POINTWISE CONVERGENCE OF CONE-LIKE RESTRICTED TWO-DIMENSIONAL $(C,1)$ MEANS OF TRIGONOMETRIC FOURIER SERIES

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ABSTRACT. The aim of this work is to generalize the more then 60 year old celebrated result of Marcinkiewicz and Zygmund on the convergence of the two-dimensional restricted $(C,1)$ means of trigonometric Fourier series. They proved for any integrable function $f \in L^1(T^2)$ the a.e. convergence

$$\sigma_{(n_1,n_2)} f \longrightarrow f$$

provided $n_1/\beta \le n_2 \le \beta n_1$, where $\beta > 1$ is fixed constant. That is, the set of indices (n_1, n_2) remains in some positive cone around the identical function. We not only generalize this theorem, but give a necessary and sufficient condition for cone-like sets (of the set of indices) in order to preserve this convergence property. For the time being there is no known result like this.

1. INTRODUCTION

The question

> *What kind of restriction implies the convergence of the two-dimensional $(C,1)$ means of trigonometric Fourier series of integrable functions?*

The only example is due to Marcinkiewicz and Zygmund [6]. It is surprising that there are no other a.e. convergence results of the two-dimensional σ_n (i.e. $(C,1)$) means of trigonometric Fourier series of integrable functions, where the set of indices is inside a cone around the identical function. We mention that Jessen, Marcinkiewicz and Zygmund also proved in [5] the a.e. convergence $\sigma_n f \to f$ without any restriction on the indices, but not for functions in L^1. They proved this for a proper subspace. Namely, for functions in $L^1 \log^+ L$.

This section contains a preliminary result and notions that are needed in formularizing the main theorems, given at the end of this section. The result presented here is an easy observation and the proof is tedious. Let $\alpha : [1, +\infty) \longrightarrow [1, +\infty)$ be a strictly monotone increasing continuous function with property $\lim_{+\infty} \alpha = +\infty$, $\alpha(1) = 1$, and $\beta : [1, +\infty) \longrightarrow [1, +\infty)$ be a monotone increasing function with property $\beta(1) > 1$.

Definition 1.1. Define the cone-like restriction sets of $\mathbb{N}^2$ as follows

$$\mathbb{N}_{\alpha,\beta,1} := \left\{ n \in \mathbb{N}^2 : \frac{\alpha(n_1)}{\beta(n_1)} \le n_2 \le \alpha(n_1)\beta(n_1) \right\},$$

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$$\mathbb{N}_{\alpha,\beta,2} := \left\{ n \in \mathbb{N}^2 : \frac{\alpha^{-1}(n_2)}{\beta(n_2)} \le n_1 \le \alpha^{-1}(n_2)\beta(n_2) \right\}.$$

For $\alpha(x) = x, \beta(x) = \beta \in (1, +\infty)$ we have

$$\mathbb{N}_{\alpha,\beta,1} = \mathbb{N}_{\alpha,\beta,2} = \left\{ n \in \mathbb{N}^2 : \frac{1}{\beta} \le \frac{n_2}{n_1} \le \beta \right\}$$

the "ordinary" restriction set used by Marcinkiewicz and Zygmund (and others).

Now, let $\beta(x) = \beta \in (1, +\infty)$ be a constant function. It is obvious that $\mathbb{N}_{\alpha,\beta_1,1} \subset \mathbb{N}_{\alpha,\beta_2,1}$ and $\mathbb{N}_{\alpha,\beta_1,2} \subset \mathbb{N}_{\alpha,\beta_2,2}$ for any $\beta_1 \le \beta_2$. Let

$$\mathbb{N}_{\alpha,i} := \{\mathbb{N}_{\alpha,\beta,i} : \beta > 1\}$$

for $i = 1, 2$. Let $i \in \{1, 2\}$. We say that $\mathbb{N}_{\alpha,i}$ is weaker than $\mathbb{N}_{\alpha,2-i}$, if for all $L \in \mathbb{N}_{\alpha,i}$ there exists an $\tilde{L} \in \mathbb{N}_{\alpha,2-i}$ such that

$$L \subset \tilde{L}.$$

This will be abbreviated by

$$\mathbb{N}_{\alpha,i} \prec \mathbb{N}_{\alpha,2-i}.$$

If $\mathbb{N}_{\alpha,1} \prec \mathbb{N}_{\alpha,2}$, and $\mathbb{N}_{\alpha,2} \prec \mathbb{N}_{\alpha,1}$, then we call $\mathbb{N}_{\alpha,1}$ and $\mathbb{N}_{\alpha,2}$ equivalent. We abbreviate this by

$$\mathbb{N}_{\alpha,1} \sim \mathbb{N}_{\alpha,2}.$$

We say that α is a cone-like restriction function (CRF), if

$$\mathbb{N}_{\alpha,1} \sim \mathbb{N}_{\alpha,2}.$$

Now let $\mathbb{N}_\alpha := \mathbb{N}_{\alpha,1} \cup \mathbb{N}_{\alpha,2}$. We say that the cone-like set $L \in \mathbb{N}_\alpha$ is based by the function α. We study the a.e. convergence of the $(C, 1)$ means $\sigma_n f$ of functions integrable that is, $f \in L^1(T^2)$, where $T := [-\pi, \pi) \times [-\pi, \pi)$. We study the convergence restricted by $n \in L, L \in \mathbb{N}_\alpha$, where α is CRF and $\wedge n \longrightarrow +\infty$. It is natural to ask: How does a cone-like restriction function look like? First we prove:

Proposition 1.2. *Function α is a cone-like restriction function if and only if there exists $\zeta, \gamma_1, \gamma_2 > 1$ such that*

$$\gamma_1\alpha(x) \le \alpha(\zeta x) \le \gamma_2\alpha(x)$$

holds for each $x \ge 1$.

Proof. First suppose that $\gamma_1\alpha(x) \le \alpha(\zeta x) \le \gamma_2\alpha(x)$ holds for each $x \ge 1$. We prove

$$\mathbb{N}_{\alpha,2} \prec \mathbb{N}_{\alpha,1}.$$

Let $L \in \mathbb{N}_{\alpha,2}$, and $n \in L$. Then $L = \mathbb{N}_{\alpha,\beta_1,2}$ for some $\beta_1 > 1$. This means

$$\frac{\alpha^{-1}(n_2)}{\beta_1} \le n_1 \le \alpha^{-1}(n_2)\beta_1.$$

This inequality is equivalent to

$$\alpha(\frac{n_1}{\beta_1}) \le n_2 \le \alpha(n_1\beta_1).$$

Since $\zeta > 1$, then there exists a $j \in \mathbb{N}$ such that $\zeta^j > \beta_1$. Thus,

$$n_2 \le \alpha(n_1\zeta^j) \le \gamma_2^j\alpha(n_1)$$

and

$$n_2 \geq \alpha(n_1/\beta_1) > \alpha(n_1/\zeta^j) \geq \frac{1}{\gamma_2^j}\alpha(n_1).$$

This implies $L \subset \mathbb{N}_{\alpha,\gamma_2^j,1}$. Thus $\mathbb{N}_{\alpha,2} \prec \mathbb{N}_{\alpha,1}$. Next, let $n \in L \in \mathbb{N}_{\alpha,1}$. Then $L = \mathbb{N}_{\alpha,\beta_1,1}$ for some $\beta_1 > 1$. This means

$$\frac{\alpha(n_1)}{\beta_1} \leq n_2 \leq \alpha(n_1)\beta_1$$

that is

$$\alpha^{-1}(n_2/\beta_1) \leq n_1 \leq \alpha^{-1}(n_2\beta_1).$$

The inequality

$$\gamma_1\alpha(x) \leq \alpha(\zeta x) \leq \gamma_2\alpha(x)$$

gives

$$\zeta\alpha^{-1}(x) \leq \alpha^{-1}(\gamma_2 x), \quad \alpha^{-1}(\gamma_1 x) \leq \zeta\alpha^{-1}(x).$$

Take $j \in \mathbb{N}$ such that $\gamma_1^j > \beta_1$.

$$n_1 \leq \alpha^{-1}(n_2\beta_1) \leq \alpha^{-1}(n_2\gamma_1^j) \leq \zeta^j\alpha^{-1}(n_2)$$

and

$$n_1 \geq \alpha^{-1}(n_2/\beta_1) \geq \alpha^{-1}(n_2/\gamma_1^j) \geq \frac{1}{\zeta^j}\alpha^{-1}(n_2).$$

That is, $n \in \mathbb{N}_{\alpha,\zeta^j,2}$, and $L \subset \mathbb{N}_{\alpha,\zeta^j,2}$. Thus, $\mathbb{N}_{\alpha,1} \prec \mathbb{N}_{\alpha,2}$. Therefore the equivalence $\mathbb{N}_{\alpha,1} \sim \mathbb{N}_{\alpha,2}$ is proved. Next, on the other hand, suppose that $\mathbb{N}_{\alpha,1} \sim \mathbb{N}_{\alpha,2}$ for some CRF α. $\mathbb{N}_{\alpha,1} \prec \mathbb{N}_{\alpha,2}$ means that for all $\beta > 1$ there exists a $\gamma > 1$ such that $\mathbb{N}_{\alpha,\beta,1} \subset \mathbb{N}_{\alpha,\gamma,2}$. Let $n \in \mathbb{N}_{\alpha,\beta,1}$ that is,

$$\frac{\alpha(n_1)}{\beta} \leq n_2 \leq \beta\alpha(n_1).$$

Then there follows

$$\alpha^{-1}(n_2/\beta) \leq n_1 \leq \alpha^{-1}(n_2\beta).$$

Since $n \in \mathbb{N}_{\alpha,\gamma,2}$, therefore

$$\frac{\alpha^{-1}(n_2)}{\gamma} \leq \alpha^{-1}(n_2/\beta) \quad \text{and} \quad \alpha^{-1}(n_2\beta) \leq \gamma\alpha^{-1}(n_2).$$

Let $x \geq 1$ be an arbitrary real number. Then

$$\alpha^{-1}(\beta x) \leq \alpha^{-1}(\beta 2\lfloor x\rfloor) \leq \alpha^{-1}(\beta^{2+\lfloor 1/\log_2\beta\rfloor}\lfloor x\rfloor) \leq \gamma^{2+\lfloor 1/\log_2\beta\rfloor}\alpha^{-1}(\lfloor x\rfloor) \leq \gamma^{2+\lfloor 1/\log_2\beta\rfloor}\alpha^{-1}(x).$$

Hence $\mathbb{N}_{\alpha,1} \prec \mathbb{N}_{\alpha,2}$ implies the existence of the real numbers $\beta_1, \gamma_1 > 1$ for which $\alpha^{-1}(\gamma_1 x) \leq \beta_1\alpha^{-1}(x)$ for $x \geq 1$. Similarly, $\mathbb{N}_{\alpha,2} \prec \mathbb{N}_{\alpha,1}$ implies the existence of the real numbers $\beta_2, \gamma_2 > 1$ for which $\alpha(\beta_2 x) \leq \gamma_2\alpha(x)$ for $x \geq 1$. Let $s := \alpha^{-1}(x)$. Thus, $\gamma_1\alpha(s) \leq \alpha(\beta_1 s)$. Since $\alpha^{-1}(1) = 1$ and α^{-1} is strictly monotone increasing we have for all $x \geq 1$ that $\gamma_1\alpha(x) \leq \alpha(\beta_1 x)$. Choose $j \in \mathbb{N}$ such that $\beta_2^j > \beta_1$.

$$\gamma_1\alpha(x) \leq \alpha(\beta_1 x) \leq \alpha(\beta_2^j x) \leq \gamma^j\alpha(x).$$

The proof of Proposition 1.2 now is complete. □

How does a CRF α look like? Every CRF α has the following structure. Let (c_n) be a bounded sequence of real numbers: $1 < \inf c_n \le \sup c_n < +\infty$, $c_1 = 1$, and let $\zeta > 1$. Take

$$\alpha(\zeta^j) := c_1 \cdots c_j$$

for $j = 0, 1, \dots,$ and α at other points be defined arbitrary taking account only that α is strictly monotone increasing and continuous. Then α is a cone-like restriction function, and every CRF can be given in this way.

