ALMOST EVERYWHERE CONVERGENCE OF FEJÉR AND LOGARITHMIC MEANS OF SUBSEQUENCES OF PARTIAL SUMS OF THE WALSH-FOURIER SERIES OF INTEGRABLE FUNCTIONS

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ABSTRACT. The aim of this paper is to prove some a.e. convergence results of Fejér and logarithmic means of subsequences of partial sums of Walsh-Fourier series of integrable functions. We prove for lacunary sequences a that the $(C, 1)$ means of the partial sums $S_{a(n)}f$ converges to f a.e. Besides, for every convex a tending to $+\infty$ and every integrable function f the logarithmic means of the partial sums $S_{a(n)}f$ converges to f a.e.

It is of main interest in the theory of Fourier series that how to reconstruct the function from the partial sums of its Fourier series. Just to mention two examples: Billard proved [3] the theorem of Carleson for the Walsh-Paley system, that is, for each function in L^2 we have the almost everywhere convergence $S_n f \to f$ and Fine proved [6] the Fejér-Lebesgue theorem, that is for each integrable function in L^1 we have the almost everywhere convergence of Fejér means $\sigma_n f \to f$.

It is also of prior interested that what can be said - with respect to this reconstruction issue - if we have only a subsequence of the partial sums. In 1936 Zalcwasser [15] asked how "rare" can be the sequence of integers $a(n)$ such that

$$\frac{1}{N}\sum_{n=1}^{N} S_{a(n)}f \to f. \tag{1}$$

This problem with respect to the trigonometric system was completely solved for continuous functions (uniform convergence) in [11, 14, 1, 5]. That is, if the sequence a is convex, then the condition $\sup_n n^{-1/2}\log a(n) < +\infty$ is necessary and sufficient for the uniform convergence for every continuous function. For the time being, this issue with respect to the Walsh-Paley system has not been solved. Only, a sufficient condition is known, which is the same as in the trigonometric case. The paper about this is written by Glukhov [7]. See the more dimensional case also by Glukhov [8].

With respect to convergence almost everywhere, and integrable functions the situation is more complicated. Belinksky proved [2] for the trigonometric system the existence of a sequence $a(n) \sim \exp(\sqrt[3]{k})$ such that the relation (1) holds a.e. for every integrable function. In this paper Belinksky also conjectured that if the sequence a is convex, then the condition $\sup_n n^{-1/2}\log a(n) < +\infty$ is necessary and sufficient again. So, that would be the answer for the problem of Zalcwasser [15] in this point of view (trigonometric system, a.e. convergence and L^1 functions). In this paper - among others - prove that this is not the case for the Walsh-Paley system. See below Theorem 1. On the other hand, this difference between the

1991 *Mathematics Subject Classification.* 42C10.

Key words and phrases. Fejér, logarithmic means, Walsh-Fourier series, subsequence of partial sums, almost everywhere convergence.

Research supported by the Hungarian National Foundation for Scientific Research (OTKA) T 048780.

Walsh-Paley and the trigonometric system is not so surprising. Because of the following. Let $v(n) := \sum_{i=0}^{\infty} |n_i - n_{i+1}|, (n = \sum_{i=0}^{\infty} n_i 2^i)$ be the variation of the natural number n expanded in the number system based 2. It is a well-known result in the literature that for each sequence a tending strictly monotone increasing to plus infinity with the property $\sup_n v(a(n)) < +\infty$ we have the a.e. convergence $S_{a(n)} f \to f$ for all integrable function f. Is it also a necessary condition? This question of Balashov was answered by Konyagin [9] in the negative. He gave an example. That is, a sequence a with property $\sup_n v(a(n)) = +\infty$ and he proved that $S_{a(n)} f \to f$ a.e. for all integrable function f.

In this paper we prove (see Theorem 1) that for each lacunar sequence a (that is $a(n+1)/a(n) \geq q > 1$) and each integrable function f the relation (1) holds a.e. This may also be interesting in the following point of view. If the sequence a is lacunar, then the a.e. relation $S_{a(n)} f \to f$ holds for all functions f in the Hardy space H. The trigonometric and the Walsh-Paley case can be found in [16] (trigonometric case) and [10] (Walsh-Paley case). But, the space H is a proper subspace of L^1. Therefore, it is of interest to investigate relation (1) for L^1 functions and lacunar sequence a.

In this paper - using the method of the proof of Theorem 1 we prove (Theorem 2) that for any convex sequence a (with $a(+\infty) = +\infty$ - of course) and for each integrable function the Riesz's logarithmic means of the function converges to the function almost everywhere. That is, we prove that the Riesz's logarithmic summability method can reconstruct the corresponding integrable function from any (convex) subsequence of the partial sums in the Walsh-Paley situation. For the time being there is no result known with respect to a.e. convergence of logarithmic means of subsequences of partial sums, neither in the trigonometric nor in the Walsh case.

Next, we give a brief introduction to the theory of the Walsh-Fourier series.

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$, and $Q := [0, 1)$. Denote the Lebesgue measure of any set $E \subset Q$ by $|E|$ and sometimes by $\operatorname{mes} E$. Denote the $L^p(Q)$ norm of any function $f : Q \to \mathbb{C}$ by $\|f\|_p$ $(1 \leq p \leq \infty)$.

Denote the dyadic expansion of $n \in \mathbb{N}$ and $x \in Q$ by $n = \sum_{j=0}^{\infty} n_j 2^j$ and $x = \sum_{j=0}^{\infty} x_j 2^{-j-1}$ (in the case of $x = \frac{k}{2^m}$ $k, m \in \mathbb{N}$ choose the expansion which terminates in zeros). n_i, x_i are the i-th coordinates of n, x, respectively. Set $e_i := 1/2^{i+1} \in Q$, the i-th coordinate of e_i is 1, the rest are zeros $(i \in \mathbb{N})$. Define the dyadic addition $+$ as

$$x + y = \sum_{j=0}^{\infty} |x_j - y_j| 2^{-j-1}.$$

The sets $I_n(x) := \{y \in Q : y_0 = x_0, ..., y_{n-1} = x_{n-1}\}$ for $x \in Q$, $I_n := I_n(0)$ for $n \in \mathbb{P}$ and $I_0(x) := Q$ are the dyadic intervals of Q. Denote by $\mathcal{I} := \{I_n(x) : x \in Q, n \in \mathbb{N}\}$ the set of the dyadic intervals on Q. $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)$ $(x \in Q)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbb{N})$.

For $n \in \mathbb{P}$ denote by $|n| := \max(j \in \mathbb{N} : n_j \neq 0)$, that is, $2^{|n|} \leq n < 2^{|n|+1}$. The Rademacher functions are defined as:

$$r_n(x) := (-1)^{x_n} \quad (x \in Q,\, n \in \mathbb{N}).$$

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k}, \quad (x \in Q,\, n \in \mathbb{N}).$$

That is, $\omega := (\omega_n, n \in \mathbb{N})$. Consider the Dirichlet, Fejér and Riesz's logarithmic kernel functions:

$$D_n := \sum_{k=0}^{n-1} \omega_k, \ K_n := \frac{1}{n}\sum_{k=1}^{n} D_k, \ H_n := \frac{1}{\log n}\sum_{k=1}^{n} \frac{D_k}{k} \ (n \in \mathbb{P}), \ D_0, K_0, H_0 := 0.$$

The Fourier coefficients, the n-th partial sum of the Fourier series, the n-th $(C,1)$ mean, the n-th Riesz's logarithmic mean of $f \in L^1(Q)$:

$$\hat{f}(n) := \int_Q f(x)\omega_n(x)\,dx \ (n \in \mathbb{N}),$$

$$S_n f(y) := \sum_{k=0}^{n-1} \hat{f}(k)\omega_k(y) = \int_Q f(x+y)D_n(x)\,dx,$$

$$\sigma_n f(y) := \frac{1}{n}\sum_{k=1}^{n} S_k f(y) = \int_Q f(x+y)K_n(x)\,dx,$$

$$G_n f(y) := \frac{1}{\log n}\sum_{k=1}^{n} \frac{S_k f(y)}{k} = \int_Q f(x+y)H_n(x)\,dx \ (n \in \mathbb{P}, y \in Q).$$

In this paper we prove the following a.e. convergence theorems with respect to the Fejér and logarithmic means of subsequences of the partial sums of the Walsh-Fourier series of integrable functions.

Theorem 1. *Let $a : \mathbb{N} \to \mathbb{N}$ be a sequence with property $\frac{a(n+1)}{a(n)} \geq q > 1 \ (n \in \mathbb{N})$. Then for all integrable function $f \in L^1(Q)$ we have the a.e. relation*

$$\frac{1}{N}\sum_{n=1}^{N} S_{a(n)} f \to f.$$

Theorem 2. *Let $a : \mathbb{N} \to \mathbb{N}$ be a convex sequence with property $a(+\infty) = +\infty$. Then for each integrable function f we have the a.e. relation*

$$\frac{1}{\log N}\sum_{n=1}^{N} \frac{S_{a(n)} f}{n} \to f.$$

In this paper in Theorem 2 above, $a : \mathbb{N} \to \mathbb{N}$ is a convex sequence with property $a(+\infty) = +\infty$. Then a is strictly monotone increasing from some natural number. Since in the point of view of convergence of logarithmic means, a finite number of partial sums have no importance, then we can suppose that the sequence a is is strictly monotone increasing on $\mathbb{N}$. We can also suppose that $a(0) = 0$. These assumptions for a can be supposed without loss of generality. The character C denotes a positive constant which may vary from line to line and can depend only on a.

