for $n \in \mathbb{P}$.

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If we define the so-called generalized number system based on m in the following way: $M_0 := 1, M_{k+1} := m_k M_k (k \in \mathbb{N})$, then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in G_{m_j} (j \in \mathbb{P})$

and only a finite number of n_j 's differ from zero. We use the following notations. Let (for $n > 0$) $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$), $n^{(k)} = \sum_{j=k}^{\infty} n_j M_j$ and $n_{(k)} := n - n^{(k)}$.

Denote by $L^p(G_m)$ the usual (one dimensional) Lebesgue spaces ($\|\cdot\|_p$ the corresponding norms) ($1 \leq p \leq \infty$).

Next we introduce on G_m an orthonormal system which is called Vilenkin system. At first define the complex valued functions $r_k(x) : G_m \rightarrow \mathbb{C}$, the Generalized Rademacher functions in this way

$$r_k(x) := \exp \frac{2\pi i x_k}{m_k} \quad (i^2 := -1, x \in G_m, k \in \mathbb{N}).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as follows.

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specially, we call this system the Walsh-Paley one if $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L^1(G_m)$ [10].

Now, introduce the usual definitions of the Fourier-analysis. If $f \in L^1(G_m)$ we can define the following definitions in the usual way:

Fourier coefficients:

$$\widehat{f}(k) := \int_{G_m} f \bar{\psi}_k d\mu \quad (k \in \mathbb{N}),$$

partial sums:

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k \quad (n \in \mathbb{P}, S_0 f := 0),$$

Fejér means:

$$\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_k f \quad (n \in \mathbb{P}),$$

Dirichlet kernels:

$$D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in \mathbb{P}),$$

Fejér kernels:

$$K_n := \frac{1}{n} \sum_{k=0}^{n-1} D_k \quad (n \in \mathbb{P}).$$

We use a further notation:

$$K_{a,b} := \sum_{j=a}^{a+b-1} D_j \quad (a, b \in \mathbb{N}).$$

2. Approximation in norm

Review some approximation results with respect to the Vilenkin system. It is well-known that the partial sums converge to the function in norm. That is,

$$\|S_n f - f\|_p \rightarrow 0$$

for all $f \in L^p$, where $1 < p < \infty$ [8]. This is not the case if $p = 1$ or $p = \infty$.

Moreover, if we use the partial sequence M_n it is also well-known ([1]) the convergence in the supremum norm for continuous functions, and in the Lebesgue norm L^1 for functions $f \in L^1$

$$\|S_{M_n} f - f\| \rightarrow 0.$$

The properties of the convergence are better using the Fejér means on bounded Vilenkin system [9]

$$\|\sigma_n f - f\|_p \rightarrow 0 \quad \forall f \in L^p \quad 1 \leq p < \infty$$

and in the supremum norm for continuous functions. On arbitrary Vilenkin system (it is a trivial consequence of the convergence of the partial sums)

$$\|\sigma_n f - f\|_p \rightarrow 0 \quad \forall f \in L^p \quad 1 < p < \infty.$$

On the other hand, in the case of unbounded Vilenkin systems and $p = 1$ the theorem above does not hold. We have the same situation in the case of the supremum norm and continuous functions.

What is the situation regarding the Fejér means in the case of special subsequence M_n ? It is interesting, that $\forall m$ unbounded sequence $\exists f \in L^1$ such that

$$\|\sigma_{M_n} f - f\|_1 \not\rightarrow 0$$

[7], but on every Vilenkin system

$$\sigma_{M_n} f \rightarrow f \quad \text{a.e. } \forall f \in L^1$$

[2]. Moreover, there exists a continuous function such that $\sigma_{M_n} f$ does not converge to f in the supremum norm [7].

3. Nörlund logarithmic means

Define the Nörlund means, in general. If $q_k \geq 0$ ($k \in \mathbb{N}$),

$$\frac{1}{Q_n} \sum_{k=1}^{n-1} q_{n-k} S_k f, \quad \text{where } Q_n = \sum_{k=1}^{n-1} q_k.$$

Móricz discussed in [6] the norm convergence on Walsh-Paley system with respect to some sequence q . The case $q_k = \frac{1}{k}$ is excluded in this paper. Among others, that is why this case, the so-called Nörlund

logarithmic means is also interesting to be discussed. The concept of Nörlund logarithmic means is as follows

$$t_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k f}{n-k}, \text{ where } l_n := \sum_{k=1}^{n-1} \frac{1}{k}.$$

It is evident that

$$t_n f = f * F_n$$

where

$$F_n = \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{D_k}{n-k}.$$

For further information with respect to Nörlund logarithmic means on Walsh-Paley system see [5].

In their paper Gát, G., Goginava, U., [4] proved (for Walsh-Paley system), that $\exists f \in L^1$

$$\|t_n f - f\|_1 \not\rightarrow 0.$$

On the other hand, the main aim of this paper is to prove that the Nörlund logarithmic means have better approximation properties on some unbounded Vilenkin groups, than the Fejér means.