The system of functions

$$e^{\imath nx} \quad (n = 0, \pm 1, \pm 2, \dots)$$

($x \in \mathbb{R}, \imath = \sqrt{-1}$) is called the trigonometric system. It is orthogonal over any interval of length 2π, specially over $T := [-\pi, \pi)$. Let $f \in L^1(T)$, that is integrable on T. The kth Fourier coefficient of f is

$$\hat{f}(k) := \frac{1}{2\pi} \int_T f(x) e^{-\imath kt} dt,$$

where k is any integer number. The nth ($n \in \mathbb{N}$) partial sum of the Fourier series of f is

$$S_n f(y) := \sum_{k=-n}^{n} \hat{f}(k) e^{\imath ky}.$$

The nth ($n \in \mathbb{N}$) Fejér or $(C, 1)$ mean of function f is defined in the following way:

$$\sigma_n f(y) := \frac{1}{n+1} \sum_{k=0}^{n} S_k f(y).$$

It is known that

$$\sigma_n f(y) = \frac{1}{\pi} \int_T f(x) K_n(y - x) dx,$$

where the function K_n is known as the nth Fejér kernel; we will now find an appropriate expression for it (see e.g. the book of Bary [1].

$$K_n(u) = \frac{1}{2(n+1)} \left(\frac{\sin(\frac{u}{2}(n+1))}{\sin(\frac{u}{2})} \right)^2 .$$

From this expression one immediately derive the following properties of the kernel. They will play an essential role later.

$$K_n(u) \ge 0.$$

$$K_n(u) \le \frac{\pi^2}{2(n+1)u^2} \quad (0 < |u| \le \pi).$$

Let f be an integrable function that is let $f \in L^1(T^2)$. The $k = (k_1, k_2)$th Fourier coefficient of f is

$$\hat{f}(k) = \hat{f}(k_1, k_2) := \frac{1}{2\pi} \int_{T \times T} f(x_1, x_2) e^{-\imath(k_1 t_1 + k_2 t_2)} d(t_1, t_2),$$

where k_1, k_2 are integers. The nth ($n \in \mathbb{N}^2$) partial sum of the Fourier series of f is

$$S_n f(y) = S_{n_1, n_2} f(y_1, y_2) := \sum_{k_1=-n_1}^{n_1} \sum_{k_2=-n_2}^{n_2} \hat{f}(k_1, k_2) e^{\imath(k_1 y_1 + k_2 y_2)}.$$

The nth ($n \in \mathbb{N}^2$) two-dimensional Fejér or $(C,1)$ mean of function f is defined in the following way:

$$\sigma_n f(y) = \sigma_{n_1,n_2} f(y) := \frac{1}{(n_1+1)(n_2+1)} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} S_k f(y),$$

where $y \in T^2$. In 1939 Marcinkiewicz and Zygmund [6] proved their celebrated theorem on the convergence of the two-dimensional restricted $(C,1)$ means of trigonometric Fourier series. They proved for any integrable function $f \in L^1(T^2)$ the a.e. convergence

$$\sigma_{(n_1,n_2)} f \longrightarrow f$$

provided $n_1/\beta \le n_2 \le \beta n_1$, where $\beta > 1$ is fixed constant. So, the set of indices (n_1,n_2) remains in some positive cone around the identical function. Actually, their proof is not a simple one. (We remark that their theorem is also valid for the two-dimensional Walsh-Paley system. For the proof of this see [3].) For the time being there is no other restriction set for the indices, which preserves this a.e. convergence relation, is known. We remark that in 1935 Jessen, Marcinkiewicz and Zygmund [5] proved the unrestricted convergence

$$\lim_{\wedge n \to \infty} \sigma_n f = f$$

a.e. But, it is proved for functions in $L^1 \log^+ L$, which is a proper subspace of $L^1(T^2)$. It is quite natural to ask what kind of cone-like restriction sets can be given preserving the a.e. convergence of the two-dimensional Fejér means of integrable functions. The aim of this paper is to prove the following two main results

Theorem 1.3. *(The convergence.(Let α be CRF, $L \in \mathbb{N}_\alpha$. Then for any $f \in L^1(T^2)$ the a.e. equality*

$$\lim_{\substack{\wedge n \to \infty,\\ n \in L}} \sigma_n f = f$$

holds.

Theorem 1.4. *(The divergence.(Let α be CRF, $\beta : [1,+\infty) \longrightarrow [1,+\infty)$ be a monotone increasing function with property $\lim_{+\infty} \beta = +\infty$, and $\delta : [1,+\infty) \longrightarrow [0,+\infty)$ be a measurable function with property $\lim_{+\infty} \delta = 0$. Let $L := \mathbb{N}_{\alpha,\beta,1}$ or $L := \mathbb{N}_{\alpha,\beta,2}$. Then there exists a function $f \in L^1 \log^+ L\delta(L)$ such that*

$$\limsup_{\substack{\wedge n \to \infty,\\ n \in L}} \sigma_n f = +\infty$$

a.e.

One might think that if we enlarge the cone based by α, then the convergence space from L^1 to $L^1 \log(L)$ (no restriction) changes somehow continuously. Theorem 1.3, and 1.4 show that there does not exist an interim space between L^1, and $L^1 \log^+(L)$. These theorems immediately give

Corollary 1.5. *Let α be CRF, $\beta : [1,+\infty) \longrightarrow [1,+\infty)$ be a monotone increasing function with property $\beta(1) > 1$, and $L := \mathbb{N}_{\alpha,\beta,1}$ or $L := \mathbb{N}_{\alpha,\beta,2}$. Then*

$$\lim_{\substack{\wedge n \to \infty,\\ n \in L}} \sigma_n f = f$$

holds a.e. for all $f \in L^1(T^2)$ if and only if the function β is bounded.

Corollary 1.5 shows that the theorem of Marcinkiewicz and Zygmund on the convergence of the two-dimensional restricted $(C,1)$ means of trigonometric Fourier series can not be sharpened, that is the cone based by the identical function can not be enlarged infinitely preserving the a.e. convergence for each integrable function. The Corollary 1.6 below also provides a simpler proof of the theorem of Marcinkiewicz and Zygmund.

Corollary 1.6. *Let $\beta : [1,+\infty) \longrightarrow [1,+\infty)$ be a monotone increasing function with property $\beta(1) > 1$, then*

$$\lim_{\substack{\wedge n\to\infty, \\ n_1/\beta(n_1)\le n_2\le n_1\beta(n_1)}} \sigma_n f = f$$

holds a.e. for all $f \in L^1(T^2)$ if and only if the function β is bounded.

The "divergence part" of this corollary for the two-dimensional Walsh-Paley system can be read in [4], and the "convergence part" in [3]. For an introductory on the trigonometric series see also the book of Zygmund [9], or the book of Bary [1], or Edwards [2].

2. A Decomposition Lemma

The dyadic subintervals of T are defined in the following way.

$$\mathcal{I}_0 := \{T\}, \quad \mathcal{I}_1 := \{[-\pi,0),[0,\pi)\},$$

$$\mathcal{I}_2 := \{[-\pi,-\pi/2),[-\pi/2,0),[0,\pi/2),[\pi/2,\pi)\},\ldots$$

$$\mathcal{I} := \bigcup_{n=0}^{\infty} \mathcal{I}_n.$$

The elements of $\mathcal{I}$ are said to be dyadic intervals. If $F \in \mathcal{I}$, then there exists a unique $n \in \mathbf{N}$ such that $F \in \mathcal{I}_n$, and consequently $\operatorname{mes}(F) = \frac{2\pi}{2^n}$. Each $\mathcal{I}_n$ has 2^n disjoint elements $(n \in \mathbb{N})$. $\mathcal{I} \times \mathcal{I}$ is the set of dyadic rectangles.

Let functions $\psi_j : [1,+\infty) \longrightarrow \mathbb{N}$ be monotone increasing and continuous from the right with property

$$\lim_{+\infty} \psi_j = +\infty \quad (j=1,2).$$

The aim of this section is to prove the following decomposition lemma on T^2 which will play a prominent role in the proof of Theorem 1.3.

Lemma 2.1. *Let $f \in L^1(T^2)$, and $\lambda > \|f\|_1/(2\pi)^2$. Then there exists a sequence of integrable functions (f_i) such that*

$$f = \sum_{i=0}^{\infty} f_i,$$

$$\|f_0\|_\infty \le C\lambda, \quad \|f_0\|_1 \le C\|f\|_1, \quad \textit{and}$$

$$\operatorname{supp} f_i \subset I^{i,1} \times I^{i,2}, \quad \textit{where}$$

$I^{i,j} \in \mathcal{I}$ are dyadic intervals,

$$\operatorname{mes}\left(I^{i,j}\right) = \frac{2\pi}{2^{\psi_j(s_i)}}, \quad \textit{for some}$$

$s_i \geq 1\,(j=1,2,\ i \in \mathbb{N}\setminus\{0\})$. *Moreover,* $\int_{T^2} f_i(x)dx = 0 (i \geq 1)$, *the dyadic rectangles* $I^{i,1}\times I^{i,2}$ *are disjoint* $(i \in \mathbb{N}\setminus\{0\}($, *and for*

$$F := \bigcup_{i=1}^{\infty}\left(I^{i,1}\times I^{i,2}\right) \quad \textit{we have} \quad \mathrm{mes}\,(F) \leq C\|f\|_1/\lambda.$$

Proof. Let $s_1 := 1$ and

$$\Omega_1 := \left\{ J = J_1 \times J_2 \in \mathfrak{I}_{\psi_1(s_1)} \times \mathfrak{I}_{\psi_2(s_1)} : \mathrm{mes}\,(J)^{-1}\int_J |f(x)|dx > \lambda \right\}.$$

Since for each $J \in \Omega_1$, we have

$$\mathrm{mes}\,(J)^{-1} = \frac{2^{\psi_1(1)+\psi_2(1)}}{4\pi^2},$$

then we also have

$$\lambda < \mathrm{mes}\,(J)^{-1}\int_J |f(x)|dx \leq 2^{\psi_1(1)+\psi_2(1)}\frac{1}{4\pi^2}\int_{T^2}|f(x)|dx < 2^{\psi_1(1)+\psi_2(1)}\lambda \leq C\lambda.$$

Let $s_2 := \inf\{s \in [1,+\infty) : \sum_{j=1}^{2}|\psi_j(s)-\psi_j(s_1))| \geq 1\}$. Since the functions ψ_1, ψ_2 are continuous from the right then we have the following three cases:

Case 1. $\psi_1(s_2) = \psi_1(s_1)+1$ and $\psi_2(s_2) = \psi_1(s_1)$,

Case 2. $\psi_1(s_2) = \psi_1(s_1)$ and $\psi_2(s_2) = \psi_1(s_1)+1$,

Case 3. $\psi_1(s_2) = \psi_1(s_1)+1$ and $\psi_2(s_2) = \psi_1(s_1)+1$.