Expand every positive integer n with respect to the binary number system as

$$n = 2^{n_{(\alpha)}} + \cdots + 2^{n_{(0)}} = \sum_{i=0}^{\infty} n_i 2^i,$$

where $n_{(\alpha)} > \cdots > n_{(0)} \geq 0$ are integers and $n_i \in \{0,1\}$ for $i \in \mathbb{N}$. Denote the lower integer part of the binary logarithm of such an n by $|n|$. That is, $2^{|n|} \leq n < 2^{|n|+1}$ $(n \geq 1)$. To tell the truth α depends on n of course, but this notation will not cause any misunderstood. If it is absolutely necessary, then we use $\alpha(n)$. That is,

$$n = \sum_{i=0}^{\alpha(n)} 2^{n_{(i)}}.$$

Moreover, we also use the notations:

$$n^j = \sum_{i=j}^{\infty} n_i 2^i, \quad n^{(j)} = \sum_{i=j}^{\alpha} 2^{n_{(i)}}.$$

For $n, i \in \mathbb{N}, n \geq 1$ set the two-dimensional sequences $\lambda_{n,i}, d^1_{n,i}, d^2_{n,i}$ as:

$$\lambda_{n,i} = \begin{cases} 0, & \text{if } i \in \left\{n_{(1)}, n_{(2)}, \ldots, n_{(\alpha)}\right\} \text{ or } i \notin [n_{(0)}, n_{(\alpha)}), \\ 1, & \text{otherwise}, \end{cases}$$

$$d^1_{n,i} = \lambda_{n,i} \omega_{n^{i+1}} D_{2^i}, \quad d^2_{n,i} = \lambda_{n,i} \omega_{n^{i+1}} D_{2^{i+1}}.$$

We prove the following version of the decomposition of the Dirichlet kernel functions.

Lemma 3. *Let n be a positive integer. Then*

$$D_n = D_{2^{|n|+1}} + \sum_{i=0}^{|n|-1} (d^1_n - d^2_n).$$

Proof. For the sake of the proof of this lemma we introduce the notation for $n, j \in \mathbb{N}, n \geq 1$:

$$D_{n,j} = r_{n_{(\alpha)}} \ldots r_{n_{(j+1)}} D_{2^{n_{(j)}}}, \quad j = 0, \ldots, \alpha - 1.$$

Moreover, $D_{n,\alpha} = D_{2^{n_{(\alpha)}}} = D_{2^{|n|}}$ and $D_{n,j} = 0$ for $j > \alpha$. Consequently, for $j = 0, \ldots, \alpha - 1$ we have

$$\begin{aligned} &D_{n,j} \\ &= r_{n_{(\alpha)}} \ldots r_{n_{(j+1)}} \left(D_{2^{n_{(j)}}} - D_{2^{n_{(j+1)}}}\right) + r_{n_{(\alpha)}} \ldots r_{n_{(j+2)}} \left(D_{2^{n_{(j+1)}+1}} - D_{2^{n_{(j+1)}}}\right) \\ &= \omega_{n^{(j+1)}} \left(D_{2^{n_{(j)}}} - D_{2^{n_{(j)}+1}}\right) - \omega_{n^{(j+2)}} \left(D_{2^{n_{(j+1)}}} - D_{2^{n_{(j+1)}+1}}\right) \\ &+ \sum_{i=n_{(j)}+1}^{n_{(j+1)}-1} \omega_{n^{(j+1)}} \left(D_{2^i} - D_{2^{i+1}}\right). \end{aligned}$$

Moreover, by [13] it is well-known that

$$D_n = \sum_{j=0}^{\alpha} D_{n,j}.$$

By this and the above we get

$$\begin{aligned}D_n &= \sum_{j=0}^{\alpha} D_{n,j} = D_{2^{|n|}} + \sum_{j=0}^{\alpha-1} D_{n,j} \\ &= D_{2^{|n|}} + \sum_{j=0}^{\alpha-1} \left(\omega_{n^{(j+1)}}\left(D_{2^{n_{(j)}}} - D_{2^{n_{(j)}+1}}\right) - \omega_{n^{(j+2)}}\left(D_{2^{n_{(j+1)}}} - D_{2^{n_{(j+1)}+1}}\right)\right) \\ &+ \sum_{j=0}^{\alpha-1} \sum_{i=n_{(j)}+1}^{n_{(j+1)}-1} \omega_{n^{(j+1)}}\left(D_{2^i} - D_{2^{i+1}}\right) =: D_{2^{|n|}} + A + B.\end{aligned}$$

First we discuss addend A. It is easy to have by its telescopic property that

$$\begin{aligned}&D_{2^{|n|}} + A \\ &= D_{2^{|n|}} + \omega_{n^{(1)}}\left(D_{2^{n_{(0)}}} - D_{2^{n_{(0)}+1}}\right) - \left(D_{2^{n_{(\alpha)}}} - D_{2^{n_{(\alpha)}+1}}\right) \\ &= D_{2^{|n|+1}} + \omega_{n^{(1)}}\left(D_{2^{n_{(0)}}} - D_{2^{n_{(0)}+1}}\right).\end{aligned}$$

Discuss addend B. Since $n^{(j+1)} = 2^{n_{(\alpha)}} + \cdots + 2^{n_{(j+1)}}$ and $i \in \left\{n_{(j)}+1, \ldots, n_{(j+1)}-1\right\}$, then we have

$$n^{i+1} = \sum_{k=i+1}^{\infty} n_k 2^k = \sum_{n_{(k)} \geq i+1} 2^{n_{(k)}} = \sum_{k=j+1}^{\alpha} 2^{n_{(k)}} = n^{(j+1)}.$$

Consequently,

$$\begin{aligned}B &= \sum_{j=0}^{\alpha-1} \sum_{i=n_{(j)}+1}^{n_{(j+1)}-1} \omega_{n^{i+1}}\left(D_{2^i} - D_{2^{i+1}}\right) \\ &= \sum_{j=0}^{\alpha-1} \sum_{i=n_{(j)}+1}^{n_{(j+1)}} \lambda_{n,i}\omega_{n^{i+1}}\left(D_{2^i} - D_{2^{i+1}}\right) \\ &= \sum_{i=n_{(0)}+1}^{n_{(\alpha)}-1} \lambda_{n,i}\omega_{n^{i+1}}\left(D_{2^i} - D_{2^{i+1}}\right).\end{aligned}$$

That is,

$$D_n = D_{2^{|n|+1}} + \sum_{i=n_{(0)}}^{n_{(\alpha)}-1} \lambda_{n,i}\,\omega_{n^{i+1}}\left(D_{2^i} - D_{2^{i+1}}\right) = D_{2^{|n|+1}} + \sum_{i=0}^{|n|-1} \lambda_{n,i}\,\omega_{n^{i+1}}\left(D_{2^i} - D_{2^{i+1}}\right).$$

This completes the proof of Lemma 3. □

Set

$$d_{n,i} = d^1_{n,i} - d^2_{n,i}, \quad \tilde{D}_n = \sum_{i=0}^{|n|-1} d_{n,i}.$$

That is,

$$D_n = \tilde{D}_n + D_{2^{|n|+1}}. \tag{2}$$

The following lemma will play a prominent role in this paper.

Lemma 4. *Let $f, g \in L^1, 1 \le n \in \mathbb{N}, i, j \in \{0, 1, \dots, |n| - 1\}$. Then*

$$\langle \phi(f * d_{n,i}), \psi(g * d_{n,j}) \rangle = 0,$$

for all $i \neq j$, where ϕ is an $\mathcal{A}_i$ and ψ is an $\mathcal{A}_j$ measurable function.

Moreover, if $1 \le m, n \in \mathbb{N}, i, j \in \mathbb{N}, i < |m|, j < |n|$ and $|m| \neq |n|$, then we have again

$$\langle \phi(f * d_{m,i}), \psi(g * d_{n,j}) \rangle = 0.$$

Proof. We begin with the proof of the first orthogonality relation. Let say $i > j$. $d_{n,j} = \lambda_{n,j}\omega_{n^{j+1}}(D_{2^j} - D_{2^{j+1}})$ and consequently, if $\lambda_{n,j} = 0$, then there is nothing left to be proved. So, $\lambda_{n,j} = 1$ can be supposed which gives $j \ge n_{(0)}$ and consequently $i > n_{(0)}$. We can also suppose that $\lambda_{n,i} \neq 0$, otherwise there would be nothing left to be proved again. This implies $i \neq n_{(0)}, n_{(1)}, \dots, n_{(\alpha)}$ and by this we get $n_i = 0$. As a consequence of the equality $\lambda_{n,j} = \lambda_{n,i} = 1$ we have

$$d_{n,i} = \omega_{n^{i+1}} (D_{2^i} - D_{2^{i+1}}) = -r_i\omega_{n^{i+1}} D_{2^i} = - \sum_{k=n^{i+1}+2^i}^{n^{i+1}+2^i+2^{i-1}+\dots+2^0} \omega_k.$$

By the same way we have

$$d_{n,j} = - \sum_{l=n^{j+1}+2^j}^{n^{j+1}+2^j+2^{j-1}+\dots+2^0} \omega_l.$$

If

$$\begin{aligned} l &= n^{j+1} + 2^j + l_{j-1}2^{j-1} + \dots + l_0 2^0 \\ &= n^{i+1} + n_i 2^i + \dots + n_{j+1}2^{j+1} + 2^j + l_{j-1}2^{j-1} + \dots + l_0 2^0 \\ &= n^{i+1} + n_{i-1}2^{i-1} + \dots + n_{j+1}2^{j+1} + 2^j + l_{j-1}2^{j-1} + \dots + l_0 2^0. \end{aligned}$$

Consequently, $l - n^{i+1} < 2^i$. Besides, for k appearing in the sum $d_{n,i}$ we have $k \ge n^{i+1} + 2^i$. As a result we have $l \neq k$. Thus, $\langle \omega_k, \omega_l \rangle = 0$ for every ω_k and ω_l appearing in the sums $d_{n,i}$ and $d_{n,j}$. since the function ϕ is $\mathcal{A}_i$ measurable, then $\phi = \sum_{l=0}^{2^i-1} \hat{\phi}(l)\omega_l$ and consequently

$$\phi(f * d_{n,i}) = - \sum_{l=0}^{2^i-1} \hat{\phi}(l)\omega_l \sum_{k=n^{i+1}+2^i}^{n^{i+1}+2^i+2^{i-1}+\dots+2^0} \hat{f}(k)\omega_k = \sum_{k=n^{i+1}+2^i}^{n^{i+1}+2^i+2^{i-1}+\dots+2^0} c_{f,\phi}(k)\omega_k$$

for some complex numbers $c_{f,\phi}(k)$. Similarly,

$$\psi(f * d_{n,j}) = \sum_{l=n^{j+1}+2^j}^{n^{j+1}+2^j+2^{j-1}+\dots+2^0} c_{f,\psi}(l)\omega_l$$

and consequently by $\langle \omega_k, \omega_l \rangle = 0$ we have $\langle \phi(f * d_{n,i}), \psi(g * d_{n,j}) \rangle = 0$. That is, the first orthogonality relation of this lemma is proved.