4. New results on Vilenkin systems

Lemma 1. Let $1 \leq j \leq M_k - 1$. Then

$$D_{M_k-j}(x) = D_{M_k}(x) - \bar{\psi}_{M_k-1}(-x) D_j(-x).$$

Proof. Using the well known equalities $\psi_{n \oplus m} = \psi_n \psi_m$, $\psi_{n \ominus m} = \psi_n / \psi_m$, $\psi_n(x) = \bar{\psi}_n(-x)$ and $\psi_n \bar{\psi}_n = 1$ as well as the simple fact $(M_k - 1) \ominus \nu = M_k - 1 - \nu$ for $\nu = 0, 1, \dots, M_k - 1$ we can get

$$\begin{aligned} D_{M_k-j}(x) &= D_{M_k}(x) - \sum_{\nu=M_k-j}^{M_k-1} \psi_\nu(x) \\ &= D_{M_k}(x) - \bar{\psi}_{M_k-1}(-x) \sum_{\nu=M_k-j}^{M_k-1} \psi_{M_k-1}(-x) \bar{\psi}_\nu(-x) \\ &= D_{M_k}(x) - \bar{\psi}_{M_k-1}(-x) \sum_{\nu=M_k-j}^{M_k-1} \psi_{M_k-1}(-x) / \psi_\nu(-x) \\ &= D_{M_k}(x) - \bar{\psi}_{M_k-1}(-x) \sum_{\nu=M_k-j}^{M_k-1} \psi_{(M_k-1) \ominus \nu}(-x) \\ &= D_{M_k}(x) - \bar{\psi}_{M_k-1}(-x) \sum_{\nu=M_k-j}^{M_k-1} \psi_{M_k-1-\nu}(-x) \end{aligned}$$

$$\begin{aligned}
 &= D_{M_k}(x) - \bar{\psi}_{M_k-1}(-x) \sum_{\nu=0}^{j-1} \psi_\nu(-x) \\
 &= D_{M_k}(x) - \bar{\psi}_{M_k-1}(-x) D_j(-x).
 \end{aligned}$$

□

The next two lemmas can be interesting even in themselves, because it was not known any general estimate of the Fejér kernel functions before for unbounded Vilenkin systems.

Lemma 2. If $\log m_n = O(n^\delta)$, then $\|K_n\|_1 = O(|n|^\delta)$, for all $\delta > 0$.

Proof. It is known [3], that

$$nK_n = \sum_{s=0}^{|n|} \sum_{j=0}^{n_s-1} K_{n^{(s+1)+jM_s, M_s}}.$$

Let $z \in I_t \setminus I_{t+1}$.

At this point divide the examination into four parts.

1. The case $|n| \geq s > t$. Then

$$K_{n^{(s)}, M_s}(z) = \begin{cases} M_t M_s \psi_{n^{(s)}}(z) \frac{1}{1-r_t(z)}, & \text{if } z - z_t e_t \in I_s \\ 0 & \text{otherwise.} \end{cases}$$

So we obtain

$$\begin{aligned}
 &\sum_{j=1}^{n_s-1} K_{n^{(s+1)+jM_s, M_s}(z)} \\
 &= \begin{cases} \psi_{n^{(s+1)}}(z) M_t M_s \frac{1}{1-r_t(z)} \frac{r_s^{n_s}(z)-1}{r_s(z)-1}, & \text{if } z - z_t e_t \in I_s \text{ and } z_s \neq 0 \\ \psi_{n^{(s+1)}}(z) M_t M_s \frac{n_s}{1-r_t(z)} & \text{if } z_s = 0 \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned}$$

Using this equality we have

$$\begin{aligned}
 &\int_{I_t \setminus I_{t+1}} \left| \sum_{j=1}^{n_s-1} K_{n^{(s+1)+jM_s, M_s}(z)} \right| d\mu(z) \\
 &= M_t M_s \sum_{z_t=1}^{m_t-1} \int_{I_s(e_t z_t)} \frac{1}{|1-r_t(z)|} \left| \frac{r_s^{n_s}(z)-1}{r_s(z)-1} \right| d\mu(z) \\
 &\leq M_t M_s m_t \log m_t \frac{1}{M_s} \log n_s = M_{t+1} \log m_t \log n_s.
 \end{aligned}$$

Let $A := |n|$. We would like to give an upper estimate for the next expression.

$$\frac{1}{n_A M_A} \sum_{t=0}^{A-1} \int_{I_t \setminus I_{t+1}} \left| \sum_{s=t+1}^A \sum_{j=0}^{n_s-1} K_{n^{(s+1)+jM_s, M_s}(z)} \right| d\mu(z)$$

$$\begin{aligned}
&\leq \frac{1}{n_A M_A} \sum_{t=0}^{A-1} \sum_{s=t+1}^A M_{t+1} \log m_t \log n_s \\
&= \frac{1}{n_A M_A} \sum_{s=1}^A \log n_s \sum_{t=0}^{s-1} M_{t+1} \log m_t := \frac{1}{n_A M_A} \sum_{s=1}^A \Gamma_s.
\end{aligned}$$

Observe the addends of this sum separately. At first let $s = A$.

$$\frac{1}{n_A M_A} \Gamma_A = \frac{1}{n_A M_A} \log n_A \sum_{t=0}^{A-1} M_{t+1} \log m_t \leq c \frac{1}{M_A} \sum_{t=0}^{A-1} M_{t+1} t^\delta \leq A^\delta.$$

Now let $s = A - 1$.

$$\frac{1}{n_A M_A} \Gamma_{A-1} \leq \frac{1}{M_A} \log m_{A-1} \sum_{t=0}^{A-2} M_{t+1} t^\delta \leq A^\delta \frac{\log m_{A-1}}{m_{A-1}} \leq A^\delta.$$