We decompose the dyadic rectangles contained in

$$\left[\mathfrak{I}_{\psi_1(s_1)} \times \mathfrak{I}_{\psi_2(s_1)}\right] \setminus \{J : J \in \Omega_1\}.$$

That is,

$$\Omega_2 := \left\{ J \in \mathfrak{I}_{\psi_1(s_2)} \times \mathfrak{I}_{\psi_2(s_2)} : \mathrm{mes}\,(J)^{-1}\int_J |f(x)|dx > \lambda \,\text{and}\, \nexists K \in \Omega_1 \,\text{such as}\, J \subset K \right\}.$$

Consequently, for all $J \in \Omega_2$ we get

$$\lambda < \mathrm{mes}\,(J)^{-1}\int_J |f(x)|dx \leq 4\lambda.$$

(In case 1 and 2 we even have 2λ, but it makes no problem to take 4λ, instead.) Generally, for $\mathbb{N} \ni n \geq 3$ $s_n := \inf\{s \in [1,+\infty) : \sum_{j=1}^{2}|\psi_j(s)-\psi_j(s_{n-1}))| \geq 1\}$. That is, $\psi_j(s_n) = \psi_j(s_{n-1})+1$ for at last one j $(j=1,2)$. If for a j this is not valid, then $\psi_j(s_n) = \psi_j(s_{n-1})$. Also take

$$\Omega_n := \left\{ J \in \mathfrak{I}_{\psi_1(s_n)} \times \mathfrak{I}_{\psi_2(s_n)} : \mathrm{mes}\,(J)^{-1}\int_J |f(x)|dx > \lambda \,\text{and}\, \nexists K \in \bigcup_{i=1}^{n-1}\Omega_i \,\text{such as}\, J \subset K \right\}.$$

Similarly, as in the case of Ω_2 we have that for each $J \in \Omega_n$ the inequalities

$$\lambda < \mathrm{mes}\,(J)^{-1}\int_J |f(x)|dx \leq 4\lambda$$

hold. Denote by $l_n \in \mathbb{N}$ the number of elements of Ω_n, and the elements of Ω_n by $J_{n,k} (k = 1,\ldots,l_n, n \in \mathbb{N})$. Since $\mathfrak{I}_{\psi_1(s_n)} \times \mathfrak{I}_{\psi_2(s_n)}$ has $2^{\psi_1(s_n)+\psi_2(s_n)}$ (disjoint) elements, then $l_n \leq$

$2^{\psi_1(s_n)+\psi_2(s_n)}$ $(n \in \mathbb{N})$. For an arbitrary set $B \subset T^2$ the characteristic function of B is denoted by 1_B. Let

$$f_{n,k} := \left(f - \operatorname{mes}(J_{n,k})^{-1} \int_{J_{n,k}} f(x)dx \right) 1_{J_{n,k}},$$

$k = 1, \ldots, l_n$, $n \in \mathbb{N}$ and $F := \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{l_n} J_{n,k}$. Since the dyadic rectangles $J_{n,k}$ are disjoint, then we have the following decomposition of the function f:

$$\begin{aligned}
f &= \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} f 1_{J_{n,k}} + f 1_{T^2 \setminus F} \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} \left(f - \operatorname{mes}(J_{n,k})^{-1} \int_{J_{n,k}} f(x)dx \right) 1_{J_{n,k}} \\
&+ \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} \left[\operatorname{mes}(J_{n,k})^{-1} \int_{J_{n,k}} f(x)dx \right] 1_{J_{n,k}} + f 1_{T^2 \setminus F} \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} f_{n,k} + f_0.
\end{aligned}$$

This means that $f_0 = \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} \left[\operatorname{mes}(J_{n,k})^{-1} \int_{J_{n,k}} f(x)dx \right] 1_{J_{n,k}} + f 1_{T^2 \setminus F}$ and the functions f_i $(i = 1, 2, \ldots)$ in the statement of Lemma 2.1 will be the functions $f_{n,k}$ $(k = 1, \ldots, l_n,\ n \in \mathbb{N})$. $\operatorname{supp} f_{n,k} \subset J_{n,k}$ are disjoint dyadic rectangles,

$$\operatorname{mes}(J_{n,k}) = \frac{4\pi^2}{2^{\psi_1(s_n)+\psi_2(s_n)}},$$

$$\int_{T^2} f_{n,k}(x)dx = \int_{J_{n,k}} f(x)dx - \operatorname{mes}(J_{n,k})^{-1} \int_{J_{n,k}} f(x)dx \cdot \operatorname{mes}(J_{n,k}) = 0.$$

$$\|f_{n,k}\|_1 \le \|f 1_{J_{n,k}}\|_1 + \operatorname{mes}(J_{n,k})^{-1} \int_{J_{n,k}} |f(x)|dx \|1_{J_{n,k}}\|_1 = 2\|f 1_{J_{n,k}}\|_1.$$

Consequently,

$$\|\sum_{n=1}^{\infty} \sum_{k=1}^{l_n} f_{n,k}\|_1 \le 2 \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} \|f 1_{J_{n,k}}\|_1 = 2 \int_F |f(x)|dx \le 2\|f\|_1.$$

This immediately gives

$$\|f_0\|_1 = \|f - \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} f_{n,k}\|_1 \le 3\|f\|_1.$$

Since F is the disjoint union of the dyadic rectangles $J_{n,k}$, then for the two-dimensional Lebesgue measure of F we get

$$\begin{aligned}\operatorname{mes}(F) &= \sum_{n=1}^{\infty}\sum_{k=1}^{l_n}\operatorname{mes}(J_{n,k}) \\ &< \sum_{n=1}^{\infty}\sum_{k=1}^{l_n}\frac{1}{\lambda}\int_{J_{n,k}}|f(x)|dx \\ &= \frac{1}{\lambda}\int_F |f(x)|dx \le \frac{1}{\lambda}\|f\|_1.\end{aligned}$$

There remains to prove $\|f_0\| \le C\lambda$. The construction of Ω_n gives the inequality

$$\operatorname{mes}(J_{n,k})^{-1}\int_{J_{n,k}}|f(x)|dx \le C\lambda$$

(in the case of $n = 1$ we have $2^{\psi_1(1)+\psi_2(1)}$, and in the case of $n \ge 2$ we have number 4 as constant C). That is,

$$\begin{aligned}\|f_0\|_\infty &\le \lambda\|\sum_{n=1}^{\infty}\sum_{k=1}^{l_n}1_{J_{n,k}}\|_\infty + \|f1_{T^2\setminus F}\|_\infty \\ &= \lambda\|1_F\|_\infty + \|f1_{T^2\setminus F}\|_\infty \\ &\le \lambda + \|f1_{T^2\setminus F}\|_\infty.\end{aligned}$$

Let $\mathcal{A}_n$ be the σ-algebra generated by the elements of $\mathcal{I}_{\psi_1(s_n)} \times \mathcal{I}_{\psi_2(s_n)}$ $(n \in \mathbb{N})$. Then we have an increasing sequence of σ algebras

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots.$$

The conditional expectation operator of the function f with respect to $\mathcal{A}_n$ at a given point $x \in T^2$ is

$$\operatorname{mes}(J)^{-1}\int_J f(t)dt,$$

where J is the unique element of $\mathcal{I}_{\psi_1(s_n)} \times \mathcal{I}_{\psi_2(s_n)}$ such that $x \in J$. Since $\lim_{+\infty}\psi_1 = \lim_{+\infty}\psi_2 = +\infty$, then the martingale convergence theorem (see e.g. the book of Neveau [7]) gives that this integral mean value converges to $f(x)$ for almost all x in T^2.

Now let $x \in T^2 \setminus F$. Then the construction of the set Ω_n gives for each $J \in \mathcal{I}_{\psi_1(s_n)} \times \mathcal{I}_{\psi_2(s_n)}$ that $\operatorname{mes}(J)^{-1}\int_J |f(t)|dt \le \lambda$ (for all $n \in \mathbb{N}$). From the lines above there follows

$$|f(x)| \le \lambda$$

for almost all $x \in T^2 \setminus F$, so

$$\|f1_{T^2\setminus F}\|_\infty \le \lambda, \quad \|f_0\|_\infty \le 2\lambda.$$

With this the proof of Lemma 2.1 is complete. □

3. The Convergence

The aim of this section is to prove Theorem 1.3, the convergence theorem. To perform this we need several lemmas. Mainly, we prove that the maximal operator $\sigma_L^* f := \sup_{n\in L} |\sigma_n f|$ ($L \in \mathbb{N}_\alpha$, α is CRF) is of weak type $(1,1)$. This means

$$\sup_{\lambda>0} \lambda \operatorname{mes}(x : \sigma_L^* f(x) > \lambda) \le C\|f\|_1$$

for all $f \in L^1(T^2)$. We give further details later. First, apply Lemma 2.1 for functions $\psi_1(s) := \lfloor \log_2(s) \rfloor$ ($\lfloor x \rfloor$ denotes the lower integer part of x) and $\psi_2(s) := \lfloor \log_2(\alpha(s)) \rfloor$, where α is CRF. Then we prove

Lemma 3.1. *Let α be CRF, $L \in \mathbb{N}_\alpha$, $f \in L^1(T^2)$, and* $\operatorname{supp} f \subset J_1 \times J_2 \in \mathcal{I} \times \mathcal{I}$ *with* $\operatorname{mes}(J_j) = \frac{2\pi}{2^{\psi_j(s)}}$ *for some $s \ge 1$ ($j = 1, 2$(. Suppose that*

$$\int_T f(x_1, x_2)\, dx_j = 0 \quad (for\ each) \quad x_{2-j} \in T,\ j = 1, 2.$$

Then it follows that

$$\int_{(T\setminus 2J_1)\times(T\setminus 2J_2)} \sigma_L^* f(x_1, x_2)\, d(x_1, x_2) \le C\|f\|_1.$$