Now, turn our attention to the second orthogonality relation. Just as in the investigation of the case of the first orthogonality we have

$$\psi(f * d_{n,j}) = \sum_{l=n^{j+1}+2^j}^{n^{j+1}+2^j+2^{j-1}+\dots+2^0} c_{f,\psi}(l)\omega_l$$

once again and also

$$\phi(f * d_{m,i}) = \sum_{k=m^{i+1}+2^i}^{m^{i+1}+2^i+2^{i-1}+\cdots+2^0} c_{f,\phi}(k)\omega_k.$$

since $i \leq |m|-1$, then $i+1 \leq |m|$, that is $|m^{i+1}| = |m|$. Similarly, $|n^{j+1}| = |n|$. Thus, $|k| = |m| \neq |n| = |l|$ and consequently $k \neq l$ which by $\langle \omega_k, \omega_l \rangle = 0$ proves the second orthogonality relation. This completes the proof of Lemma 4. □

For each $\lambda > 0$ real (there is no connection between the real λ and the zero-one sequence $\lambda_{n,i}$) define the following stopping time [13, 4]

$$\nu_\lambda(x) := \inf\{n \in \mathbb{N} : E_n(|f|)(x) > \lambda\} \quad (\inf \emptyset = +\infty).$$

Then the function $1_{\{\nu_\lambda > i\}}$ is $\mathcal{A}_i$ measurable. (1_X is the characteristic function of the set X.) We prove the inquality

$$1_{\{\nu_\lambda > i\}} \leq 1_{\{\nu_{2\lambda} > i+1\}}. \tag{3}$$

to prove (3) we suppose that $1_{\{\nu_\lambda > i\}}(x) = 1$. Then we get

$$E_0(|f|)(x), \ldots, E_i(|f|)(x) \leq \lambda.$$

Thus,

$$E_{i+1}(|f|)(x) = 2^{i+1}\int_{I_{i+1}(x)} |f|(y)dy \leq 2 \cdot 2^i \int_{I_i(x)} |f|(y)dy = 2E_i(|f|)(x) \leq 2\lambda.$$

This means $1_{\{\nu_{2\lambda} > i+1\}}(x) = 1$. That is, (3) is proved. Next, we prove the following inequality with respect to the kernels $d_{n,i}$. ($g * h$ is the dyadic convolution of functions g and h.)

Lemma 5. *Let $f \in L^1(Q), 1 \leq n \in \mathbb{N}, \lambda > 0$. Then we have*

$$\sum_{i=0}^{|n|-1} \|1_{\{\nu_\lambda > i\}}(f * d_{n,i})\|_2^2 \leq C\lambda \|f\|_1.$$

Proof. In the statement of Lemma 5 we have to take account only the addends for which $\lambda_{n,i} \neq 0$ of course. This means that $n_i = 0$ or $i = n_{(0)}$. If $i = n_{(0)}$, then apply (3):

$$\begin{aligned}
&\|1_{\{\nu_\lambda > n_{(0)}\}}(f * d_{n,n_{(0)}})\|_2^2 \\
&\leq \left\| 1_{\{\nu_{2\lambda} > n_{(0)}+1\}} \left(E_{n_{(0)}}|f| + E_{n_{(0)}+1}|f| \right)^2 \right\|_1 \\
&\leq \left\| 4\lambda \left(E_{n_{(0)}}|f| + E_{n_{(0)}+1}|f| \right) \right\|_1 \leq 8\lambda \|f\|_1.
\end{aligned}$$

In the other cases (that is, $n_i = 0$) we have

$$\begin{aligned}
f * d_{n,i} &= \omega_{n^{i+1}}(E_i(f\omega_{n^{i+1}}) - E_{i+1}(f\omega_{n^{i+1}})) \\
&= \omega_{n^i}(E_i(f\omega_{n^i}) - E_{i+1}(f\omega_{n^i})) = \omega_n(E_i(f\omega_n) - E_{i+1}(f\omega_n)).
\end{aligned}$$

Let $g = f\omega_n$ and apply the inequality of Burkholder [4] (note that $|g| = |f|$ and consequently the stopping time ν_λ for f and g coincides)

$$\begin{aligned}
&\sum_{i=0, i\neq n_{(0)}}^{|n|-1} \|1_{\{\nu_\lambda > i\}} |f * d_{n,i}|^2\|_1 \\
&\leq \sum_{i=0, i\neq n_{(0)}}^{|n|-1} \|1_{\{\nu_{2\lambda} > i+1\}} |f * d_{n,i}|^2\|_1 \\
&= \sum_{i=0, i\neq n_{(0)}}^{|n|-1} \|1_{\{\nu_{2\lambda} > i+1\}} |E_i(g) - E_{i+1}(g)|^2\|_1 \\
&\leq \sum_{i=0}^{\infty} \|1_{\{\nu_{2\lambda} > i+1\}} |E_i(g) - E_{i+1}(g)|^2\|_1 \\
&\leq C2\lambda\|g\|_1 = C\lambda\|f\|_1.
\end{aligned}$$

This completes the proof of Lemma 5. □

With the application of Lemmas 3, 4 and 5 we are ready to prove Theorem 1.

The proof of Theorem 1. First of all, we suppose that $q \geq 2$. At the end of the proof of this theorem we turn back to the case $2 > q > 1$. Use the following notations

$$\tilde{S}_k f = f * \tilde{D}_k, \quad |a(n)| = A(n) = a_\alpha(n), \quad a(n) = 2^{a_{(\alpha)}(n)} + \cdots + 2^{a_{(0)}(n)}$$

and investigate the integral

$$I := \left\| \sum_{n=1}^{N} \sum_{i=0}^{A(n)-1} 1_{\{\nu_\lambda > i\}} (f * d_{a(n),i}) \right\|_2^2 .$$

If $m > n$, then $A(m) = |a(m)| > |a(n)| = A(n)$ since $a(m) \geq 2a(n)$. Consequently, by the second orthogonality relation of Lemma 4 we have

$$\langle 1_{\{\nu_\lambda > i\}} (f * d_{a(m),i}), 1_{\{\nu_\lambda > j\}} (f * d_{a(n),j}) \rangle = 0$$

for all $i < A(m), j < A(n)$ integers. This lemma can be applied since $1_{\{\nu_\lambda > i\}}$ is $\mathcal{A}_i$ measurable and $1_{\{\nu_\lambda > j\}}$ is $\mathcal{A}_j$ measurable. That is, we get

$$I = \sum_{n=1}^{N} \sum_{i=0}^{A(n)-1} \left\| 1_{\{\nu_\lambda > i\}} (f * d_{a(n),i}) \right\|_2^2 ,$$

which is bounded by $CN\lambda\|f\|_1$ as it comes from Lemma 5.

After set the Fejér means with respect to our subsequence of the partial sums of the Fourier series as

$$\sigma_N f := \frac{1}{N} \sum_{n=1}^{N} S_{a(n)} f.$$

By the help of Lemma 3 we have

$$\sigma_N f = \frac{1}{N} \sum_{n=1}^{N} S_{2^{A(n)+1}} f + \frac{1}{N} \sum_{n=1}^{N} f * \tilde{D}_{a(n)} =: II + III.$$

It is well-known that the power of two indexed partial sums converge to the function almost everywhere (see e.g. [13]) and therefore the a.e. relation $\lim_{N\to\infty} II = f$ is trivial. Consequently, we have to discuss term III only. Recall the definition of the stopping time ν_λ. $\nu_\lambda(x) := \inf\{n \in \mathbb{N} : E_n(|f|)(x) > \lambda\}$ $(\inf \emptyset = +\infty)$. In [12] one can find the well-known inequality

$$\operatorname{mes}\{\nu_\lambda < \infty\} = \operatorname{mes}\{|f|^* > \lambda\} \le \frac{C}{\lambda}\|f\|_1,$$

where $h^* = \sup |E_n(h)|$. Therefore

$$\begin{aligned}
&\operatorname{mes}\left\{x \in Q : \frac{1}{N}\left|\sum_{n=1}^{N} f * \tilde{D}_{a(n)}(x)\right| > \lambda\right\}\\
&\le \operatorname{mes}\{\nu_\lambda < \infty\} + \operatorname{mes}\left\{x \in Q : \nu_\lambda(x) = \infty, \frac{1}{N}\left|\sum_{n=1}^{N}\sum_{i=0}^{A(n)-1} (f * d_{a(n),i})(x)\right| > \lambda\right\}\\
&=: III_1 + III_2.
\end{aligned}$$