If $s = A - 2$, then

$$\frac{1}{n_A M_A} \Gamma_{A-2} \leq \frac{1}{M_A} \log m_{A-2} \sum_{t=0}^{A-3} M_{t+1} t^\delta \leq A^\delta \frac{\log m_{A-2}}{m_{A-1} m_{A-2}} \leq \frac{1}{2} A^\delta,$$

and so on

$$\frac{1}{n_A M_A} \Gamma_{A-3} \leq \frac{1}{2^2} A^\delta, \dots, \frac{1}{n_A M_A} \Gamma_1 \leq \frac{1}{2^{A-2}} A^\delta.$$

To sum up these we obtain

$$\frac{1}{n_A M_A} \sum_{s=1}^A \Gamma_s \leq c A^\delta.$$

2. The case $|n| > s = t$. From [3]

$$\begin{aligned}
\sum_{j=0}^{n_t-1} K_{n^{(t+1)+jM_t, M_t}}(z) &= \frac{M_t(M_t-1)}{2} \psi_{n^{(t+1)}}(z) \frac{r_t^{n_t}(z) - 1}{r_t(z) - 1} \\
&+ M_t^2 \psi_{n^{(t+1)}}(z) \frac{r_t^{n_t}(z) - 1}{(r_t(z) - 1)^2} - M_t^2 \psi_{n^{(t+1)}}(z) n_t \frac{1}{r_t(z) - 1} \\
&=: \Theta_1(z) + \Theta_2(z) + \Theta_3(z).
\end{aligned}$$

See the integral of the absolute value of these three functions on $I_t \setminus I_{t+1}$.

At first

$$\begin{aligned}
\int_{I_t \setminus I_{t+1}} |\Theta_1(z)| d\mu(z) &\leq c M_t^2 \frac{1}{M_{t+1}} \sum_{z_t=1}^{m_t-1} \frac{|\sin(\pi n_t z_t / m_t)|}{|\sin(\pi z_t / m_t)|} \\
&\leq c M_t \log n_t \leq c M_{t+1}.
\end{aligned}$$

After

$$\int_{I_t \setminus I_{t+1}} |\Theta_2(z)| d\mu(z) \leq c M_t^2 \frac{1}{M_{t+1}} \sum_{z_t=1}^{m_t-1} \frac{m_t^2}{z_t^2} \leq c M_{t+1},$$

and at the end

$$\begin{aligned} \int_{I_t \setminus I_{t+1}} |\Theta_3(z)| d\mu(z) &\leq M_t^2 n_t \frac{1}{M_{t+1}} \sum_{z_t=1}^{m_t-1} \frac{1}{|\sin(\pi z_t/m_t)|} \\ &\leq c M_t n_t \log m_t \leq c M_{t+1} \log m_t. \end{aligned}$$

>From these three inequalities

$$\begin{aligned} &\frac{1}{n_A M_A} \sum_{t=0}^{A-1} \int_{I_t \setminus I_{t+1}} \left| \sum_{j=0}^{n_t-1} K_{n^{(t+1)+jM_t, M_t}}(z) \right| d\mu(z) \\ &\leq \frac{1}{n_A M_A} \sum_{t=0}^{A-1} c M_{t+1} \log m_t \leq c \frac{1}{M_A} \sum_{t=0}^{A-1} M_{t+1} t^\delta \leq c A^\delta. \end{aligned}$$

3. The case $|n| \geq t > s$. It is easy to see, that if $z \in I_t \setminus I_{t+1}$, then

$$D_k(z) = \psi_k(z) \left(\sum_{j=0}^{t-1} k_j M_j + M_t \sum_{s=m_t-k_t}^{m_t-1} r_t^s \right),$$

so $|D_k(z)| \leq (k_t + 1)M_t$. It yields

$$\int_{I_t \setminus I_{t+1}} |D_k(z)| d\mu(z) \leq k_t + 1 \leq \begin{cases} 2m_t, & \text{if } t < A \\ 2n_t, & \text{if } t = A \end{cases} =: \alpha(n, t, a).$$

>From this we get

$$\int_{I_t \setminus I_{t+1}} |K_{n^{(s+1)+jM_s, M_s}}(z)| d\mu(z) \leq M_s \alpha(n, t, a),$$

and therefore

$$\int_{I_t \setminus I_{t+1}} \left| \sum_{j=0}^{n_s-1} K_{n^{(s+1)+jM_s, M_s}}(z) \right| d\mu(z) \leq M_{s+1} \alpha(n, t, a).$$

Using this result we get

$$\begin{aligned} &\frac{1}{n_A M_A} \sum_{t=1}^A \int_{I_t \setminus I_{t+1}} \left| \sum_{s=0}^{t-1} \sum_{j=0}^{n_s-1} K_{n^{(s+1)+jM_s, M_s}}(z) \right| d\mu(z) \\ &\leq \frac{1}{n_A M_A} \sum_{t=1}^{A-1} \sum_{s=0}^{t-1} 2M_{s+1} m_t + \frac{1}{n_A M_A} \sum_{s=0}^{A-1} 2M_{s+1} n_A \leq c. \end{aligned}$$

4. The case $|n| = t = s$. It is known from [3]: if $z \in I_t \setminus I_{t+1}$, then

$$K_{n^{(t+1)+jM_t, M_t}}(z) = K_{jM_A, M_A}(z) =: \Theta_4(z) + \Theta_5(z),$$

where $\Theta_4(z) \leq M_A^2$, so

$$\frac{1}{n_A M_A} \int_{I_A \setminus I_{A+1}} \sum_{j=0}^{n_A-1} |\Theta_4(z)| d\mu(z) \leq c.$$