Proof. We remark that $2J_1$ means the double of the interval J_1 with the same center. Let $u_j \in T$ be the center of the dyadic interval J_j ($j = 1, 2$). Then we have

$$\begin{aligned}
&\int_{(T\setminus 2J_1)\times(T\setminus 2J_2)} \sigma_L^* f(x_1, x_2)\, d(x_1, x_2) \\
&= \int_{T\setminus[u_1 - \frac{2\pi}{2^{\psi_1(s)}}, u_1 + \frac{2\pi}{2^{\psi_1(s)}})} \int_{T\setminus[u_2 - \frac{2\pi}{2^{\psi_2(s)}}, u_2 + \frac{2\pi}{2^{\psi_2(s)}})} \sup_{n\in L} \Bigg| \int_{u_1 - \frac{\pi}{2^{\psi_1(s)}}}^{u_1 + \frac{\pi}{2^{\psi_1(s)}}} \int_{u_2 - \frac{\pi}{2^{\psi_2(s)}}}^{u_2 + \frac{\pi}{2^{\psi_2(s)}}} f(x_1, x_2) \\
&\times K_{n_1}(y_1 - x_1) K_{n_2}(y_2 - x_2) d(x_1, x_2) \Bigg| d(y_1, y_2) \\
&= \int_{T\setminus[-\frac{2\pi}{2^{\psi_1(s)}}, \frac{2\pi}{2^{\psi_1(s)}})} \int_{T\setminus[-\frac{2\pi}{2^{\psi_2(s)}}, \frac{2\pi}{2^{\psi_2(s)}})} \sup_{n\in L} \Bigg| \int_{-\frac{\pi}{2^{\psi_1(s)}}}^{\frac{\pi}{2^{\psi_1(s)}}} \int_{-\frac{\pi}{2^{\psi_2(s)}}}^{\frac{\pi}{2^{\psi_2(s)}}} f(x_1 + u_1, x_2 + u_2) \\
&\times K_{n_1}(y_1 - x_1) K_{n_2}(y_2 - x_2) d(x_1, x_2) \Bigg| d(y_1, y_2).
\end{aligned}$$

This equality shows that without loss of generality we can suppose the center of both intervals J_1 and J_2 be 0. That is, we suppose

$$J_j = \left[-\frac{\pi}{2^{\psi_j(s)}}, \frac{\pi}{2^{\psi_j(s)}}\right) \quad (j = 1, 2).$$

Then we have

$$\begin{aligned}
&\int_{(T\setminus 2J_1)\times(T\setminus 2J_2)} \sigma_L^* f(x_1,x_2)\, d(x_1,x_2) \\
&= \sum_{\substack{i=-2^{\psi_1(s)}\\ i\neq -2,-1,0,1}}^{2^{\psi_1(s)}-1} \sum_{\substack{j=-2^{\psi_2(s)}\\ j\neq -2,-1,0,1}}^{2^{\psi_2(s)}-1} \int_{\frac{\pi i}{2^{\psi_1(s)}}}^{\frac{\pi(i+1)}{2^{\psi_1(s)}}} \int_{\frac{\pi j}{2^{\psi_2(s)}}}^{\frac{\pi(j+1)}{2^{\psi_2(s)}}} \sup_{n\in L} \Bigg| \int_{-\frac{\pi}{2^{\psi_1(s)}}}^{\frac{\pi}{2^{\psi_1(s)}}} \int_{-\frac{\pi}{2^{\psi_2(s)}}}^{\frac{\pi}{2^{\psi_2(s)}}} f(x_1,x_2) \\
&\times K_{n_1}(y_1-x_1)K_{n_2}(y_2-x_2)d(x_1,x_2)\Bigg| d(y_1,y_2).
\end{aligned}$$

Let the real number $s_{ij}\geq 1$ be defined later. By the help of the following inequality we discuss the maximal function σ_L^*.

$$\sup_{n\in L}|\sigma_n f| \leq \sup_{\substack{n\in L\\ n_1\geq s_{ij}}} |\sigma_n f| + \sup_{\substack{n\in L\\ n_1< s_{ij}}} |\sigma_n f|.$$

See the first part on the right hand side. In the book of Bary [1] one can find that

$$0\leq K_n(u)\leq \frac{\pi^2}{2(n+1)u^2} \quad (0<|u|\leq \pi, n\in\mathbb{N}).$$

Since

$$y_2\in\left[\frac{\pi j}{2^{\psi_2(s)}}, \frac{\pi(j+1)}{2^{\psi_2(s)}}\right) \quad \text{and} \quad x_2\in\left[-\frac{\pi}{2^{\psi_2(s)}}, \frac{\pi}{2^{\psi_2(s)}}\right),$$

then we have

$$\frac{1}{|y_2-x_2|}\leq C\frac{2^{\psi_2(s)}}{j},$$

thus

$$0\leq K_{n_2}(y_2-x_2)\leq C\frac{4^{\psi_2(s)}}{n_2 j^2}, \quad \text{and similarly} \quad 0\leq K_{n_1}(y_1-x_1)\leq C\frac{4^{\psi_1(s)}}{n_1 i^2}.$$

Since $L\in\mathbb{N}_\alpha, \alpha$ is CRF, then without loss of generality, $L=\mathbb{N}_{\alpha,\beta,1}$ can be supposed for some $\beta>1$. Let $n_1\geq s_{ij}$, $n\in L$. Then $n_1\geq 2^{\log_2(s_{ij})}\geq 2^{\psi_1(s_{ij})}$ and $n_2\geq \frac{\alpha(n_1)}{\beta}\geq\frac{\alpha(s_{ij})}{\beta}\geq\frac{1}{\beta}2^{\psi_2(s_{ij})}$. This gives

$$\sup_{\substack{n\in L\\ n_1\geq s_{ij}}} |K_{n_1}(y_1-x_1)K_{n_2}(y_2-x_2)|\leq C\frac{4^{\psi_1(s)+\psi_2(s)}}{2^{\psi_1(s_{ij})+\psi_2(s_{ij})}i^2j^2}.$$

This gives

$$\begin{aligned}
&\sum_{\substack{i=-2^{\psi_1(s)}\\ i\neq -2,-1,0,1}}^{2^{\psi_1(s)}-1} \sum_{\substack{j=-2^{\psi_2(s)}\\ j\neq -2,-1,0,1}}^{2^{\psi_2(s)}-1} \int_{\frac{\pi i}{2^{\psi_1(s)}}}^{\frac{\pi(i+1)}{2^{\psi_1(s)}}} \int_{\frac{\pi j}{2^{\psi_2(s)}}}^{\frac{\pi(j+1)}{2^{\psi_2(s)}}} \sup\left\{|\sigma_n f(x)| : n\in L, n_1\geq s_{ij}\right\} dx \\
&\leq C\|f\|_1 \sum_{\substack{i=-2^{\psi_1(s)}\\ i\neq -2,-1,0,1}}^{2^{\psi_1(s)}-1} \sum_{\substack{j=-2^{\psi_2(s)}\\ j\neq -2,-1,0,1}}^{2^{\psi_2(s)}-1} \frac{2^{\psi_1(s)+\psi_2(s)}}{2^{\psi_1(s_{ij})+\psi_2(s_{ij})}i^2j^2} \\
&=: C\|f\|_1 A.
\end{aligned}$$

Later, we give an upper bound for A. Now, we discuss the second part, that is,

$$\sup\left\{|\sigma_n f(x)| : n\in L, n_1< s_{ij}\right\}.$$

It is well known that

$$\sigma_n f(y_1,y_2)=\sum_{|k|=0}^{n_1}\sum_{|l|=0}^{n_2}\left(1-\frac{|k|}{n_1+1}\right)\left(1-\frac{|l|}{n_2+1}\right)\hat f(k,l)e^{\imath k y_1+\imath l y_2}.$$

We give upper bounds for the Fourier coefficients $\hat f(k,l)$:

$$\begin{aligned}&\hat f(k,l)\\&=\frac{1}{4\pi^2}\int_{J_1}\int_{J_2}f(x_1,x_2)e^{-\imath(kx_1+lx_2)}d(x_1,x_2)\\&=\frac{1}{4\pi^2}\int_{J_1}\int_{J_2}f(x_1,x_2)\left(e^{-\imath kx_1}-1\right)\left(e^{-\imath lx_2}-1\right)d(x_1,x_2).\end{aligned}$$

This follows from the equalities $\int_{J_1}f(x_1,x_2)dx_1=0$ $(x_2\in T)$ and $\int_{J_2}f(x_1,x_2)dx_2=0$ $(x_1\in T)$. It is simple to have

$$\left|\left(e^{-\imath kx_1}-1\right)\left(e^{-\imath lx_2}-1\right)\right|\le Cklx_1x_2\le Ckl\frac{1}{2^{\psi_1(s)+\psi_2(s)}}.$$

That is, for the Fourier coefficients we have

$$|\hat f(k,l)|\le C\|f\|_1\frac{kl}{2^{\psi_1(s)+\psi_2(s)}},$$

which implies

$$\begin{aligned}&|\sigma_n f(y_1,y_2)|\\&\le\sum_{|k|=0}^{n_1}\sum_{|l|=0}^{n_2}\left(1-\frac{|k|}{n_1+1}\right)\left(1-\frac{|l|}{n_2+1}\right)C\|f\|_1\frac{kl}{2^{\psi_1(s)+\psi_2(s)}}\\&\le C\|f\|_1\frac{n_1^2n_2^2}{2^{\psi_1(s)+\psi_2(s)}}.\end{aligned}$$

If $n_1<s_{ij}$ and $n\in L$, then $n_2\le\beta\alpha(n_1)\le\beta\alpha(s_{ij})$. That is, $n_1<s_{ij}$ and $n_2\le\beta\alpha(s_{ij})$. This gives

$$\begin{aligned}&\sum_{\substack{i=-2^{\psi_1(s)}\\ i\ne-2,-1,0,1}}^{2^{\psi_1(s)}-1}\sum_{\substack{j=-2^{\psi_2(s)}\\ j\ne-2,-1,0,1}}^{2^{\psi_2(s)}-1}\int_{\frac{\pi i}{2^{\psi_1(s)}}}^{\frac{\pi(i+1)}{2^{\psi_1(s)}}}\int_{\frac{\pi j}{2^{\psi_2(s)}}}^{\frac{\pi(j+1)}{2^{\psi_2(s)}}}\sup\left\{|\sigma_n f(x)|:n\in L,n_1<s_{ij}\right\}dx\\&\le C\|f\|_1\sum_{\substack{i=-2^{\psi_1(s)}\\ i\ne-2,-1,0,1}}^{2^{\psi_1(s)}-1}\sum_{\substack{j=-2^{\psi_2(s)}\\ j\ne-2,-1,0,1}}^{2^{\psi_2(s)}-1}\frac{4^{\psi_1(s_{ij})+\psi_2(s_{ij})}}{4^{\psi_1(s)+\psi_2(s)}}.\end{aligned}$$