$III_1 \le \frac{C}{\lambda}\|f\|_1$ as it has proven. Since in the case $\nu_\lambda(x) = \infty$ we have $1_{\{\nu_\lambda > i\}}(x) = 1$ for every i, x, then it follows

$$\begin{aligned}
&III_2\\
&\le \operatorname{mes}\left\{x \in Q : \frac{1}{N}\left|\sum_{n=1}^{N}\sum_{i=0}^{A(n)-1} 1_{\{\nu_\lambda > i\}}(x)(f * d_{a(n),i})(x)\right| > \lambda\right\}\\
&\le \frac{1}{\lambda^2}\frac{1}{N^2}\left\|\sum_{n=1}^{N}\sum_{i=0}^{A(n)-1} 1_{\{\nu_\lambda > i\}}(f * d_{a(n),i})\right\|_2^2\\
&\le C\frac{1}{\lambda^2}\frac{1}{N}\lambda\|f\|_1,
\end{aligned}$$

where the last inequality is implied by the inequality given for I at the beginning of the proof of this theorem. In other words, we proved for the operator $\tilde{\sigma}_N f := \frac{1}{N}\sum_{n=1}^{N} f * \tilde{D}_{a(n)}$ that

$$\operatorname{mes}\{\nu_\lambda = \infty, |\tilde{\sigma}_N f| > \lambda\} \le \frac{C}{N\lambda}\|f\|_1.$$

This last inequality immediately gives

$$\operatorname{mes}\left\{\nu_\lambda = \infty, \sup_k |\tilde{\sigma}_{k^2} f| > \lambda\right\} \le \sum_{k=1}^{\infty} \operatorname{mes}\{\nu_\lambda = \infty, |\tilde{\sigma}_{k^2} f| > \lambda\} \le \frac{C}{\lambda}\|f\|_1.$$

Let now $k^2 \le m < (k+1)^2$. Then

$$|\tilde{\sigma}_m f| \le |\tilde{\sigma}_{k^2} f| + \frac{1}{k^2}\left|\sum_{n=k^2+1}^{m} \tilde{S}_{a(n)} f\right|.$$

Have a look at the beginning the proof of this theorem. More precisely, the investigation of term I by the two orthogonality relations of Lemma 4. That is,

$$\begin{aligned}
&\operatorname{mes}\left\{x \in Q : \frac{1}{k^2}\left|\sum_{n=k^2+1}^{m}\sum_{i=0}^{A(n)-1} 1_{\{\nu_\lambda > i\}}(x)(f * d_{a(n),i})(x)\right| > \lambda\right\}\\
&\frac{1}{\lambda^2}\frac{1}{k^4}\left\|\sum_{n=k^2+1}^{m}\sum_{i=0}^{A(n)-1} 1_{\{\nu_\lambda > i\}}(f * d_{a(n),i})\right\|_2^2\\
&\leq \frac{1}{\lambda^2}\frac{1}{k^4}\sum_{n=k^2+1}^{m}\sum_{i=0}^{A(n)-1}\left\|1_{\{\nu_\lambda > i\}}(f * d_{a(n),i})\right\|_2^2\\
&\leq \frac{1}{\lambda^2}\frac{1}{k^4}C(m-k^2)\lambda\|f\|_1 \leq \frac{C}{\lambda k^3}\|f\|_1.
\end{aligned}$$

This implies

$$\begin{aligned}
&\operatorname{mes}\left\{\sup_m |\tilde{\sigma}_m f| > \lambda\right\}\\
&\leq \operatorname{mes}\{\nu_\lambda < \infty\} + \operatorname{mes}\left\{\nu_\lambda = \infty, \sup_k |\tilde{\sigma}_{k^2} f| > \lambda\right\}\\
&+\sum_{k=1}^{\infty}\sum_{m=k^2}^{(k+1)^2-1}\operatorname{mes}\left\{\frac{1}{k^2}\left|\sum_{n=k^2+1}^{m}\sum_{i=0}^{A(n)-1} 1_{\{\nu_\lambda > i\}}(f * d_{a(n),i})\right| > \lambda\right\}\\
&\leq \frac{C}{\lambda}\|f\|_1 + \sum_{k=1}^{\infty}\sum_{m=k^2}^{(k+1)^2-1}\frac{C}{\lambda k^3}\|f\|_1\\
&\leq \frac{C}{\lambda}\|f\|_1.
\end{aligned}$$

Since $\sigma_N f = \frac{1}{N}\sum_{n=1}^{N} S_{2^{A(n)+1}} f + \tilde{\sigma}_N f$ and since the first addend of the right side is bounded by f^*, then we have that the maximal operator $\sup|\sigma_N|$ is of weak type (L^1, L^1), that is,

$$\operatorname{mes}\left\{\sup_N |\sigma_N f| > \lambda\right\} \leq \frac{C}{\lambda}\|f\|_1.$$

Since for each Walsh polynomial $P = \sum_{l=0}^{L} c_l \omega_l$ we have $\lim_N \sigma_N P = P$ everywhere (because $S_u P = P$ for $u > L$), then by the standard density argument (the set of Walsh polynomials is dense in $L^1(Q)$) and by the weak (L^1, L^1) typeness of the maximal operator $\sup|\sigma_N|$ we complete the proof of Theorem 1 in the case of $q \geq 2$.

Finally, we turn the attention to the case $2 > q > 1$. Recall that $a(n+1)/a(n) \geq q$. Set $\beta = \lceil\frac{\log 2}{\log q}\rceil$ ($\lceil x\rceil$ denotes the upper integer part of x). Then $1 \leq \beta \in \mathbb{N}$ and $q^\beta \geq 2$. This follows that for every $l = 1, \ldots, \beta$ we have $a((k+1)\beta + l)/a(k\beta + l) \geq q^\beta \geq 2$ and consequently the above written gives

$$\frac{1}{N}\sum_{k=0}^{N-1} S_{a(k\beta+l)} f \to f$$

a.e. for each $f \in L^1$. Now, let $1 \leq M \in \mathbb{N}$ and $M = N\beta + j$, where $N, j \in \mathbb{N}$ and $1 \leq j \leq \beta$. Thus,

$$\begin{aligned}
&\frac{1}{M}\sum_{k=1}^{M} S_{a(k)}f \\
&= \frac{1}{M}\sum_{l=1}^{j}\sum_{k=0}^{N} S_{a(k\beta+l)}f + \frac{1}{M}\sum_{l=j+1}^{\beta}\sum_{k=0}^{N-1} S_{a(k\beta+l)}f \\
&= \frac{N}{M}\left(\sum_{l=1}^{j}\frac{1}{N}\sum_{k=0}^{N} S_{a(k\beta+l)}f + \sum_{l=j+1}^{\beta}\frac{1}{N}\sum_{k=0}^{N-1} S_{a(k\beta+l)}f\right) \to f
\end{aligned}$$

a.e. because $\frac{N}{M} \to \frac{1}{\beta}$ as $M \to \infty$. This completes the proof of Theorem 1. □

Denote by $F_a(x) := \max\{n : a(n) < x\}$ or simple $F(x)$ the distribution function of a. That is, $a(F(l)) < l$ and $a(F(l)+1) \geq l$. Later, we need the following concavity-type inequality with respect to F.

Lemma 6. *$F(y) \leq \frac{y}{t}(F(t)+1)$ for all $y \geq t > 0$.*

Proof. Let $y > t > 0$. For the sake of the proof of this lemma we introduce the function $\tilde{a} : [0, +\infty) \to [0, +\infty)$ as $\tilde{a}(n) = a(n)$ for $n \in \mathbb{N}$ and $\tilde{a}(z) = (a(n+1) - a(n))(z-n) + a(n)$ for $n < z < n+1$. That is, $\tilde{a}$ is linear on the interval $[n, n+1]$ ($n \in \mathbb{N}$). It is obvious that $\tilde{a}$ is a convex, strictly monotone increasing bijective function. Its distribution function $F_{\tilde{a}}(x) := \max\{n : \tilde{a}(n) < x\} = \max\{n : a(n) < x\} = F_a(x) = F(x)$ is the same as the distribution function of a. The definition of the distribution function F gives

$$\tilde{a}(F(y)) < y, \quad \tilde{a}(F(t)+1) \geq t.$$

Since $\tilde{a}$ is convex, then for $x = \tilde{a}^{-1}(t)$ we have

$$\tilde{a}\left(\frac{y}{t}x\right) \geq \frac{y}{t}\tilde{a}(x) = y.$$

Consequently, the strictly monotone increasing property of $\tilde{a}$ gives

$$F(y) < \tilde{a}^{-1}(y) \leq \tilde{a}^{-1}\left(\tilde{a}\left(\frac{y}{t}x\right)\right) = \frac{y}{t}x = \frac{y}{t}\tilde{a}^{-1}(t) \leq \frac{y}{t}[F(t)+1].$$

This completes the proof of Lemma 6. □

Moreover, define the sequence of natural numbers b as $b_l := \min\left\{n : F(2^n) \geq 2^l\right\}$. Since $\lim_{+\infty} F = +\infty$, then b is well-defined. Also define the operator T_L as

$$T_L f := \frac{1}{\log F(2^{b_{L+1}})}\sum_{l=0}^{L}\sum_{n=F(2^{b_l})+1}^{F(2^{b_{l+1}})}\frac{1}{n}\sum_{i=0}^{A(n)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(n),i}).$$

In the sequel we prove a lemma which will play a prominent role in the proof of the Theorem 2 with respect to the logarithmic means of the partial sums $S_{a(n)}f$.