On the other hand, [3]

$$\sum_{j=0}^{n_A-1} \Theta_5(z) = M_A^2 \frac{r_A^{n_A}(z) - 1}{(r_A(z) - 1)^2} + M_A^2 n_A \frac{1}{1 - r_A(z)} =: \Theta_{5,1}(z) + \Theta_{5,2}(z).$$

At first observe $\Theta_{5,2}(z)$.

$$\begin{aligned} & \frac{1}{n_A M_A} \int_{I_A \setminus I_{A+1}} |\Theta_{5,2}(z)| d\mu(z) \\ &= M_A \int_{I_A \setminus I_{A+1}} \frac{1}{|1 - r_A(z)|} d\mu(z) \leq c \log m_A \leq cA^\delta. \end{aligned}$$

At last

$$|\Theta_{5,1}(z)| \leq 2M_A^2 \frac{|\sin(\pi n_A z_A / m_A)|}{|\sin(\pi z_A / m_A)|^2} \leq cM_A^2 \frac{n_A z_A}{m_A} \frac{1}{\sin^2(\pi z_A / m_A)},$$

so

$$\begin{aligned} \frac{1}{n_A M_A} \int_{I_A \setminus I_{A+1}} |\Theta_{5,1}(z)| d\mu(z) &\leq c \frac{1}{n_A M_A} \sum_{z_A=1}^{m_A-1} \frac{1}{M_{A+1}} M_A^2 n_A \frac{z_A}{m_A} \frac{m_A^2}{z_A^2} \\ &= c \sum_{z_A=1}^{m_A-1} \frac{1}{z_A} = c \log m_A \leq cA^\delta. \end{aligned}$$

Summarizing the reasons of the last four cases

$$\sum_{t=0}^A \int_{I_t \setminus I_{t+1}} |K_n(z)| d\mu(z) \leq cA^\delta.$$

At the end, based on if $z \in I_{A+1}$, then $|K_n(z)| \leq cn$ we get

$$\int_{I_{A+1}} |K_n(z)| d\mu(z) \leq \frac{1}{M_{A+1}} n \leq c.$$

Consequently

$$\|K_n(z)\|_1 \leq cA^\delta = c|n|^\delta.$$

□

>From now we use the following notation.

$$\alpha_A := \frac{1}{M_A} \sum_{t=0}^{A-1} M_{t+1} \log m_t,$$

where $M_A \leq n < M_{A+1}$ (it means $|n|=A$).

Lemma 3.

$$\|K_n\|_1 \leq c \sum_{i=0}^{A+1} \frac{\alpha_{A+1-i}}{2^i}.$$

Proof. During the proof we use some method and result from the proof of Lemma 2. Analogously, we have four cases.

1. The case $|n| \geq s > t$. With the notations of Lemma 2. the case $s = A$

$$\frac{\Gamma_A}{n_A M_A} = \frac{1}{n_A M_A} \log n_A \sum_{t=0}^{A-1} M_{t+1} \log m_t \leq c \alpha_A.$$

Similarly, $s = A - 1$, $s = A - 2$

$$\begin{aligned} \frac{\Gamma_{A-1}}{n_A M_A} &\leq \frac{1}{M_A} \log m_{A-1} \sum_{t=0}^{A-2} M_{t+1} \log m_t \leq \alpha_{A-1}, \\ \frac{\Gamma_{A-2}}{n_A M_A} &\leq \alpha_{A-2} 2^{-1}, \quad \frac{\Gamma_{A-3}}{n_A M_A} \leq \alpha_{A-3} 2^{-2}, \end{aligned}$$

and so on. In this way we obtain

$$\frac{1}{n_A M_A} \sum_{i=1}^A \Gamma_i \leq c \sum_{i=1}^A \frac{\alpha_{A-i}}{2^i}$$

2. the case $|n| > s = t$. We use the same estimation we obtained in the 2nd case of the proof of Lemma 2.

$$\begin{aligned} \frac{1}{n_A M_A} \sum_{t=0}^{A-1} \int_{I_t \setminus I_{t+1}} \left| \sum_{j=0}^{n_t-1} K_{n^{(t+1)+j} M_t, M_t}(z) \right| d\mu(z) \\ \leq \frac{1}{n_A M_A} \sum_{t=0}^{A-1} c M_{t+1} \log m_t \leq c \alpha_A. \end{aligned}$$

3. The case $|n| \geq t > s$. As in the 3rd case of the proof Lemma 2,

$$\frac{1}{n_A M_A} \sum_{t=1}^A \int_{I_t \setminus I_{t+1}} \left| \sum_{s=0}^{t-1} \sum_{j=0}^{n_s-1} K_{n^{(s+1)+j} M_s, M_s}(z) \right| d\mu(z) \leq c.$$

4. The case $|n| = t = s$. Lemma 2. yields that

$$\int_{I_{A+1}} |K_n| \leq c.$$

If we summarize the results of the last four case we get

$$\|K_n\|_1 \leq c \sum_{i=0}^A \frac{\alpha_{A-i}}{2^i} + c + c \alpha_{A+1} \leq c \sum_{i=0}^{A+1} \frac{\alpha_{A+1-i}}{2^i},$$

where $|n| = A$. □

Lemma 4.

$$\|F_{M_n}\|_1 \leq c \frac{\sum_{j=0}^{n-1} \log^2 m_j}{\log M_n}$$

Proof. From Lemma 1 we get

$$\sum_{k=1}^{M_n-1} \frac{D_k(x)}{M_n - k} = \sum_{k=1}^{M_n-1} \frac{1}{k} (D_{M_n}(x) - \bar{\psi}_{M_n-1}(-x) D_k(-x))$$

$$= D_{M_n}(x)l_{M_n} - \bar{\psi}_{M_{n-1}}(-x) \sum_{k=1}^{M_{n-1}} \frac{1}{k} D_k(-x) =: \Lambda_1 - \Lambda_2$$

Observe case Λ_2 . Since

$$\|\Lambda_2\|_1 = \left\| \bar{\psi}_{M_{n-1}}(-x) \sum_{k=1}^{M_{n-1}} \frac{1}{k} D_k(-x) \right\|_1 = \left\| \sum_{k=1}^{M_{n-1}} \frac{1}{k} D_k(x) \right\|_1,$$

we consider this expression instead of the original one.