Let $1/2<\delta<1$ be an arbitrary real number. Since the function $s\alpha(s)$ is a continuous monotone strictly increasing function, and $\lim_{+\infty}s\alpha(s)=+\infty$ then for each $i,j\in\mathbb{Z}\setminus\{0\}$

we have an $s_{ij} \geq 1$ so that

$$\begin{aligned}
&\frac{2^{\psi_1(s)+\psi_2(s)}}{|ij|^\delta}\\
&\leq \frac{s\alpha(s)}{|ij|^\delta}\\
&= \frac{s_{ij}\alpha(s_{ij})}{\alpha(1)}\\
&\leq 4\frac{2^{\psi_1(s)+\psi_2(s)}}{|ij|^\delta}.
\end{aligned}$$

because $|i| < 2^{\psi_1(s)}$, $|j| \leq 2^{\psi_2(s)}$. Consequently,

$$1 < \frac{2^{\psi_1(s)+\psi_2(s)}}{|ij|} \leq \frac{2^{\psi_1(s)+\psi_2(s)}}{|ij|^\delta},$$

that is

$$\frac{2^{\psi_1(s)+\psi_2(s)}}{2^{\psi_1(s_{ij})+\psi_2(s_{ij})}} \leq C|ij|^\delta \quad \text{and} \quad \frac{2^{\psi_1(s_{ij})+\psi_2(s_{ij})}}{2^{\psi_1(s)+\psi_2(s)}} \leq \frac{C}{|ij|^\delta}.$$

By these inequalities (taking also account the line where the double sum A is defined) we get

$$\begin{aligned}
&\int_{(T\setminus 2J_1)\times(T\setminus 2J_2)} \sigma_L^* f(x_1,x_2)\, d(x_1,x_2)\\
&\leq C\|f\|_1 \sum_{\substack{i=-2^{\psi_1(s)}\\ i\neq -2,-1,0,1}}^{2^{\psi_1(s)}-1} \sum_{\substack{j=-2^{\psi_2(s)}\\ j\neq -2,-1,0,1}}^{2^{\psi_2(s)}-1} \left[\frac{2^{\psi_1(s)+\psi_2(s)}}{2^{\psi_1(s_{ij})+\psi_2(s_{ij})}i^2j^2} + \frac{4^{\psi_1(s_{ij})+\psi_2(s_{ij})}}{4^{\psi_1(s)+\psi_2(s)}}\right]\\
&\leq C\|f\|_1 \sum_{\substack{i=-2^{\psi_1(s)}\\ i\neq -2,-1,0,1}}^{2^{\psi_1(s)}-1} \sum_{\substack{j=-2^{\psi_2(s)}\\ j\neq -2,-1,0,1}}^{2^{\psi_2(s)}-1} \left(|ij|^{\delta-2} + |ij|^{-2\delta}\right) \leq C\|f\|_1.
\end{aligned}$$

We recall that, $1/2 < \delta < 1$ thus $\delta - 2, -2\delta < -1$. This completes the proof of Lemma 3.1. □

Lemma 3.2. *Let α be CRF, $L \in \mathbb{N}_\alpha$, $g \in L^1(T)$, and* $\operatorname{supp} g \subset J_1 \in \mathfrak{I}$, $J_2 \in \mathfrak{I}$, $\operatorname{mes}(J_j) = \frac{2\pi}{2^{\psi_j(s)}}$ *for some $s \geq 1$ ($j = 1, 2$(. Suppose that*

$$\int_T g(x)\, dx = 0.$$

Then there follows

$$\int_{(T\setminus 2J_1)\times(T\setminus 2J_2)} \sigma_L^* \left(\frac{2^{\psi_1(s)}}{2\pi} 1_{J_1} \times g\right)(x_1,x_2)\, d(x_1,x_2) \leq C\|g\|_1.$$

Proof. Do the same procedure as in the proof of Lemma 3.1. We can suppose that the center of J_1 and J_2 is 0. Besides,

$$\sum_{\substack{i=-2^{\psi_1(s)}\\ i\neq -2,-1,0,1}}^{2^{\psi_1(s)}-1}\sum_{\substack{j=-2^{\psi_2(s)}\\ j\neq -2,-1,0,1}}^{2^{\psi_2(s)}-1}\int_{\frac{\pi i}{2^{\psi_1(s)}}}^{\frac{\pi(i+1)}{2^{\psi_1(s)}}}\int_{\frac{\pi j}{2^{\psi_2(s)}}}^{\frac{\pi(j+1)}{2^{\psi_2(s)}}}\sup\left\{\left|\sigma_n\left(\frac{2^{\psi_1(s)}}{2\pi}1_{J_1}\times g\right)(x)\right| : n\in L, n_1\geq s_{i,j}\right\}d(x_1,x_2)$$
$$\leq C\|g\|_1\sum_{\substack{i=-2^{\psi_1(s)}\\ i\neq -2,-1,0,1}}^{2^{\psi_1(s)}-1}\sum_{\substack{j=-2^{\psi_2(s)}\\ j\neq -2,-1,0,1}}^{2^{\psi_2(s)}-1}\frac{2^{\psi_1(s)+\psi_2(s)}}{2^{\psi_1(s_{ij})+\psi_2(s_{ij})}i^2j^2}$$
$$=: C\|g\|_1 A.$$

The real numbers $s_{ij}\geq 1$ will be defined later in the proof of this lemma. What can be said in the case of $n_1 < s_{ij}$?

$$\sigma_n\left(\frac{2^{\psi_1(s)}}{2\pi}1_{J_1}\times g\right)=\frac{2^{\psi_1(s)}}{2\pi}\sigma_{n_1}\left(1_{J_1}\right)\times\sigma_{n_2}g.$$

$$\sigma_{n_1}\left(1_{J_1}\right)(y_1)=\int_{-\frac{\pi}{2^{\psi_1(s)}}}^{\frac{\pi}{2^{\psi_1(s)}}}K_{n_1}(y_1-x_1)dx_1\leq\frac{Cn_1}{2^{\psi_1(s)}}\leq\frac{Cs_{ij}}{s}.$$

On the other hand,

$$\hat{g}(l)=\int_{J_2}g(x)\left(e^{-\imath lx_2}-1\right)dx_2.$$

This gives

$$|\hat{g}(l)|\leq\frac{Cl}{2^{\psi_2(s)}}\|g\|_1,$$

therefore

$$\left|\sigma_n\left(\frac{2^{\psi_1(s)}}{2\pi}1_{J_1}\times g\right)(y_1,y_2)\right|$$
$$\leq C\sum_{|l|=0}^{n_2}\left(1-\frac{|l|}{n_2+1}\right)\frac{l}{2^{\psi_2(s)}}\|g\|_1 2^{\psi_1(s)}\frac{s_{ij}}{s}$$
$$\leq C\|g\|_1\frac{2^{\psi_1(s)}n_2^2 s_{ij}}{2^{\psi_2(s)}s}.$$

Since $L\in\mathbb{N}_\alpha$, where α is CRF then, without loss of generality, $L=\mathbb{N}_{\alpha,\beta,1}$ can be supposed for some $\beta>1$. If $n_1<s_{ij}$ and $n\in L$, then $n_2\leq\beta\alpha(n_1)\leq\beta\alpha(s_{ij})\leq C2^{\psi_2(s_{ij})}$. This gives

$$\sum_{\substack{i=-2^{\psi_1(s)}\\ i\neq -2,-1,0,1}}^{2^{\psi_1(s)}-1}\sum_{\substack{j=-2^{\psi_2(s)}\\ j\neq -2,-1,0,1}}^{2^{\psi_2(s)}-1}\int_{\frac{\pi i}{2^{\psi_1(s)}}}^{\frac{\pi(i+1)}{2^{\psi_1(s)}}}\int_{\frac{\pi j}{2^{\psi_2(s)}}}^{\frac{\pi(j+1)}{2^{\psi_2(s)}}}\sup\left\{\left|\sigma_n\left(\frac{2^{\psi_1(s)}}{2\pi}1_{J_1}\times g\right)(x)\right| : n\in L, n_1< s_{i,j}\right\}d(x_1,x_2)$$
$$\leq C\|g\|_1\sum_{\substack{i=-2^{\psi_1(s)}\\ i\neq -2,-1,0,1}}^{2^{\psi_1(s)}-1}\sum_{\substack{j=-2^{\psi_2(s)}\\ j\neq -2,-1,0,1}}^{2^{\psi_2(s)}-1}\frac{4^{\psi_2(s_{ij})}}{4^{\psi_2(s)}}\frac{s_{ij}}{s}$$
$$=: C\|g\|_1 B.$$

We give the construction of a double sequence (s_{ij}) such that both sums A and B will be finite. Let $0<\epsilon<\delta<1$ be real numbers defined later. ϵ "will be near" 0 and δ "will be near" 1. Define (s_{ij}) in a way that

$$\alpha(1)s\alpha(s)=s_{ij}\alpha(s_{ij})|ij|^{\delta}$$

for all i and j. This can be done since α is continuous and strictly monotone increasing with property $\lim_{+\infty}\alpha=+\infty$, and

$$1\alpha(1)|ij|^{\delta}\le 1\alpha(1)|ij|\le\alpha(1)2^{\psi_1(s)}2^{\psi_2(s)}\le\alpha(1)s\alpha(s).$$

We give an $\epsilon>0$ such that

$$\frac{\alpha(s_{ij})}{\alpha(s)}\le|ij|^{-\epsilon-1+\delta}\gamma_2^{\tau}$$

for all i and j ($\tau\in\mathbb{N}$ discussed later, and depends on γ_2 and ζ). On the contrary, suppose that for all $\epsilon>0$ there exist an i and j such that

$$\alpha(s_{ij})>|ij|^{-\epsilon-1+\delta}\alpha(s)\gamma_2^{\tau}.$$

Let $\eta:=-\epsilon-1+\delta$. Then since α is CRF we have

$$\alpha(s_{ij})>\alpha(s)\gamma_2^{\eta\log_{\gamma_2}|ij|}\gamma_2^{\tau}\ge\alpha(s)\gamma_2^{\lfloor\eta\log_{\gamma_2}|ij|\rfloor+\tau}\ge\alpha(s\zeta^{\lfloor\eta\log_{\gamma_2}|ij|\rfloor+\tau}).$$

This implies

$$s_{ij}\ge s\zeta^{\lfloor\eta\log_{\gamma_2}|ij|\rfloor+\tau}\ge s\zeta^{\eta\log_{\gamma_2}|ij|+\tau-1},$$

$$s_{ij}\alpha(s_{ij})\ge\zeta^{\eta\log_{\gamma_2}|ij|+\tau-1}\gamma_2^{\tau}|ij|^{\eta}s\alpha(s).$$

Thus,

$$\alpha(1)|ij|^{-\delta}\ge|ij|^{\eta}|ij|^{\eta\log_{\gamma_2}\zeta}\zeta^{\tau-1}\gamma_2^{\tau}.$$

Let τ be defined in a way that $\alpha(1)<\zeta^{\tau-1}\gamma_2^{\tau}$. This gives $|ij|^{-\delta}\ge|ij|^{\eta(1+\log_{\gamma_2}\zeta)}$, $\delta\le(\epsilon+1-\delta)(1+\log_{\gamma_2}\zeta)$. This does not hold for all δ and ϵ. To see this let $\delta\nearrow 1$ and $\epsilon\searrow 0$. We found that there exists an $\epsilon>0$ such that

$$\frac{\alpha(s_{ij})}{\alpha(s)}\le C|ij|^{-\epsilon-1+\delta}$$

for all i and j. Discuss expression A and B.