Lemma 7. *$\|T_L f\|_2^2 \leq \frac{C\lambda}{L}\|f\|_1$ for each integrable function f.*

Proof. For every $n_j \in (F(2^{b_{l_j}}), F(2^{b_{l_j+1}})]$, $j = 1, 2$, $l_1 \neq l_2$ we have

$$2^{b_{l_j}} \leq a(F(2^{b_{l_j}}) + 1) \leq a(n_j) \leq a(F(2^{b_{l_j+1}})) < 2^{b_{l_j+1}},$$

that is, $A(n_j) = |a(n_j)| \in [b_{l_j}, b_{l_j+1})$ for $j = 1, 2$ and if - say - $l_1 < l_2$, then

$$A(n_1) < b_{l_1+1} \leq b_{l_2} \leq A(n_2).$$

Consequently, $A(n_1) \neq A(n_2)$. This by the second orthogonality relation in lemma 4 gives

$$\|T_L f\|_2^2 = \frac{1}{\log^2 F(2^{b_{L+1}})} \sum_{l=0}^{L} \left\| \sum_{n=F(2^{b_l})+1}^{F(2^{b_{l+1}})} \frac{1}{n} \sum_{i=0}^{A(n)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(n),i}) \right\|_2^2 .$$

Next, we investigate the integral in the sum of the right side of this equation.

$$\begin{aligned}
&\left\| \sum_{n=F(2^{b_l})+1}^{F(2^{b_{l+1}})} \frac{1}{n} \sum_{i=0}^{A(n)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(n),i}) \right\|_2^2 \\
&\leq \sum_{n,m=F(2^{b_l})+1}^{F(2^{b_{l+1}})} \left| \left\langle \frac{1}{n} \sum_{i=0}^{A(n)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(n),i}), \frac{1}{m} \sum_{i=0}^{A(m)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(m),i}) \right\rangle \right| \\
&\leq \sum_{n,m=F(2^{b_l})+1}^{F(2^{b_{l+1}})} \left\| \frac{1}{n} \sum_{i=0}^{A(n)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(n),i}) \right\|_2 \cdot \left\| \frac{1}{m} \sum_{i=0}^{A(m)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(m),i}) \right\|_2 \\
&\leq \sum_{n,m=F(2^{b_l})+1}^{F(2^{b_{l+1}})} \frac{1}{2} \left\| \frac{1}{n} \sum_{i=0}^{A(n)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(n),i}) \right\|_2^2 + \frac{1}{2} \left\| \frac{1}{m} \sum_{i=0}^{A(m)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(m),i}) \right\|_2^2 \\
&= \sum_{n,m=F(2^{b_l})+1}^{F(2^{b_{l+1}})} \frac{1}{n^2} \left\| \sum_{i=0}^{A(n)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(n),i}) \right\|_2^2 .
\end{aligned}$$

The first orthogonality relation in in lemma 4 and Lemma 5 give that

$$\left\| \sum_{i=0}^{A(n)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(n),i}) \right\|_2^2 = \sum_{i=0}^{A(n)-1} \|1_{\{\mu_\lambda > i\}}(f * d_{a(n),i})\|_2^2 \leq C\lambda \|f\|_1.$$

Since by Lemma 6 $F(2^{b_{l+1}}) \leq 2F(2^{b_{l+1}-1}) + 2 \leq C2^l$, $F(2^{b_l}) \geq 2^l$ and

$$\begin{aligned}
&\sum_{n,m=F(2^{b_l})+1}^{F(2^{b_{l+1}})} \frac{1}{n^2} \\
&= [F(2^{b_{l+1}}) - F(2^{b_l})] \sum_{n=F(2^{b_l})+1}^{F(2^{b_{l+1}})} \frac{1}{n^2} \\
&\leq C[F(2^{b_{l+1}}) - F(2^{b_l})] \left(\frac{1}{F(2^{b_l})} - \frac{1}{F(2^{b_{l+1}})} \right) \leq C,
\end{aligned}$$

then we have

$$\left\| \sum_{n=F(2^{b_l})+1}^{F(2^{b_{l+1}})} \frac{1}{n} \sum_{i=0}^{A(n)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(n),i}) \right\|_2^2 \leq C\lambda \|f\|_1.$$

This inequality immediately gives for the L^2 norm of the operator T_L that

$$\|T_L f\|_2^2 \leq \frac{1}{\log^2 F(2^{b_{L+1}})} \sum_{l=0}^{L} C\lambda \|f\|_1 \leq \frac{C\lambda}{L} \|f\|_1$$

because $F(2^{b_{L+1}}) \geq 2^{L+1}$. This completes the proof of Lemma 7. □

We apply Lemma 7 in order to get a bound for the maximal operator of T_L.

Lemma 8. $\| \sup_L |T_L f| \|_2^2 \leq C\lambda \|f\|_1$.

Proof. Let $M^2 \leq L < (M+1)^2$. Then

$$\begin{aligned} T_L f &:= \frac{1}{\log F(2^{b_{L+1}})} \sum_{l=0}^{M^2} \sum_{n=F(2^{b_l})+1}^{F(2^{b_{l+1}})} \frac{1}{n} \sum_{i=0}^{A(n)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(n),i}) \\ &+ \frac{1}{\log F(2^{b_{L+1}})} \sum_{l=M^2+1}^{L} \sum_{n=F(2^{b_l})+1}^{F(2^{b_{l+1}})} \frac{1}{n} \sum_{i=0}^{A(n)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(n),i}) \\ &=: T_L^1 f + T_L^2 f. \end{aligned}$$

That is, $|T_L^1 f| \leq \sup_M |T_{M^2} f|$ and by Lemma 7 we get

$$\|T_L^1 f\|_2^2 \leq \|T_{M_2} f\|_2^2 \leq \frac{C\lambda}{M^2} \|f\|_1.$$

Hence

$$\| \sup_L |T_L f| \|_2^2 \leq C \| \sup_M |T_{M_2} f| \|_2^2 + C \| \sup_L |T_L^2 f| \|_2^2.$$

Besides, following the proof of Lemma 7 we can give an upper bound for the L^2 norm of $T_L^2 f$.

$$\begin{aligned} \|T_L^2 f\|_2^2 &= \frac{1}{\log^2 F(2^{b_{L+1}})} \sum_{l=M^2+1}^{L} \left\| \sum_{n=F(2^{b_l})+1}^{F(2^{b_{l+1}})} \frac{1}{n} \sum_{i=0}^{A(n)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(n),i}) \right\|_2^2 \\ &\leq \frac{1}{\log^2 F(2^{b_{L+1}})} \sum_{l=M^2+1}^{L} \sum_{n,m=F(2^{b_l})+1}^{F(2^{b_{l+1}})} \frac{1}{n^2} \left\| \sum_{i=0}^{A(n)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(n),i}) \right\|_2^2 \\ &\leq \frac{C}{L^2} \sum_{l=M^2+1}^{L} \lambda \|f\|_1 \leq C\lambda \|f\|_1 \frac{1}{L^2} [(M+1)^2 - M^2] \leq \frac{C}{M^3} \lambda \|f\|_1. \end{aligned}$$

Therefore, $|T^1_{A^2+B}f| \le |T_{A^2}f|$ $(B \le 2A)$ implies

$$\begin{aligned}\|\sup_{B\le 2A}|T^1_{A^2+B}f|\|_2^2 &\le \|\sup_A |T_{A^2}f|\|_2^2\\ &\le C\int_0^1 \sum_{A=1}^\infty |T_{A^2}f|^2 \le C\sum_{A=1}^\infty \int_0^1 |T_{A^2}f|^2\\ &\le C\sum_{A=1}^\infty \frac{\lambda\|f\|_1}{A^2} \le C\lambda\|f\|_1.\end{aligned}$$

On the other hand,

$$\|\sup_{B\le 2A}|T^2_{A^2+B}f|\|_2^2 \le \int_0^1 \sum_{A=1}^\infty\sum_{B=0}^{2A}|T^2_{A^2+B}f|^2 \le \sum_{A=1}^\infty\sum_{B=0}^{2A}\frac{C}{A^3}\lambda\|f\|_1 \le C\lambda\|f\|_1.$$

Consequently,

$$\begin{aligned}&\|\sup_L |T_Lf|\|_2^2 = \|\sup_{B\le 2A}|T_{A^2+B}f|\|_2^2\\ &\le C\|\sup_{B\le 2A}|T^1_{A^2+B}f|\|_2^2 + C\|\sup_{B\le 2A}|T^2_{A^2+B}f|\|_2^2\\ &\le C\lambda\|f\|_1.\end{aligned}$$

This completes the proof of Lemma 8. □

For $K, L \in \mathbb{N}$ set

$$\begin{aligned}N_{K,L} := \Big\{&n\in\mathbb{N} : |n| = K, \exists i \in \{0,\dots,K-1\} \text{ such that } n_i = 1, n_0 = \dots = n_{i-1} = 0\\ &\text{and } |\{n_j : n_i = 0, i < j < K\}| \ge L\Big\}.\end{aligned}$$

Recall that for a positive integer n the number $|n|$ is the lower integer part of the binary logarithm of n and for a finite set X the number $|X|$ is the number of the elements of this set. The coordinate i is the minimal index for which $n_i = 1$ and there are at least L zeros among $n_{i+1}, n_{i+2}, \dots, n_{K-1}$. Of course if $K \le L$, then $N_{K,L} = \emptyset$ and if $K \ge L+1$, then $N_{K,L} \ne \emptyset$.

The next lemma will also be another important key tool in the proof of our theorem with respect to the logarithmic means of the partial sums $S_{a(n)}f$.