Using Abel transformation we obtain

$$\begin{aligned} \left| \sum_{k=1}^{M_{n-1}} \frac{1}{k} D_k \right| &= \left| \sum_{k=1}^{M_{n-2}} \left(\frac{1}{k} - \frac{1}{k+1} \right) \sum_{j=1}^k D_j + \frac{1}{M_{n-1}} \sum_{j=1}^{M_{n-1}} D_j \right| \\ &\leq \sum_{k=1}^{M_{n-2}} \frac{|K_{k+1}|}{k} + c|K_{M_n}| =: \Lambda_3 + \Lambda_4. \end{aligned}$$

Start with Λ_3 .

$$\begin{aligned} \|\Lambda_3\|_1 &\leq c \sum_{k=1}^{M_{n-1}} \frac{1}{k} \|K_k\|_1 \\ &\leq c \sum_{j=0}^{n-1} \sum_{a=1}^{m_j-1} \sum_{s=aM_j}^{(a+1)M_j-1} \frac{1}{aM_j} (\alpha_{j+1} + \alpha_j 2^{-1} + \dots + \alpha_0 2^{-j-1}) \\ &\leq c \sum_{j=0}^{n-1} \log m_j (\alpha_{j+1} + \alpha_j 2^{-1} + \dots + \alpha_0 2^{-j-1}) \end{aligned}$$

It is easy to see that

$$\log m_A \leq \alpha_{A+1} \leq \log m_A + 2^{-1} \log m_{A-1} + \dots + 2^{-A} \log m_0.$$

Since

$$\begin{aligned} \alpha_{j+1} + \alpha_j 2^{-1} + \dots + \alpha_0 2^{-j-1} &= \sum_{b=0}^{j+1} 2^{b-j-1} \alpha_b \\ &\leq \sum_{b=0}^{j+1} 2^{b-j-1} (\log m_{b-1} + 2^{-1} \log m_{b-2} + \dots + 2^{-b+1} \log m_0) \\ &= \sum_{b=0}^{j+1} 2^{b-j-1} \sum_{u=0}^{b-1} 2^{-b+u+1} \log m_u = 2^{-j} \sum_{b=0}^{j+1} \sum_{u=0}^{b-1} 2^u \log m_u \\ &= 2 \sum_{u=0}^j \frac{j+1-u}{2^{j+1-u}} \log m_u = 2 \sum_{u=1}^{j+1} \frac{u}{2^u} \log m_{j+1-u}, \end{aligned}$$

so, using that $\sum_{u=1}^{\infty} \frac{u}{2^u} < \infty$

$$\|\Lambda_3\|_1 \leq c \sum_{j=0}^{n-1} \log m_j \sum_{u=1}^{j+1} \frac{u}{2^u} \log m_{j+1-u} = c \sum_{j=0}^{n-1} \sum_{u=1}^{j+1} \frac{u}{2^u} \log m_j \log m_{j+1-u}$$

$$\begin{aligned}
 &\leq c \sum_{j=0}^{n-1} \left(\sum_{u=1}^{j+1} \frac{u}{2^u} \log^2 m_j + \sum_{u=1}^{j+1} \frac{u}{2^u} \log^2 m_{j+1-u} \right) \\
 &\leq c \sum_{j=0}^{n-1} \log^2 m_j + c \sum_{j=0}^{n-1} \sum_{u=1}^{j+1} \frac{u}{2^u} \log^2 m_{j+1-u} \\
 &= c \sum_{j=0}^{n-1} \log^2 m_j + c \sum_{j=0}^{n-1} \log^2 m_j \sum_{u=1}^{n-j} \frac{u}{2^u} \\
 &\leq c \sum_{j=0}^{n-1} \log^2 m_j.
 \end{aligned}$$

Proceed to Λ_4 .

$$\|\Lambda_4\|_1 \leq \left\| \frac{1}{M_n - 1} \sum_{j=0}^{M_n-2} D_j \right\|_1 + \frac{1}{M_n - 1} \|D_{M_n-1}\|_1$$

It is known that $\|D_{M_n-1}\|_1 \leq c$. On the other hand,

$$\begin{aligned}
 \left\| \frac{1}{M_n - 1} \sum_{j=0}^{M_n-2} D_j \right\|_1 &= \|K_{M_n-1}\|_1 \leq \alpha_n + 2^{-1}\alpha_{n-1} + \dots + 2^{-n}\alpha_0 \\
 &\leq (\log m_{n-1} + 2^{-1} \log m_{n-2} + \dots + 2^{-n+1} \log m_0) \\
 &\quad + (2^{-1} \log m_{n-2} + 2^{-2} \log m_{n-3} + \dots + 2^{-n+1} \log m_0) \\
 &\quad + \dots + (2^{-n+2} \log m_1 + 2^{-n+1} \log m_0) + (2^{-n+1} \log m_0) \\
 &= \sum_{j=0}^{n-1} \frac{n-j}{2^{n-1-j}} \log m_j \leq \sum_{j=0}^{n-1} \log m_j \\
 &= \log M_n
 \end{aligned}$$