$$\begin{aligned}A&=\sum_{\substack{i=-2^{\psi_1(s)}\\ i\ne-2,-1,0,1}}^{2^{\psi_1(s)}-1}\ \sum_{\substack{j=-2^{\psi_2(s)}\\ j\ne-2,-1,0,1}}^{2^{\psi_2(s)}-1}\frac{2^{\psi_1(s)+\psi_2(s)}}{2^{\psi_1(s_{ij})+\psi_2(s_{ij})}i^2j^2}\\&\le C\sum_{\substack{i=-2^{\psi_1(s)}\\ i\ne-2,-1,0,1}}^{2^{\psi_1(s)}-1}\ \sum_{\substack{j=-2^{\psi_2(s)}\\ j\ne-2,-1,0,1}}^{2^{\psi_2(s)}-1}\frac{|ij|^{\delta}}{|ij|^2}\\&<\infty,\end{aligned}$$

because $0 < \delta < 1$, and

$$\begin{aligned} B &= \sum_{\substack{i=-2^{\psi_1(s)} \\ i\neq -2,-1,0,1}}^{2^{\psi_1(s)}-1} \sum_{\substack{j=-2^{\psi_2(s)} \\ j\neq -2,-1,0,1}}^{2^{\psi_2(s)}-1} \frac{4^{\psi_2(s_{ij})}}{4^{\psi_2(s)}} \frac{s_{ij}}{s} \\ &\leq C \sum_{\substack{i=-2^{\psi_1(s)} \\ i\neq -2,-1,0,1}}^{2^{\psi_1(s)}-1} \sum_{\substack{j=-2^{\psi_2(s)} \\ j\neq -2,-1,0,1}}^{2^{\psi_2(s)}-1} \frac{s_{ij}\alpha(s_{ij})}{s\alpha(s)} \frac{\alpha(s_{ij})}{\alpha(s)} \\ &\leq C \sum_{\substack{i=-2^{\psi_1(s)} \\ i\neq -2,-1,0,1}}^{2^{\psi_1(s)}-1} \sum_{\substack{j=-2^{\psi_2(s)} \\ j\neq -2,-1,0,1}}^{2^{\psi_2(s)}-1} |ij|^{-\epsilon-1} \\ &< \infty. \end{aligned}$$

The proof of Lemma 3.2 is complete. □

Lemma 3.3. *Let α be CRF, $L \in \mathbb{N}_\alpha$, $g \in L^1(T)$, and* $\operatorname{supp} g \subset J_2 \in \mathfrak{I}$, $J_2 \in \mathfrak{I}$, *with* $\operatorname{mes}(J_j) = \frac{2\pi}{2^{\psi_j(s)}}$ *for some $s \geq 1$ $(j = 1, 2($. Suppose that*

$$\int_T g(x)\, dx = 0.$$

Then there follows

$$\int_{(T\setminus 2J_1)\times(T\setminus 2J_2)} \sigma_L^* \left(g \times \frac{2^{\psi_2(s)}}{2\pi} 1_{J_2} \right)(x_1, x_2)\, d(x_1, x_2) \leq C\|g\|_1.$$

Proof. The proof is similar to the proof of Lemma 3.2, and therefore it is left to the reader. □

Later on we will need the following lemma corresponding to the maximal function of the one-dimensional Fejér kernels.

Lemma 3.4.

$$\int_{T\setminus[-\frac{\pi}{a},\frac{\pi}{a})} \sup_{n\geq b} K_n(t) dt \leq C\frac{a}{b} \quad (a, b \in \mathbb{N}\setminus\{0\}).$$

Proof. Once again we refer to Bary's book [1], one can find there

$$0 \leq K_n(u) \leq \frac{\pi^2}{2(n+1)u^2} \quad (0 < |u| \leq \pi, n \in \mathbb{N}).$$

$$\left[\frac{\pi}{a}, \pi\right) \subset \left[\frac{\pi}{a}, \frac{2\pi}{a}\right) \cup \left[\frac{2\pi}{a}, \frac{4\pi}{a}\right) \cup \cdots \cup \left[\frac{2^j\pi}{a}, \pi\right),$$

where $j = \lfloor \log_2(a) \rfloor$. Then

$$\int_{\left[\frac{2^l\pi}{a},\frac{2^{l+1}\pi}{a}\right)} \sup_{n\geq b} K_n(t) dt \leq \frac{C}{b}\frac{a^2}{4^l} \operatorname{mes}\left(\left[\frac{2^l\pi}{a}, \frac{2^{l+1}\pi}{a}\right)\right) \leq \frac{Ca}{b2^l}.$$

This gives,

$$\int_{\left[\frac{\pi}{a},\pi\right)} \sup_{n\geq b} K_n(t) dt \leq \sum_{l=0}^{\infty} \frac{Ca}{b2^l} \leq C\frac{a}{b}.$$

In the same way we also get

$$\int_{\left[-\pi,-\frac{\pi}{a}\right)} \sup_{n\geq b} K_n(t)dt \leq \sum_{l=0}^{\infty} \frac{Ca}{b2^l} \leq C\frac{a}{b}.$$

This completes the proof of Lemma 3.4. □

Next we prove that the operator σ_L^* ($L \in \mathbb{N}_\alpha$, α is CRF) is a quasi-local-like (for the exact definition of local and quasi-local operators see e.g. the book of Schipp, Wade and Simon [8]) one. That is, we prove

Lemma 3.5. *Let α be CRF, $L \in \mathbb{N}_\alpha$, $f \in L^1(T^2)$, and* $\operatorname{supp} f \subset J_1 \times J_2 \in \mathfrak{I} \times \mathfrak{I}$, *with* $\operatorname{mes}(J_j) = \frac{2\pi}{2^{\psi_j(s)}}$ *for some $s \geq 1$ ($j = 1,2$(. Suppose that*

$$\int_{T^2} f(x_1,x_2)\, d(x_1,x_2) = 0.$$

Then there follows

$$\int_{T^2\setminus(2J_1\times 2J_2)} \sigma_L^* f(y_1,y_2)\, d(y_1,y_2) \leq C\|f\|_1.$$

Proof. Since $L \in \mathbb{N}_\alpha, \alpha$ is CRF, then without loss of generality, $L = \mathbb{N}_{\alpha,\beta,1}$ can be supposed for some $\beta > 1$.

$T^2 \setminus (2J_1 \times 2J_2) = [(T \setminus 2J_1) \times (T \setminus 2J_2)] \cup [2J_1 \times (T \setminus 2J_2)] \cup [(T \setminus 2J_1) \times 2J_2] =: A_1 \cup A_2 \cup A_3.$

We discuss the integral on the set A_1 first. Let

$$g(x_1,x_2) := f(x_1,x_2) - 1_{J_1}(x_1)\frac{2^{\psi_1(s)}}{2\pi}\int_{J_1} f(t,x_2)dt - 1_{J_2}(x_2)\frac{2^{\psi_2(s)}}{2\pi}\int_{J_2} f(x_1,t)dt$$

for $(x_1,x_2) \in T^2$. Then Lemma 3.1 can be applied for function g:

$$\int_{A_1} \sigma_L^* g(x_1,x_2)d(x_1,x_2) \leq C\|g\|_1 \leq C\|f\|_1.$$

Similarly, for the function

$$1_{J_1}(x_1)\frac{2^{\psi_1(s)}}{2\pi}\int_{J_1} f(t,x_2)dt$$

we apply Lemma 3.2, and for the function

$$1_{J_2}(x_2)\frac{2^{\psi_2(s)}}{2\pi}\int_{J_2} f(x_1,t)dt$$

apply Lemma 3.3. This, and the sublinearity of the operator σ_L^* gives

$$\int_{A_1} \sigma_L^* f(x_1,x_2)d(x_1,x_2) \leq C\|f\|_1.$$

We discuss the integral of $\sigma_L^* f(x_1,x_2)$ on the set A_3. As in the proof of Lemma 3.1 we can suppose that the center of J_1 and J_2 is 0. First, investigate the integral

$$\int_{A_3} \sup\left\{|\sigma_n f(y_1,y_2)| : n \in L, n_1 < s\right\} d(y_1,y_2).$$

This integral is less or equal than

$$\sum_{j=0}^{\lceil \log_\zeta(s)\rceil} \int_{A_3} \sup\left\{|\sigma_n f(y_1,y_2)| : n\in L, n_1\in\left[\frac{s}{\zeta^{j+1}},\frac{s}{\zeta^j}\right)\right\} d(y_1,y_2) =: \sum_{j=0}^{\lceil \log_\zeta(s)\rceil} B_{3,j}.$$

Give an upper bound for $B_{3,j}$. Let $\tau := \lceil \log_\zeta(4)\rceil$ (in the proof of this lemma, only). For any $j\in\left\{0,\dots,\lceil \log_\zeta(s)\rceil\right\}$ we have

$$\begin{aligned} A_3 &= (T\setminus 2J_1)\times 2J_2 \subset T\times 2J_2\\ &= \left(T\setminus\left[-\frac{\pi\zeta^{j+\tau}}{s},\frac{\pi\zeta^{j+\tau}}{s}\right)\right)\times 2J_2 \cup \left(\left[-\frac{\pi\zeta^{j+\tau}}{s},\frac{\pi\zeta^{j+\tau}}{s}\right)\right)\times 2J_2\\ &=: A_{3,1,j}\cup A_{3,2,j}. \end{aligned}$$

In order to give an upper bound for $B_{3,j}$ discuss the following integral, first.

$$B^1_{3,j} := \int_{T\setminus\left[-\frac{\pi\zeta^{j+\tau}}{s},\frac{\pi\zeta^{j+\tau}}{s}\right)} \int_{2J_2} \sup_{\substack{n\in L,\\ n_1\in\left[\frac{s}{\zeta^{j+1}},\frac{s}{\zeta^j}\right)}} |\sigma_n f(y)|\,dy.$$

Since $x_1\in J_1, x_2\in J_2$, $y_1\in T\setminus\left[-\frac{\pi\zeta^{j+\tau}}{s},\frac{\pi\zeta^{j+\tau}}{s}\right)$ and $y_2\in 2J_2$, then $y_2-x_2\in 4J_2$, $y_1-x_1\in \left[-\frac{\pi\zeta^j}{s},\frac{\pi\zeta^j}{s}\right)$, because e.g. $y_1-x_2\ge\frac{\pi(\zeta^{j+\tau}-2)}{s}\ge\frac{\pi\zeta^j}{s}$ $(\beta^j(\zeta^\tau-2)\ge 2\zeta^j\ge 2)$. By this and by the theorem of Fubini we have have

$$B^1_{3,j} \le \|f\|_1 \int_{T\setminus\left[-\frac{\pi\zeta^j}{s},\frac{\pi\zeta^j}{s}\right)} \int_{4J_2} \sup_{\substack{n\in L,\\ n_1\in\left[\frac{s}{\zeta^{j+1}},\frac{s}{\zeta^j}\right)}} K_{n_1}(t_1)K_{n_2}(t_2)\,d(t_1,t_2).$$

Since $n_1\le\frac{s}{\zeta^j}$ and $n\in L$, then we have $n_2\le\beta\alpha(\frac{s}{\zeta^j})$, and

$$\int_{4J_2}\sup_{n_2\le\beta\alpha(\frac{s}{\zeta^j})} K_{n_2}(t_2)\,dt_2 \le C\frac{\beta\alpha(\frac{s}{\zeta^j})}{\alpha(s)}.$$