Lemma 9. *Let $K, L, M \in \mathbb{N}, F(2^K) < M \le F(2^{K+1})$. Then there exists a disjoint decomposition*

$$N_{K,L} = \bigcup_{\underline{j}\in\mathbb{P}^L} \Omega^{K,L}_{\underline{j}}$$

such that for $n \in \Omega_{\underline{j}}^{K,L}$ *denoting the* L*-tuple* $(j_{1,n}, \ldots, j_{L,n})$ *by* $\underline{j}$ *we have*

$$\begin{aligned}
&\sum_{n=F(2^K)+1}^{M} \frac{1}{n} \sum_{i=0}^{A(n)-1} 1_{\{\nu_\lambda > i\}}(f * d_{a(n),i}) \\
&= \sum_{\{n:\, A(n)=K,\, a(n)\notin N_{K,L},\, n\le M\}} \frac{1}{n} \sum_{i=0}^{A(n)-1} 1_{\{\nu_\lambda > i\}}(f * d_{a(n),i}) \\
&+ \sum_{\{n:\, a(n)\in N_{K,L},\, n\le M\}} \frac{1}{n} \sum_{k=1}^{L} 1_{\{\nu_\lambda > K-j_{1,n}-\cdots-j_{k,n}\}}(f * d_{a(n),K-j_{1,n}-\cdots-j_{k,n}}) \\
&+ \sum_{\{n:a(n)\in N_{K,L},\, n\le M\}} \frac{1}{n} \sum_{i=0}^{a_{(0)}(n)} 1_{\{\nu_\lambda > i\}}(f * d_{a(n),i}) \\
&+ \sum_{\{n:a(n)\in N_{K,L},\, n\le M\}} \frac{1}{n} \sum_{i=a_{(0)}(n)+1}^{A(n)-j_{1,n}-\cdots-j_{L,n}-1} 1_{\{\nu_\lambda > i\}}(f * d_{a(n),i}) \\
&=: I + II + III + IV,
\end{aligned}$$

and

$$|I| + |II| + |III| \le CL|f|^*, \quad \|IV\|_2^2 \le C\lambda \|f\|_1 \left(\frac{1}{3^L} + \frac{1}{F(2^K)} \right).$$

Proof. The definition of $N_{K,L}$ and $\lambda_{n,i}$ gives that $n \in N_{K,L}$ if and only if there are at least $L+1$ pieces of i such that $\lambda_{n,i} = 1$. Recall that $\lambda_{n,n_{(0)}} = 1$, where $n_{(0)}$ is the minimal coordinate of n which equals with 1. Let

$$\Omega_{j_1}^{K,L} := \{n \in N_{K,L} : \lambda_{n,K-1} = 0, \ldots, \lambda_{n,K-j_1+1} = 0, \lambda_{n,K-j_1} = 1\}.$$

Of course the set $\Omega_{j_1}^{K,L}$ may be $\emptyset$ if j_1 is too big - say - $j_1 > K - L$. This means

$$N_{K,L} = \bigcup_{j_1=1}^{\infty} \Omega_{j_1}^{K,L},$$

where this union is a disjoint one. $\Omega_{j_1}^{K,L}$ can also be decomposed in the same way, i.e.

$$\Omega_{j_1}^{K,L} = \bigcup_{j_2=1}^{\infty} \Omega_{j_1,j_2}^{K,L},$$

where $n \in \Omega_{j_1,j_2}^{K,L}$ means that

$$\begin{aligned}
&n \in N_{K,L}, \lambda_{n,K-1} = \cdots = \lambda_{n,K-j_1+1} = 0, \lambda_{n,K-j_1} = 1, \\
&\lambda_{n,K-j_1-1} = \cdots = \lambda_{n,K-j_1-j_2+1} = 0, \lambda_{n,K-j_1-j_2} = 1.
\end{aligned}$$

Going further we can write

$$N_{K,L} = \bigcup_{j_L=1}^{\infty} \cdots \bigcup_{j_1=1}^{\infty} \Omega_{j_1,\ldots,j_L}^{K,L} = \bigcup_{\underline{j}\in\mathbb{P}^L} \Omega_{\underline{j}}^{K,L}.$$

That is, (we emphasize this because will be important later in the proof) for $n \in \Omega_{\underline{j}}^{K,L}$ we have $n \in N_{K,L}$, $\lambda_{n,K-j_1} = \cdots = \lambda_{n,K-j_1-j_2-\cdots-j_L} = 1$ and for the other i's between $K-j_1$ and $K-j_1-\cdots-j_L$ (not belonging to the set $\{K-j_1, K-j_1-j_2, \ldots, K-j_1-\cdots-j_L\}$) we have $\lambda_{n,i} = 0$. That is, we decomposed the set $N_{K,L}$ into disjoint sets and also proved that the decomposition $I+II+III$ in the statement of this lemma fulfills. We can go further in the proof of this lemma. Next, we prove the inequality for $|I|+|II|+|III|$.

First, discuss I. since $A(n) = K$ and $a(n) \notin N_{K,L}$, then there are at most L pieces of index i in the set $\{0, \ldots, A(n)-1\}$ such that $\lambda_{a(n),i} \neq 0$. The definition of $d_{a(n),i}$ immediately gives

$$|f * d_{a(n),i}| \leq 2 \sup_{k\in\mathbb{N}} E_k|f| = 2|f|^*,$$

that is,

$$|I| \leq \sum_{\left\{n:\, A(n)=K,\, a(n)\notin N_{K,L},\, n\leq M\right\}} \frac{1}{n} L2|f|^*.$$

In the same way we have

$$|II| \leq \sum_{\left\{n:\, a(n)\in N_{K,L},\, n\leq M\right\}} \frac{1}{n} L2|f|^*.$$

Consequently,

$$|I|+|II| \leq C\left(\sum_{n=F(2^K)+1}^{F(2^{K+1})} \frac{1}{n} L|f|^*\right) \leq CL|f|^*$$

since by Lemma 6 we have $F(2^{K+1}) \leq 2(F(2^K)+1)$. The definition of $\lambda_{n,i}$ gives that for $i < a_{(0)}(n)$ we have $\lambda_{a(n),i} = 0$ and also that $\lambda_{a(n),a_{(0)}(n)} = 1$. This follows

$$III = \sum_{\left\{n:a(n)\in N_{K,L},\, n\leq M\right\}} \frac{1}{n} 1_{\left\{\nu_\lambda > a_{(0)}(n)\right\}}(f * d_{a(n),a_{(0)}(n)}),$$

$$|III| \leq C|f|^* \sum_{\left\{n:a(n)\in N_{K,L},\, n\leq M\right\}} \frac{1}{n} \leq C|f|^* \sum_{n=F(2^K)+1}^{F(2^{K+1})} \frac{1}{n} \leq C|f|^*.$$

That is, it is remained to discuss IV. it is obvious by the above that

$$IV = \sum_{j_L=1}^{\infty} \cdots \sum_{j_1=1}^{\infty} \sum_{\left\{n\,:\,a(n)\in\Omega_{\underline{j}}^{K,L},\, n\leq M\right\}} \frac{1}{n} \sum_{i=a_{(0)}(n)+1}^{A(n)-j_1-\cdots-j_L-1} 1_{\{\nu_\lambda > i\}}(f * d_{a(n),i}).$$

Let $\underline{j}$ and $\underline{\tilde{j}}$ be different L tuples, $a(n) \in \Omega_{\underline{j}}^{K,L}$ and $a(m) \in \Omega_{\underline{\tilde{j}}}^{K,L}$. We are to prove the following orthogonality relation

$$\langle 1_{\{\mu_\lambda > i\}}(f * d_{a(n),i}), 1_{\{\mu_\lambda > \tilde{\imath}\}}(f * d_{a(m),\tilde{\imath}})\rangle = 0 \tag{4}$$

for every

$$i \in \left\{a_{(0)}(n)+1, \ldots, A(n)-j_1-\cdots-j_L-1\right\},$$
$$\tilde{\imath} \in \left\{a_{(0)}(m)+1, \ldots, A(m)-\tilde{j}_1-\cdots-\tilde{j}_L-1\right\}.$$

If this scalar product differs from zero, then the equality $\lambda_{a(n),i} = \lambda_{a(m),\tilde{\imath}} = 1$ must hold. Suppose this. Since $i > a_{(0)}(n), \tilde{\imath} > a_{(0)}(m)$, then $a_i(n) = a_{\tilde{\imath}}(m) = 0$ as it comes from the definition of $\lambda_{n,i}$. This also follows $d_{a(n),i} = \omega_{a^{i+1}(n)}(D_{2^i} - D_{2^{i+1}})$. The function $1_{\{\nu_\lambda > i\}}$ is $\mathcal{A}_i$ measurable and consequently

$$1_{\{\mu_\lambda > i\}}(f * d_{a(n),i}) = \sum_{k=a^{i+1}(n)+2^i}^{a^{i+1}(n)+2^{i+1}-1} c_k\omega_k, \quad 1_{\{\mu_\lambda > \tilde{\imath}\}}(f * d_{a(n),\tilde{\imath}}) = \sum_{k=a^{\tilde{\imath}+1}(n)+2^{\tilde{\imath}}}^{a^{\tilde{\imath}+1}(n)+2^{\tilde{\imath}+1}-1} \tilde{c}_k\omega_k$$

for some $c_k, c_{\tilde{k}}$ complex numbers.