Originally, we started with

$$\begin{aligned}
 \|F_{M_n}\|_1 &= \frac{1}{l_{M_n}} \left\| \sum_{k=1}^{M_n-1} \frac{D_k}{M_n - k} \right\|_1 \\
 &\leq \frac{\|\Lambda_1\|_1 + c \sum_{j=0}^{n-1} \log^2 m_j + c + \log M_n}{\log M_n} \\
 &\leq c \frac{\sum_{j=0}^{n-1} \log^2 m_j}{\log M_n}.
 \end{aligned}$$

□

Theorem 1. If $f \in L^p$ ($1 \leq p < \infty$) and

$$\limsup_{n \in \mathbb{N}} \frac{\sum_{k=0}^{n-1} \log^2 m_k}{\log M_n} < \infty,$$

then

$$\|t_{M_n} f - f\|_p \rightarrow 0.$$

In the case $f \in C$ the convergence holds in the supremum norm.

Proof. Using the theorem of Fubini and Lemma 4 we obtain

$$\begin{aligned} \|t_{M_n}f\|_1 &= \int_{G_m} |t_{M_n}f(y)|d\mu(y) \\ &= \int_{G_m} \left| \int_{G_m} f(x)F_{M_n}(y-x)d\mu(x) \right| d\mu(y) \\ &\leq \int_{G_m} \int_{G_m} |f(x)||F_{M_n}(y-x)|d\mu(y)d\mu(x) \\ &\leq \int_{G_m} |f(x)||F_{M_n}\|_1d\mu(x) \leq c\|f\|_1 \end{aligned}$$

that t_{M_n} operator is $(1, 1)$ type, uniformly in n . Similarly,

$$\begin{aligned} \|t_{M_n}f\|_\infty &= \sup_{y \in G_m} |t_{M_n}f(y)| \\ &= \sup_{y \in G_m} \left| \int_{G_m} f(x)F_{M_n}(y-x)d\mu(x) \right| \\ &\leq \sup_{y \in G_m} \int_{G_m} |f(x)||F_{M_n}(y-x)|d\mu(y) \\ &\leq \sup_{y \in G_m} \int_{G_m} \|f(x)\|_1 \|F_{M_n}(y-x)\|d\mu(x) \leq \|f\|_\infty \|F_{M_n}\|_1 \leq c\|f\|_\infty \end{aligned}$$

that is the operator t_{M_n} is of type (∞, ∞) (uniformly in n).

The Marcinkiewicz interpolation theorem implies that the operator t_{M_n} is of type (p, p) ($1 \leq p \leq \infty$). From this we get the proof of the theorem in the usual way. \square

Lemma 5. Let $p_A = M_{2A} + M_{2A-2} + \dots + M_0$. If $\log m_n = O(n^\delta)$ for some $0 < \delta < 1/2$, then

$$\|F_{p_A}\|_1 \geq c_\beta A^\beta.$$

for every $0 < \beta < 1 - \delta$.

Proof. Introduce the following notation

$$G_n := \sum_{k=1}^{n-1} \frac{D_{n-k}}{k}.$$

We use this decomposition of G_{p_A}

$$\begin{aligned} G_{p_A} &= \sum_{k=1}^{M_{2A-2}+\dots+M_0-1} \frac{1}{k} D_{p_A-k} + \sum_{k=M_{2A-2}+\dots+M_0}^{p_A-1} \frac{1}{k} D_{p_A-k} \\ &=: B_1 + B_2. \end{aligned}$$

Start with B_2 . Let $k' := k - (M_{2A-2} + \dots + M_0)$. In this case $0 \leq k' \leq M_{2A} - 1$. Consequently we can use Lemma 1.

$$D_{p_A-k}(x) = D_{M_{2A}-k'}(x) = D_{M_{2A}}(x) - \bar{\psi}_{M_{2A}-1}(-x)D_{k'}(-x).$$

So,

$$\begin{aligned}
 B_2(x) &= \sum_{k=M_{2A-2}+\dots+M_0}^{p_A-1} \frac{1}{k} (D_{M_{2A}}(x) - \bar{\psi}_{M_{2A-1}}(-x)D_{k'}(-x)) = \\
 &\quad \sum_{k'=0}^{M_{2A}-1} \frac{1}{k} (D_{M_{2A}}(x) - \bar{\psi}_{M_{2A-1}}(-x)D_{k'}(-x)) = \\
 &\quad \sum_{k'=0}^{M_{2A}-1} \frac{1}{M_{2A-2} + \dots + M_0 + k'} (D_{M_{2A}}(x) - \bar{\psi}_{M_{2A-1}}(-x)D_{k'}(-x)) = \\
 &\quad D_{M_{2A}}(x)c_2(A) - \bar{\psi}_{M_{2A-1}}(-x) \sum_{k'=0}^{M_{2A}-1} \frac{1}{M_{2A-2} + \dots + M_0 + k'} D_{k'}(-x) \\
 &\quad =: B_{2,1}(x) + B_{2,2}(x).
 \end{aligned}$$