On the other hand, by Lemma 3.4 it follows that

$$\int_{T\setminus\left[-\frac{\pi\zeta^j}{s},\frac{\pi\zeta^j}{s}\right)}\sup_{n_1\ge\frac{s}{\zeta^{j+1}}} K_{n_1}(t_1)\,dt_1\le C.$$

Thus,

$$B^1_{3,j}\le C\|f\|_1\frac{\beta\alpha(\frac{s}{\zeta^j})}{\alpha(s)}.$$

Meanwhile,

$$\begin{aligned} &\int_{\left[-\frac{\pi\zeta^{j+\tau}}{s},\frac{\pi\zeta^{j+\tau}}{s}\right)}\int_{2J_2}\sup_{\substack{n\in L\\ n_1\le s/\zeta^j}}|\sigma_n f|\\ &\le C\|f\|_1\int_{\left[-\frac{\pi\zeta^{j+\tau}}{s},\frac{\pi\zeta^{j+\tau}}{s}\right)}\int_{2J_2}\sup_{\substack{n\in L\\ n_1\le s/\zeta^j}} n_1n_2\\ &\le C\|f\|_1\frac{\zeta^j}{s}\frac{1}{\alpha(s)}\frac{s}{\zeta^j}\beta\alpha(\frac{s}{\zeta^j})\\ &\le C\|f\|_1\frac{\alpha(\frac{s}{\zeta^j})}{\alpha(s)}. \end{aligned}$$

That is,

$$B_{3,j} \leq C\|f\|_1 \frac{\alpha(\frac{s}{\zeta^j})}{\alpha(s)}.$$

Consequently,

$$\begin{aligned}
&\int_{A_3} \sup\{|\sigma_n f(y_1,y_2)| : n \in L, n_1 < s\}\, d(y_1,y_2) \\
&\leq C \sum_{j=0}^{\lceil \log_\zeta(s) \rceil} \|f\|_1 \frac{\alpha(\frac{s}{\zeta^j})}{\alpha(s)} \\
&\leq C \frac{\|f\|_1}{\alpha(s)} \left(\alpha(s) + \frac{1}{\gamma_1}\alpha(s) + \frac{1}{\gamma_1^2}\alpha(s) + \dots \right) \\
&\leq C\|f\|_1
\end{aligned}$$

(recall that $\alpha(s/\zeta) \leq \frac{1}{\gamma_1}\alpha(s)$). Thus,

$$\int_{A_3} \sigma_L^* f \leq C\|f\|_1 + \int_{A_3} \sup\{|\sigma_n f(y_1,y_2)| : n \in L, n_1 \geq s\}\, d(y_1,y_2)$$

That is, the rest to prove that the second addable is also bounded by $C\|f\|_1$. If $n_1 \in [s\zeta^j, s\zeta^{j+1}), n \in L, j \in \mathbb{N}$, then we have

$$n_2 \in \left[\frac{\alpha(s\zeta^j)}{\beta}, \beta\alpha(s\zeta^{j+1}) \right).$$

By Lemma 3.4 we have

$$\int_{T\setminus 2J_1} \sup_{n_1 \in [s\zeta^j, s\zeta^{j+1})} K_{n_1}(t_1) dt_1 \leq \frac{C}{\zeta^j}$$

and

$$\begin{aligned}
&\int_T \sup_{n_2 \in \left[\frac{\alpha(s\zeta^j)}{\beta}, \beta\alpha(s\zeta^{j+1})\right)} K_{n_2}(t_2) dt_2 \\
&\leq \int_{\frac{-\pi}{\beta\alpha(s\zeta^{j+1})}}^{\frac{\pi}{\beta\alpha(s\zeta^{j+1})}} C\beta\alpha(s\zeta^{j+1}) dt_2 + \int_{T\setminus[\frac{-\pi}{\beta\alpha(s\zeta^{j+1})}, \frac{\pi}{\beta\alpha(s\zeta^{j+1})})} \sup_{n_2 \geq \frac{\alpha(s\zeta^j)}{\beta}} K_{n_2}(t_2) dt_2 \\
&\leq C + \frac{C\alpha(s\zeta^{j+1})}{\alpha(s\zeta^j)} \leq C + C\gamma_2 \leq C.
\end{aligned}$$

We have proved

$$\int_{A_3} \sigma_L^* f \leq C\|f\|_1.$$

In the same way one also can have

$$\int_{A_2} \sigma_L^* f \leq C\|f\|_1.$$

That is, the inequality

$$\int_{T^2\setminus(2J_1 \times 2J_2)} \sigma_L^* f(y_1,y_2)\, d(y_1,y_2) \leq C\|f\|_1$$

is proved. This completes the proof of Lemma 3.5. □

Now, we are ready to prove

Theorem 3.6. *Let α be CRF, $L \in \mathbb{N}_\alpha$. Then the operator σ_L^* is of weak type* $(1,1)$.

Proof. The fact that the operator σ_L^* is of type (∞,∞) (this means $\|\sigma_L^* f\|_\infty \le C\|f\|_\infty$ for all $f \in L^\infty(T^2)$) easily follows from the well known inequality

$$\int_{-\pi}^{\pi} |K_n(x)|dx = \pi \quad (n \in \mathbb{N}).$$

Let $f \in L^1(T^2)$, and $\lambda > \|f\|_1/(4\pi^2)$. By Lemma 2.1 we have a sequence of functions (f_i) such that

$$f = \sum_{i=0}^{\infty} f_i,$$

$$\|f_0\|_\infty \le C\lambda, \quad \|f_0\|_1 \le C\|f\|_1 \quad \text{and}$$

$$\operatorname{supp} f_i \subset I^{i,1} \times I^{i,2}, \quad \text{where}$$

$I^{i,j} \in \mathfrak{I}$ are dyadic intervals

$$\operatorname{mes}\left(I^{i,j}\right) = \frac{2\pi}{2^{\psi_j(s_i)}}, \quad \text{for some}$$

$s_i \ge 1\,(j = 1,2,\ i \in \mathbb{N})$. Moreover, $\int_{T^2} f_i(x)dx = 0 (i \ge 1)$, the dyadic rectangles $I^{i,1} \times I^{i,2}$ are disjoint $(i \in \mathbb{N} \setminus \{0\})$, and for

$$F := \bigcup_{i=1}^{\infty} \left(I^{i,1} \times I^{i,2}\right) \quad \text{we have} \quad \operatorname{mes}(F) \le C\|f\|_1/\lambda.$$

It is obvious that

$$\operatorname{mes}\left(\sigma_L^* f > \tilde{C}\lambda\right) \le \operatorname{mes}\left(\sigma_L^* f_0 > \frac{1}{2}\tilde{C}\lambda\right) + \operatorname{mes}\left(\sigma_L^*(\sum_{i=1}^{\infty} f_i) > \frac{1}{2}\tilde{C}\lambda\right).$$

The inequality

$$\|\sigma_L^* f_0\|_\infty \le \pi^2 \|f_0\|_\infty \le C\pi^2\lambda$$

shows that if we choose $\tilde{C} > 2C\pi^2$, then

$$\operatorname{mes}\left(\sigma_L^* f_0 > \frac{1}{2}\tilde{C}\lambda\right) = 0.$$

On the other hand, by Lemma 3.5 it follows that

$$\begin{aligned}
&\operatorname{mes}\left(\sigma_L^*(\sum_{i=1}^{\infty} f_i) > \frac{1}{2}\tilde{C}\lambda\right) \\
&\leq \operatorname{mes}\left(\bigcup_{i=1}^{\infty}(2I_{i,1} \times 2I_{i,1})\right) + \operatorname{mes}\left(\left\{x \in T^2 \setminus \bigcup_{i=1}^{\infty}(2I_{i,1} \times 2I_{i,1}) : \sigma_L^*(\sum_{i=1}^{\infty} f_i)(x) > \frac{1}{2}\tilde{C}\lambda\right\}\right) \\
&\leq C\|f\|_1/\lambda + \frac{2}{\tilde{C}\lambda}\int_{T^2\setminus\bigcup_{i=1}^{\infty}(2I_{i,1}\times 2I_{i,1})} \sigma_L^*(\sum_{i=1}^{\infty} f_i)(x)dx \\
&\leq C\|f\|_1/\lambda + \frac{2}{\tilde{C}\lambda}\sum_{i=1}^{\infty}\int_{T^2\setminus(2I_{i,1}\times 2I_{i,1})} \sigma_L^* f_i(x)dx \\
&\leq C\|f\|_1/\lambda + \frac{C}{\lambda}\sum_{i=1}^{\infty}\|f_i\|_1 \\
&\leq C\|f\|_1/\lambda.
\end{aligned}$$

The proof of Lemma 3.6 is complete. □

Corollary 3.7. *Let α be CRF, $L \in \mathbb{N}_\alpha$. Then the operator σ_L^* is of type (p,p) for all $1 < p \leq \infty$.*

Proof. Apply the interpolation lemmma of Marcinkiewicz (see e.g. the book of Schipp, Wade and Simon [8]), and the fact that the operator σ_L^* is sublinear. □

The proof of Proposition 1.3. (The Convergence.(It is known that the set of two-dimensional trigonometric polynomials

$$\mathcal{P} := \left\{ P = \sum_{\substack{k\leq n_1\\ l\leq n_2}} c_{k,l}e^{\imath k x_1}e^{\imath l x_2} : c_{k,l} \in \mathbb{C}, k = 0,\ldots,n_1,\ l = 0,\ldots,n_2,\ n_1,n_2 \in \mathbb{N} \right\}$$

is dense in $L^1(T^2)$. Let $f \in L^1(T^2)$, and let $\epsilon > 0$ be arbitrary. $\delta > 0$ is discussed later. There exists a $P \in \mathcal{P}$ such that

$$\|f - P\|_1 \leq \delta.$$

$$\begin{aligned}
&\operatorname{mes}\left(\limsup_{\substack{\wedge n\to\infty\\ n\in L}} |\sigma_n f - f| > \epsilon\right) \\
&\leq \operatorname{mes}\left(\sup_{n\in L}|\sigma_n f - P| > \epsilon/3\right) + \operatorname{mes}\left(\limsup_{\substack{\wedge n\to\infty\\ n\in L}}|\sigma_n P - P| > \epsilon/3\right) + \operatorname{mes}\left(|P - f| > \epsilon/3\right) \\
&\leq \frac{C}{\epsilon}\|f - P\|_1 + \frac{3}{\epsilon}\|P - f\|_1 \\
&< \frac{C}{\epsilon}\delta
\end{aligned}$$

since for all $P \in \mathcal{P}$ we have $\sigma_n P \to P$ $(\wedge n \to \infty)$. This relation holds for all $\delta > 0$. Thus for all $\epsilon > 0$ we have

$$\operatorname{mes}\left(\limsup_{\substack{\wedge n\to\infty\\ n\in L}}|\sigma_n f - f| > \epsilon\right) = 0.$$

This gives

$$\lim_{\substack{\wedge n\to\infty\\ n\in L}} \sigma_n f = f$$

almost everywhere. The proof of Theorem 1.3 is complete.