Say, $k = a^{i+1}(n) + 2^i + k_{i-1} + \cdots + k_0$, $\tilde{k} = a^{\tilde{\imath}+1}(n) + 2^{\tilde{\imath}} + \tilde{k}_{\tilde{\imath}-1} + \cdots + \tilde{k}_0$. Can k be equivalent with $\tilde{k}$? Since $j \neq \tilde{\jmath}$, then we find a $s \leq L$ such that

$$j_1 = \tilde{\jmath}_1, \ldots, j_{s-1} = \tilde{\jmath}_{s-1}, j_s \neq \tilde{\jmath}_s.$$

Say, $j_s < \tilde{\jmath}_s$. Then

$$\begin{aligned}
&\lambda_{a(n),K-j_1} = \cdots = \lambda_{a(n),K-j_1-\cdots-j_{s-1}} = \lambda_{a(n),K-j_1-\cdots-j_{s-1}-j_s} = 1,\\
&\lambda_{a(m),K-j_1} = \cdots = \lambda_{a(m),K-j_1-\cdots-j_{s-1}} = 1,\\
&\lambda_{a(m),K-j_1-\cdots-j_{s-1}-1} = \cdots = \lambda_{a(m),K-j_1-\cdots-j_{s-1}-j_s} = 0.
\end{aligned}$$

Thus, $a_{K-j_1-\cdots-j_s}(n) = 0$ and consequently $i+1 \leq K-j_1-\cdots-j_s-\cdots-j_L \leq K-j_1-\cdots-j_s$ gives

$$a^{i+1}_{K-j_1-\cdots-j_s}(n) = a_{K-j_1-\cdots-j_s}(n) = 0.$$

On the other hand,

$$\tilde{\imath} + 1 \leq K - \tilde{\jmath}_1 - \cdots - \tilde{\jmath}_L = K - j_1 - \ldots j_{s-1} - \tilde{\jmath}_s - \cdots - \tilde{\jmath}_L < K - j_1 - \ldots j_{s-1} - j_s.$$

Consequently,

$$a^{\tilde{\imath}+1}_{K-j_1-\cdots-j_s}(m) = a_{K-j_1-\cdots-j_s}(m) = 1.$$

(Recall that $\lambda_{a(m),K-j_1-\cdots-j_s} = 0$ and $K > K - j_1 - \cdots - j_s \geq K - j_1 - \cdots - j_L > a_{(0)}(n)$.) This means for $k = a^{i+1}(n) + 2^i + k_{i-1} + \cdots + k_0$ and $\tilde{k} = a^{\tilde{\imath}+1}(n) + 2^{\tilde{\imath}} + \tilde{k}_{\tilde{\imath}-1} + \cdots + \tilde{k}_0$ that $k_{K-j_1-\cdots-j_s} = a^{i+1}_{K-j_1-\cdots-j_s}(n) = 0$ and $\tilde{k}_{K-j_1-\cdots-j_s} = a^{\tilde{\imath}+1}_{K-j_1-\cdots-j_s}(m) = 1$. That is, $k \neq \tilde{k}$. This means that the orthogonality relation (4) is proved. This implies

$$\|IV\|_2^2 = \sum_{j_L=1}^{\infty} \cdots \sum_{j_1=1}^{\infty} \left\| \sum_{\left\{n\,:\,a(n)\in\Omega_{\underline{j}}^{K,L},\, n\leq M\right\}} \frac{1}{n} \sum_{i=a_{(0)}(n)}^{A(n)-j_1-\cdots-j_L-1} 1_{\{\nu_\lambda > i\}}(f * d_{a(n),i}) \right\|_2^2 .$$

Investigate the integral

$$\left\| \sum_{\left\{n\,:\,a(n)\in\Omega_{\underline{j}}^{K,L},\, n\leq M\right\}} \frac{1}{n} \sum_{i=a_{(0)}(n)+1}^{A(n)-j_1-\cdots-j_L-1} 1_{\{\nu_\lambda > i\}}(f * d_{a(n),i}) \right\|_2^2 =: \left\| \sum_{\left\{n\,:\,a(n)\in\Omega_{\underline{j}}^{K,L},\, n\leq M\right\}} \frac{1}{n} g_n \right\|_2^2 .$$

It is trivial that

$$\left\| \sum_{\left\{n\,:\,a(n)\in\Omega_{\underline{j}}^{K,L},\, n\leq M\right\}} \frac{1}{n} g_n \right\|_2^2 = \sum_{\left\{n\,:\,a(n)\in\Omega_{\underline{j}}^{K,L},\, n\leq M\right\}} \sum_{\left\{m\,:\,a(m)\in\Omega_{\underline{j}}^{K,L},\, n\leq M\right\}} \langle \frac{1}{n} g_n, \frac{1}{m} g_m \rangle .$$

Since

$$|\langle \frac{1}{n} g_n, \frac{1}{m} g_m \rangle| \leq \|\frac{g_n}{n}\|_2 \|\frac{g_m}{m}\|_2 \leq \frac{1}{2n^2}\|g_n\|_2^2 + \frac{1}{2m^2}\|g_m\|_2^2,$$

then it follows

$$\left\| \sum_{\left\{n\,:\,a(n)\in\Omega_{\underline{j}}^{K,L},\, n\le M\right\}} \frac{1}{n} g_n \right\|_2^2 \le \sum_{\left\{m\,:\,a(m)\in\Omega_{\underline{j}}^{K,L}\right\}} \sum_{\left\{n\,:\,a(n)\in\Omega_{\underline{j}}^{K,L}\right\}} \frac{1}{n^2}\|g_n\|_2^2.$$

The first orthogonality relation in Lemma 4 and Lemma 5 give

$$\begin{aligned} \|g_n\|_2^2 &= \left\| \sum_{i=a_{(0)}(n)+1}^{A(n)-j_{1,n}-\cdots-j_{L,n}-1} 1_{\{\nu_\lambda>i\}}(f*d_{a(n),i}) \right\|_2^2 \\ &\le \sum_{i=0}^{A(n)-j_{1,n}-\cdots-j_{L,n}-1} \|1_{\{\nu_\lambda>i\}}(f*d_{a(n),i})\|_2^2 \le C\lambda\|f\|_1. \end{aligned}$$

Thus,

$$\|IV\|_2^2 \le \sum_{j_L=1}^{\infty}\cdots\sum_{j_1=1}^{\infty} C\lambda\|f\|_1 \left[\sum_{\left\{n\,:\,a(n)\in\Omega_{\underline{j}}^{K,L}\right\}} \frac{1}{n^2} \right] \left| \left\{ n \,:\, a(n)\in\Omega_{\underline{j}}^{K,L} \right\} \right|.$$

Since for $a(n)\in\Omega_{\underline{j}}^{K,L}$ we have $A(n)=K$, then $n\ge F(2^K)$ and consequently

$$\|IV\|_2^2 \le C \sum_{j_L=1}^{\infty}\cdots\sum_{j_1=1}^{\infty} \left| \left\{ n \,:\, a(n)\in\Omega_{\underline{j}}^{K,L} \right\} \right|^2 \lambda\|f\|_1 \frac{1}{F^2(2^K)}$$

and therefore it is necessary to investigate the cardinality of the set $\left\{ n \,:\, a(n)\in\Omega_{\underline{j}}^{K,L} \right\}$. If

$n\in\Omega_{\underline{j}}^{K,L}$, then

$$\begin{aligned} n &= 2^K + \sum_{i=K-j_1-\cdots-j_L}^{K-1} \lambda_{n,i}2^i + \sum_{i=0}^{K-j_1-\cdots-j_L-1} n_i 2^i \\ &= 2^K + \sum_{i\in\{K-j_1,\ldots,K-j_1-\cdots-j_L\}} 2^i + \sum_{i=0}^{K-j_1-\cdots-j_L-1} n_i 2^i \\ &=: 2^K + x + \sum_{i=0}^{K-j_1-\cdots-j_L-1} n_i 2^i, \end{aligned}$$

where x - of course - depends on $\Omega_{\underline{j}}^{K,L}$ but does not depend on n. This gives

$$2^K + x \le \min\Omega_{\underline{j}}^{K,L} \le \max\Omega_{\underline{j}}^{K,L} \le 2^K + x + 2^{K-j_1-\cdots-j_L} - 1.$$

This by Lemma 6 implies

$$\begin{aligned}
&\left|\left\{n \,:\, a(n) \in \Omega_{\underline{j}}^{K,L}\right\}\right| = F(\max \Omega_{\underline{j}}^{K,L} + 1) - F(\min \Omega_{\underline{j}}^{K,L}) \\
&\le F(2^K + x + 2^{K-j_1-\cdots-j_L}) - F(2^K + x) \\
&= F\left(\frac{2^K + x + 2^{K-j_1-\cdots-j_L}}{2^K + x}(2^K + x)\right) - F(2^K + x) \\
&\le \frac{2^K + x + 2^{K-j_1-\cdots-j_L}}{2^K + x}[F(2^K + x) + 1] - F(2^K + x) \\
&= \frac{2^{K-j_1-\cdots-j_L}}{2^K + x}\left(F(2^K + x) + 1\right) + 1 \\
&\le 2^{-j_1-\cdots-j_L}\left(F(2^{K+1}) + 1\right) + 1
\end{aligned}$$

There are two cases. If $2^{-j_1-\cdots-j_L}\left(F(2^{K+1}) + 1\right) \ge \frac{1}{2}$, then

$$\left|\left\{n \,:\, a(n) \in \Omega_{\underline{j}}^{K,L}\right\}\right| \le C2^{-j_1-\cdots-j_L}\left(F(2^{K+1}) + 1\right). \tag{5}$$

Meanwhile, in the case of $2^{-j_1-\cdots-j_L}\left(F(2^{K+1}) + 1\right) < \frac{1}{2}$ we get $\left|\left\{n \,:\, a(n) \in \Omega_{\underline{j}}^{K,L}\right\}\right| < \frac{3}{2}$, thus

$$\left|\left\{n \,:\, a(n) \in \Omega_{\underline{j}}^{K,L}\right\}\right| \in \{0, 1\}. \tag{6}$$

Taking account both cases (5) and (6) we have

$$\begin{aligned}
\|IV\|_2^2 &\le C\lambda\|f\|_1 \frac{1}{F^2(2^K)} \sum_{\left\{\underline{j}\in\mathbb{P}^L : \left|\left\{n : a(n)\in\Omega_{\underline{j}}^{K,L}\right\}\right| \ge 2\right\}} 4^{-j_1-\cdots-j_L}\left(F(2^{K+1}) + 1\right)^2 \\
&+ C\lambda\|f\|_1 \frac{1}{F^2(2^K)} \sum_{\left\{\underline{j}\in\mathbb{P}^L : \left|\left\{n : a(n)\in\Omega_{\underline{j}}^{K,L}\right\}\right| \le 1\right\}} \left|\left\{n : a(n) \in \Omega_{\underline{j}}^{K,L}\right\}\right|^2 \\
&\le C\lambda\|f\|_1 \frac{F^2(2^{K+1})}{F^2(2^K)} \frac{1}{3^L} + C\lambda\|f\|_1 |N_{K,L}| \frac{1}{F^2(2^K)}.
\end{aligned}$$