Investigate $c_2(A)$

$$\begin{aligned}
 0 < c_2(A) &= \sum_{k'=0}^{M_{2A}-1} \frac{1}{M_{2A-2} + \dots + M_0 + k'} \\
 &< \log \left(\frac{M_{2A}}{M_{2A-2}} \right) = \log(m_{2A-1}m_{2A-2}) \leq cA^\delta.
 \end{aligned}$$

Consequently $c_2(A)/A \rightarrow 0$ if $A \rightarrow \infty$, that is $c_2(A) = o(A)$. This gives

$$\|B_{2,1}\|_1 = \int_{G_m} |D_{M_{2A}}| o(A) = o(A).$$

Using Abel transform and bringing Fejér kernels observe $B_{2,2}$

$$\begin{aligned}
 |B_{2,2}(x)| &\leq \\
 &\sum_{k'=0}^{M_{2A}-2} \left(\frac{1}{M_{2A-2} + \dots + M_0 + k'} - \frac{1}{M_{2A-2} + \dots + M_0 + k' + 1} \right) \cdot \\
 &\quad \cdot (k' + 1) |K_{k'+1}(-x)| + \frac{1}{M_{2A} + \dots + M_0} (M_{2A}) |K_{M_{2A}}(-x)|.
 \end{aligned}$$

So we can obtain using Lemma 2

$$\begin{aligned}
 \|B_{2,2}\|_1 &\leq \sum_{k'=0}^{M_{2A}-2} \frac{1}{(M_{2A-2} + \dots + M_0 + k')^2} (k' + 1) \|K_{k'+1}\|_1 + \|K_{M_{2A}}\|_1 \\
 &\leq \sum_{k'=0}^{M_{2A}-2} \frac{1}{M_{2A-2} + \dots + M_0 + k'} \|K_{k'+1}\|_1 + \|K_{M_{2A}}\|_1 \\
 &\leq cA^\delta \log \left(\frac{M_{2A}}{M_{2A-2}} \right) = cA^\delta \log(m_{2A-1}m_{2A-2}) \leq cA^{2\delta}.
 \end{aligned}$$

This gives

$$\|B_{2,2}\|_1 = o(A),$$

consequently we have

$$\|B_2\|_1 = o(A).$$

Focus on B_1 . Easy to see that in case of $0 \leq j < M_n$,

$$\begin{aligned} D_{M_n+j} &= D_{M_n} + \sum_{k=M_n}^{M_n+j-1} \psi_k = D_{M_n} + \sum_{l=0}^{j-1} \psi_{l+M_n} \\ &= D_{M_n} + \sum_{l=0}^{j-1} \psi_{l \oplus M_n} = D_{M_n} + \psi_{M_n} \sum_{l=0}^{j-1} \psi_l = D_{M_n} + r_n D_j \end{aligned}$$

holds. Thus,

$$\begin{aligned} B_1 &= \sum_{k=1}^{M_{2A-2}+\dots+M_0-1} \frac{1}{k} D_{p_A-k} \\ &= \sum_{k=1}^{M_{2A-2}+\dots+M_0-1} \frac{1}{k} (D_{M_{2A}} + r_{2A} D_{M_{2A-2}+\dots+M_0-k}) \\ &= D_{M_{2A}} (\log(p_{A-1}) + o(A)) + r_{2A} \sum_{k=1}^{M_{2A-2}+\dots+M_0-1} \frac{1}{k} D_{M_{2A-2}+\dots+M_0-k} \\ &= D_{M_{2A}} (\log(p_{A-1}) + o(A)) + r_{2A} G_{M_{2A-2}+\dots+M_0}. \end{aligned}$$

>From B_1 and B_2 we got

$$\|G_{M_{2A}+\dots+M_0}\|_1 \geq \|D_{M_{2A}} \log(p_{A-1}) + r_{2A} G_{M_{2A-2}+\dots+M_0}\|_1 - o(A).$$

Let us divide this norm of the sum to two pieces, like $\|\cdot\|_1 = \int_{I_{2A}} |\cdot| + \int_{I \setminus I_{2A}} |\cdot|$. First

$$\begin{aligned} &\int_{I_{2A}} |D_{M_{2A}} \log(p_{A-1}) + r_{2A} G_{M_{2A-2}+\dots+M_0}| \\ &\geq \log(p_{A-1}) - \frac{1}{M_{2A}} G_{M_{2A-2}+\dots+M_0}(0). \end{aligned}$$

After

$$\begin{aligned} &\int_{I \setminus I_{2A}} |D_{M_{2A}} \log(M_{2A}) + r_{2A} G_{M_{2A-2}+\dots+M_0}| \\ &= \int_{I \setminus I_{2A}} |G_{M_{2A-2}+\dots+M_0}| \\ &= \|G_{M_{2A-2}+\dots+M_0}\|_1 - \int_{I_{2A}} |G_{M_{2A-2}+\dots+M_0}| \\ &= \|G_{M_{2A-2}+\dots+M_0}\|_1 - \frac{1}{M_{2A}} G_{M_{2A-2}+\dots+M_0}(0). \end{aligned}$$

These inequalities yield

$$\begin{aligned} &\|G_{M_{2A}+\dots+M_0}\|_1 \\ &\geq \log(p_{A-1}) + \|G_{M_{2A-2}+\dots+M_0}\|_1 - 2 \frac{1}{M_{2A}} G_{M_{2A-2}+\dots+M_0}(0) - o(A). \end{aligned}$$

We will use the estimation

$$G_n(0) = \sum_{k=1}^n \frac{n-k}{k} = n \sum_{k=1}^n \frac{1}{k} - n + 1 = n \log(n) + O(n).$$