Since $\frac{F^2(2^{K+1})}{F^2(2^K)} \le C$ and $|N_{K,L}| \le F(2^{K+1})$, the inequality above follows

$$\|IV\|_2^2 \le C\lambda\|f\|_1 \left(\frac{1}{3^L} + \frac{1}{F(2^K)}\right).$$

That is, the proof of Lemma 9 is complete. □

We use Lemma 9 in order to prove the next one which will directly imply that the maximal operator of the logarithmic means of the partial sums $S_{a(n)}f$ is of weak type (L^1, L^1). For $L, M \in \mathbb{N}, M \le F(2^{b_{L+1}})$ let

$$T_{L,M}f := \frac{1}{L} \sum_{n=F(2^{b_L})+1}^{M} \frac{1}{n} \sum_{i=0}^{A(n)-1} 1_{\{\mu_\lambda > i\}}(f * d_{a(n),i}), \quad T_0^* f := \sup_{L\in\mathbb{P}} \sup_{F(2^{b_L}) < M \le F(2^{b_{L+1}})} |T_{L,M}f|.$$

Lemma 10. *The operator T_0^* is of weak type (L^1, L^1), that is,*

$$\operatorname{mes}\{T_0^* f > \lambda\} \le \frac{C}{\lambda}\|f\|_1$$

for each $\lambda > 0$, $f \in L^1(Q)$.

Proof. Apply Lemma 9 with $K = b_L$. Then,

$$\operatorname{mes}\{T_0^* f > \lambda\} \le \operatorname{mes}\left\{\sup_{L\in\mathbb{P}} \frac{1}{L} CL|f|^* > \frac{\lambda}{2}\right\}$$
$$+\sum_{L=1}^{\infty} \sum_{M=F(2^{b_L})+1}^{F(2^{b_L+1})} \frac{1}{L^2}\frac{1}{\lambda^2}\left\| \sum_{\{n\,:\,a(n)\in N_{b_L,L},\, n\le M\}} \frac{1}{n} \sum_{i=a_{(0)}(n)+1}^{A(n)-j_{1,a(n)}-\cdots-j_{L,a(n)}-1} 1_{\{\nu_\lambda > i\}}(f * d_{a(n),i})\right\|_2^2$$
$$\le \frac{C}{\lambda}\|f\|_1 + \frac{C}{\lambda^2}\sum_{L=1}^{\infty} \sum_{M=F(2^{b_L})+1}^{F(2^{b_L+1})} \frac{1}{L^2}\lambda\|f\|_1\left(\frac{1}{3^L} + \frac{1}{F(2^{b_L})}\right).$$

Since $F(2^{b_L}) \ge 2^L$ and $F(2^{b_L-1}) < 2^L$, then

$$2^L \le F(2^{b_L}) \le 2(F(2^{b_L-1})+1) < 2^{L+1}+2 \le C2^L.$$

This implies

$$\sum_{M=F(2^{b_L})+1}^{F(2^{b_L+1})} \left(\frac{1}{3^L} + \frac{1}{F(2^{b_L})}\right) \le \frac{F(2^{b_{L+1}})}{3^L} + \frac{F(2^{b_{L+1}})}{F(2^{b_L})} \le \frac{C2^L}{3^L} + \frac{C2^L}{2^L} \le C.$$

Consequently,

$$\operatorname{mes}\{T_0^* f > \lambda\} \le \frac{C}{\lambda}\|f\|_1 + \frac{C}{\lambda}\|f\|_1 \sum_{L=1}^{\infty}\frac{1}{L^2} \le \frac{C}{\lambda}\|f\|_1.$$

This completes the proof of Lemma 10. □

Set for $f \in L^1(Q)$

$$R_N f := \frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\sum_{i=0}^{A(n)-1} 1_{\{\nu_\lambda > i\}}(f * d_{a(n),i}), \quad R^* f := \sup_{N\in\mathbb{P}} |R_N f|.$$

Lemma 11. *The operator R^* is of weak type (L^1, L^1).*

Proof. Basically, the proof of this lemma is the application of Lemma 8 and Lemma 10. Let L be the lower integer part of the binary logarithm of N. That is, $2^L \le N < 2^{L+1}$. The definition of operator T_L gives

$$\begin{aligned}|R_N f| &\le \frac{1}{L}\left|\sum_{l=0}^{L-1}\sum_{n=F(2^{b_l})+1}^{F(2^{b_{l+1}})}\frac{1}{n}\sum_{i=0}^{A(n)-1}1_{\{\mu_\lambda>i\}}(f*d_{a(n),i})\right|\\ &+\frac{1}{L}\left|\sum_{n=1}^{F(2^{b_0})}\frac{1}{n}\sum_{i=0}^{A(n)-1}1_{\{\mu_\lambda>i\}}(f*d_{a(n),i})\right|\\ &+\frac{1}{L}\left|\sum_{n=F(2^{b_L})+1}^{N}\frac{1}{n}\sum_{i=0}^{A(n)-1}1_{\{\mu_\lambda>i\}}(f*d_{a(n),i})\right|\\ &\le C\sup_L|T_L f|+C|f|^*+T_0^*f.\end{aligned}$$

Lemma 8 says $\|\sup_L|T_Lf|\|_2^2\le C\lambda\|f\|_1$ and Lemma 10 says that the operator T_0^* is of weak type (L^1,L^1). Consequently, the weak (L^1,L^1) typeness of $f\to|f|^*$ gives

$$\begin{aligned}&\operatorname{mes}\{R^*f>\lambda\}\le\operatorname{mes}\left\{\sup_L|T_Lf|>\frac{\lambda}{3}\right\}+\operatorname{mes}\left\{|f|^*>\frac{\lambda}{3}\right\}+\operatorname{mes}\left\{|T_0^*f|>\frac{\lambda}{3}\right\}\\ &\le\frac{C}{\lambda^2}\|\sup_L|T_Lf|\|^2+\frac{C}{\lambda}\|f\|_1+\frac{C}{\lambda}\|f\|_1\le\frac{C}{\lambda}\|f\|_1.\end{aligned}$$

This completes the proof of Lemma 11. □

Recall the definition of the logarithmic means and its maximal operator.

$$G_Nf=\frac{1}{\log N}\sum_{n=1}^{N}\frac{S_{a(n)}f}{n},\quad G^*f:=\sup_{N\in\mathbb{P}}|G_Nf|.$$

Lemma 12. *The operator G^* is of weak type (L^1,L^1).*

Proof. Recall the definition of the stopping time

$$\nu_\lambda(x):=\inf\{n\in\mathbb{N}:E_n(|f|)(x)>\lambda\}\ \ (\inf\emptyset=+\infty).$$

The fact that $\operatorname{mes}\{\nu_\lambda < \infty\} = \operatorname{mes}\{|f|^* > \lambda\} \le \frac{C}{\lambda}\|f\|_1$ and Lemma 3 with the application of Lemma 11 imply

$$\begin{aligned}
&\operatorname{mes}\{G^*f > \lambda\} \le \operatorname{mes}\left\{\sup_N \frac{1}{\log N}\left|\sum_{n=1}^N \frac{1}{n} S_{2^{A(n)+1}} f\right| > \frac{\lambda}{2}\right\} \\
&+ \operatorname{mes}\{\nu_\lambda < \infty\} + \operatorname{mes}\left\{\nu_\lambda = \infty, \sup_N \frac{1}{\log N}\left|\sum_{n=1}^N \frac{1}{n}\sum_{i=0}^{A(n)-1} f * d_{a(n),i}\right| > \frac{\lambda}{2}\right\} \\
&\le \operatorname{mes}\left\{\sup_n |E_n f| > \frac{\lambda}{2}\right\} + \frac{C}{\lambda}\|f\|_1 \\
&+ \operatorname{mes}\left\{\sup_N \frac{1}{\log N} 1_{\{\nu_\lambda=\infty\}}\left|\sum_{n=1}^N \frac{1}{n}\sum_{i=0}^{A(n)-1} f * d_{a(n),i}\right| > \frac{\lambda}{2}\right\} \\
&\le \frac{C}{\lambda}\|f\|_1 + \operatorname{mes}\left\{\sup_N \frac{1}{\log N}\left|\sum_{n=1}^N \frac{1}{n}\sum_{i=0}^{A(n)-1} 1_{\{\nu_\lambda > i\}}(f * d_{a(n),i})\right| > \frac{\lambda}{2}\right\} \\
&\le \frac{C}{\lambda}\|f\|_1.
\end{aligned}$$

The last inequality is implied by Lemma 11, that is by the weak (L^1, L^1) typeness of operator R^* and by the fact that if $1_{\{\nu_\lambda=\infty\}}(x) = 1$ for some $x \in Q$, then $1_{\{\nu_\lambda>i\}}(x) = 1$ for all $i < A(n), n \le N$. This completes the proof of this lemma. □

The proof of Theorem 2. Since the set of Walsh polynomials is dense in $L^1(Q)$ and for each Walsh polynomial P we have $S_{a(n)}P = P$ for n "large enough" and consequently, $G_N P \to P$ almost everywhere (even everywhere), then the previous lemma, that is the weak (L^1, L^1) typeness of the maximal operator G^* by the standard density argument (see e.g. [13]) completes the proof of Theorem 2. □

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