By simple calculations we get

$$M_{2A} < p_A = M_{2A} + \dots + M_0 < M_{2A} + M_{2A-1} \leq \frac{3}{2} M_{2A}.$$

Since

$$\begin{aligned} & 2 \frac{1}{M_{2A}} G_{M_{2A-2}+\dots+M_0}(0) \\ &= 2 \frac{1}{M_{2A}} (M_{2A-2} + \dots + M_0) \log(M_{2A-2} + \dots + M_0) + c \\ &\leq 2 \frac{1}{M_{2A}} \frac{3}{2} M_{2A-2} \log(p_{A-1}) + c \\ &= \frac{3 \log(p_{A-1})}{m_{2A-2} m_{2A-1}} + c \\ &\leq \frac{3 \log(p_{A-1})}{4} + c, \end{aligned}$$

we have

$$\begin{aligned} & \|G_{M_{2A}+\dots+M_0}\|_1 \\ &\geq \|G_{M_{2A-2}+\dots+M_0}\|_1 + \log(p_{A-1}) - \frac{3 \log(p_{A-1})}{4} - c \\ &= \|G_{M_{2A-2}+\dots+M_0}\|_1 + \frac{\log(p_{A-1})}{4} - c. \end{aligned}$$

This yields

$$\begin{aligned} & \|G_{M_{2A}+\dots+M_0}\|_1 \\ &\geq \frac{1}{4} \cdot \frac{3}{2} \sum_{k=0}^{A-1} \log(M_{2k}) - O(A). \end{aligned}$$

Easy to see that

$$\begin{aligned} \sum_{k=1}^K \log M_{2A-2k} &= \sum_{k=1}^K \left(\log M_{2A-2} - \log \left(\prod_{h=1}^{k-1} m_{2A-2h-1} m_{2A-2h-2} \right) \right) \\ &\geq K \log M_{2A} - K \log(m_{2A-1} m_{2A-2}) \\ &\quad - \sum_{k=1}^K \sum_{h=1}^{k-1} (\log m_{2A-2h-1} + \log m_{2A-2h-2}) \\ &\geq K \log M_{2A} - c A^\delta K^2, \end{aligned}$$

if $K \in \{1, \dots, A\}$. Since $\|F_{p_A}\|_1 \geq c \frac{\|G_{p_A}\|_1}{\log p_A}$, let us observe

$$\frac{\sum_{k=0}^{A-1} \log M_{2k}}{\log p_A}$$

$$\begin{aligned} &\geq \frac{2 \sum_{k=1}^K \log M_{2A-2k}}{3 \log M_{2A}} \geq \frac{2}{3}K - \frac{2cA^\delta K^2}{3A \log 4} \\ &\geq c \left(K - \frac{K^2}{A^{1-\delta}} \right). \end{aligned}$$

Finally, let us choose $0 < \beta < 1 - \delta$ and $A^\beta - 1 \leq K < A^\beta$. In this case $\lim_{A \rightarrow \infty} K = \infty$ and $K - \frac{K^2}{A^{1-\delta}} = O(A^\beta)$, so

$$\|F_{p_A}\|_1 \geq c_\beta A^\beta. \quad \square$$

Theorem 2. If $\log m_n = O(n^\delta)$ for some $0 < \delta < 1/2$, then $\exists f \in L^1$ that

$$\|t_n f - f\|_1 \not\rightarrow 0.$$

Proof. The proof of this theorem is based on Lemma 5 and some standard method. See for instance [7]. \square

Some conditions in our theorems can seem a bit strange, so we show a simple example.

Example. Let

$$m_k = \begin{cases} \left\lceil \exp(k^{\frac{1}{4}}) \right\rceil, & \text{if } k = j^2 \text{ for any } j \in \mathbb{N}^+, \\ 2 & \text{otherwise} \end{cases}$$

In this case it is true that

$$\text{a. } \limsup_{k \in \mathbb{N}} m_k = \infty$$

$$\text{b. } \log m_k = O(k^{\frac{1}{4}})$$

$$\text{c. } \limsup_{n \in \mathbb{N}} \frac{\sum_{k=0}^{n-1} \log^2 m_k}{\log M_n} < \infty. \text{ (It implies } \|F_{M_n}\|_1 \leq c.)$$

Proof. It is easy to see that in the case of a. and b. the equalities hold. Investigate the case c. Since

$$\sum_{k=0}^{n-1} \log^2 m_k \leq \sum_{k \neq j^2} \log^2 2 + \sum_{k=j^2} k^{\frac{1}{2}} \leq (n - [\sqrt{n}]) \log^2 2 + \sum_{j=1}^{[\sqrt{n}]} j \leq n$$

and $\log M_n \geq n \log 2$, we obtain

$$\frac{\sum_{k=0}^{n-1} \log^2 m_k}{\log M_n} < \frac{3}{2}. \quad \square$$

Remark

We formerly thought, that the Fejér means have "better properties", than the logarithmic ones. Although, what is the situation in case of unbounded Vilenkin system? $\forall m$ unbounded sequence $\exists f \in L^1$ that

$$\|\sigma_{M_n} f - f\|_1 \not\rightarrow 0,$$

(it was known before), but we realized: $\exists m$ unbounded sequence, that $\forall f \in L^1$

$$\|t_{M_n} f - f\|_1 \rightarrow 0.$$

We have the same situation with respect to the space of continuous functions and the supremum norm.

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