4. The Divergence

The main aim of this section is to prove Theorem 1.4, that is to prove the theorem of divergence. Let α be CRF, $\beta : [1,+\infty) \longrightarrow [1,+\infty)$ be a monotone increasing function with property $\lim_{+\infty}\beta = +\infty$, and let $\delta : [1,+\infty) \longrightarrow [0,+\infty)$ be a measurable function with the property $\lim_{+\infty}\delta = 0$. Let $L := \mathbb{N}_{\alpha,\beta,1}$ (or $L := \mathbb{N}_{\alpha,\beta,2}$). We prove the existence of such a function $f \in L^1\log^+ L\delta(L)$, that is

$$\int_{T^2} |f(x)|\log^+|f(x)|\delta(|f(x)|)dx < \infty$$

such that

$$\sup_{n\in L}\sigma_n f = +\infty$$

almost everywhere, that is the relation $\lim_{\substack{\wedge n\to\infty,\\ n\in L}} \sigma_n f = f$ may hold on a set of measure zero.

We suppose that $L = \mathbb{N}_{\alpha,\beta,1}$. The case $L = \mathbb{N}_{\alpha,\beta,2}$ can be discussed in the same way, therefore it is left to the reader. Let $x \in T^2$, $n \in \mathbb{N}^2$. Denote by

$$I_n(x) = I_{n_1}(x_1) \times I_{n_2}(x_2) \in \mathcal{I}\times\mathcal{I}$$

the two-dimensional dyadic rectangle for which $x \in I_n(x)$ and

$$\operatorname{mes}\left(I_{n_j}(x_j)\right) = \frac{2\pi}{2^{n_j}} \quad (j = 1,2).$$

For $n, a \in \mathbb{N}^2$ define the following subset of $\mathcal{I}\times\mathcal{I}$:

$$\mathcal{I}_{n,a}(x) := \left\{I_{n_1+j}(x_1) \times I_{n_2+a_2-j}(x_2) : j = 0,1,\dots,\wedge a\right\}.$$

It is easy to get

$$\bigcap \mathcal{I}_{n,a}(x) = \left\{I_{n_1+\wedge a}(x_1) \times I_{n_2+a_2}(x_2)\right\},$$

$$\operatorname{mes}\left(\bigcap \mathcal{I}_{n,a}(x)\right) = \frac{4\pi^2}{2^{n_1+n_2+\wedge a+a_2}}.$$

$F \in \mathcal{I}_{n,a}(x)$ implies $\operatorname{mes}(F) = \frac{4\pi^2}{2^{n_1+n_2+a_2}}$. Next we prove

Lemma 4.1.

$$\operatorname{mes}\left(\bigcup \mathcal{I}_{n,a}(x)\right) = \frac{4\pi^2(1+\wedge a/2)}{2^{n_1+n_2+a_2}}.$$

Proof. Denote (only for the sake of this proof)

$$\mu_k := \operatorname{mes}\left(\bigcup_{j=0}^{k}\left(I_{n_1+j}(x_1) \times I_{n_2+a_2-j}(x_2)\right)\right)$$

for $k=0,1,...,\wedge a$. Then, $\mu_0 = \frac{4\pi^2}{2^{n_1+n_2+a_2}}$, and for $k>0$ we have

$$\begin{aligned}
\mu_k &= \mu_{k-1} + \operatorname{mes}\left(I_{n_1+k}(x_1)\times I_{n_2+a_2-k}(x_2)\right)\\
&\quad - \operatorname{mes}\left(\bigcup_{j=0}^{k-1}\left(I_{n_1+j}(x_1)\times I_{n_2+a_2-j}(x_2)\right)\cap\left(I_{n_1+k}(x_1)\times I_{n_2+a_2-k}(x_2)\right)\right)\\
&\quad = \mu_{k-1} + \frac{4\pi^2}{2^{n_1+n_2+a_2}} - \operatorname{mes}\left(\bigcup_{j=0}^{k-1}\left(I_{n_1+k}(x_1)\times I_{n_2+a_2-j}(x_2)\right)\right)\\
&\quad = \mu_{k-1} + \frac{4\pi^2}{2^{n_1+n_2+a_2}} - \operatorname{mes}\left(I_{n_1+k}(x_1)\times I_{n_2+a_2-k+1}(x_2)\right)\\
&\quad = \mu_{k-1} + \frac{4\pi^2}{2^{n_1+n_2+a_2}} - \frac{4\pi^2}{2^{n_1+n_2+a_2+1}}\\
&\quad = \mu_{k-1} + \frac{4\pi^2}{2^{n_1+n_2+a_2+1}}.
\end{aligned}$$

This gives

$$\operatorname{mes}\left(\bigcup \mathfrak{I}_{n,a}(x)\right) = \mu_{\wedge a} = \mu_0 + \frac{4\wedge a\pi^2}{2^{n_1+n_2+a_2+1}} = \frac{4\pi^2(1+\wedge a/2)}{2^{n_1+n_2+a_2}}.$$

This completes the proof of Lemma 4.1. □

For $t\in T^2$, $k\in\mathbb{N}$, $a,b\in N^2$ define the sets $J^k_{b,a}(t)$, $\Omega^k_{b,a}(t)$ recursively:

$$J^0_{b,a}(t) := \{t\}, \quad \Omega^0_{b,a}(t) := \bigcup \mathfrak{I}_{b,a}(t).$$

Suppose that the sets $J^j_{b,a}(t)$ and $\Omega^j_{b,a}(t)$ are defined for $j<k$. Then consider

$$\left(I_{b_1}(t_1)\times I_{b_2}(t_2)\right)\setminus\bigcup_{j=0}^{k-1}\Omega^j_{b,a}(t)$$

as the disjoint union of dyadic rectangles of the form $I_{b_1+ka_1}(x_1)\times I_{b_2+ka_2}(x_2)$. Take from each rectangle an element as representative. The set of x's corresponding to these rectangles is $J^k_{b,a}(t)$. That is,

$$\left(I_{b_1}(t_1)\times I_{b_2}(t_2)\right)\setminus\bigcup_{j=0}^{k-1}\Omega^j_{b,a}(t) = \bigcup_{x\in J^k_{b,a}(t)}\left[I_{b_1+ka_1}(x_1)\times I_{b_2+ka_2}(x_2)\right].$$

Then take

$$\Omega^k_{b,a}(t) := \bigcup_{x\in J^k_{b,a}(t)}\bigcup \mathfrak{I}_{b+ka,a}(x).$$

This gives the a.e. equality

$$I_b(t) = \bigcup_{j=0}^{\infty}\Omega^j_{b,a}(t).$$

T^2 is the disjoint union of $2^{b_1+b_2}$ pieces of dyadic rectangles of the form $I_b(t)$, that is denoting

$$e_{k,m} := -\pi + \frac{2\pi m}{2^k} \quad (m=0,\dots,2^k-1,\ k\in\mathbb{N}),$$

we have

$$T^2 = \bigcup_{\substack{m_j=0,\dots,2^{b_j-1} \\ j=1,2}} \bigcup_{j=0}^{\infty} \Omega_{b,a}^j(e_{b_1,m_1}, e_{b_2,m_2}).$$

Denote $t_m := (e_{b_1,m_1}, e_{b_2,m_2})$ the pairs, which determine the decomposition of T^2. Define the functions $f_{b,a} : T^2 \to [0,+\infty)$ $(a, b \in \mathbb{N}^2)$ as

$$f_{b,a}(x) = 2^{\wedge a} \sum_{\substack{m_j=0,\dots,2^{b_j-1} \\ j=1,2}} \sum_{k=0}^{\infty} \sum_{y \in J_{b,a}^k(t_m)} 1_{I_{b_1+ka_1+\wedge a} \times I_{b_2+(k+1)a_2}(y)}(x).$$

Next we prove that the functions $f_{b,a}$ are in $L^1 \log^+ L$, that is we prove

Lemma 4.2. *For all $a, b \in \mathbb{N}^2$ we have*

$$\int_{T^2} |f_{b,a}(x)| \log^+ |f_{b,a}(x)| dx \le 80.$$

Proof.

$$\begin{aligned}
&\int_{T^2} |f_{b,a}(x)| \log^+(|f_{b,a}(x)|) dx \\
&= 2^{\wedge a} \log(2^{\wedge a}) \sum_{\substack{m_j=0,\dots,2^{b_j-1} \\ j=1,2}} \sum_{k=0}^{\infty} \sum_{y \in J_{b,a}^k(t_m)} \operatorname{mes}\left(1_{I_{b_1+ka_1+\wedge a} \times I_{b_2+(k+1)a_2}(y)}(x) = 1\right) \\
&= 2^{\wedge a} \log(2^{\wedge a}) \sum_{\substack{m_j=0,\dots,2^{b_j-1} \\ j=1,2}} \sum_{k=0}^{\infty} \sum_{y \in J_{b,a}^k(t_m)} \operatorname{mes}\left(\bigcap \mathcal{I}_{b+ka,a}(y)\right) \\
&= 2^{\wedge a} \log(2^{\wedge a}) \sum_{\substack{m_j=0,\dots,2^{b_j-1} \\ j=1,2}} \sum_{k=0}^{\infty} \sum_{y \in J_{b,a}^k(t_m)} \frac{\operatorname{mes}\left(\bigcup \mathcal{I}_{b+ka,a}(y)\right)}{2^{\wedge a}(1+\wedge a/2)} \\
&\le \frac{\log(2^{\wedge a})}{1+\wedge a/2} \operatorname{mes}\left(T^2\right) \le 80.
\end{aligned}$$

□

Recall that

$$\gamma_1 \alpha(x) \le \alpha(\zeta x) \le \gamma_2 \alpha(x)$$

holds for each $x \ge 1$.

One can suppose that $\log_{\gamma_2}(\zeta)$ is a power of 2, where the exponent is an integer. Namely, let $\tau \in \mathbb{Z}$ be defined as

$$\gamma_2^{2^\tau} \le \zeta < \gamma_2^{2^{\tau+1}}.$$

Then, if we substitute ζ by $\gamma_2^{2^{\tau+1}}$ and γ_2 by γ_2^2 we have that the inequality above holds even now, and $\log_{\gamma_2}(\zeta) = 2^\tau$.

That is, let both a_1 and $\log_{\gamma_2}(\zeta)$ be powers of 2, and let $a_2 := \frac{a_1}{\log_{\gamma_2}(\zeta)}$ also a positive integer and a power of two. Let $b_1 \in \mathbb{N}$ be defined in a way that

$$\beta(2^{b_1-2}) \ge \tilde{C} 2^{a_2},$$

where $\tilde{C}$ is specified later, and $b_2 \in \mathbb{N}$ be defined by

$$2^{b_2-1} \leq \alpha(2^{b_1}) \leq 2^{b_2}.$$

We prove

$$L_k = L_{k,1} \times L_{k,2} := [2^{b_1+ka_1-2}, 2^{b_1+(k+1)a_1}] \times [2^{b_2+ka_2-2}, 2^{b_2+(k+1)a_2}] \subset \mathbb{N}_{\alpha,\beta,1}.$$

Let $n_1 \in L_{k,1}$. We have to prove

$$L_{k,2} \subset [\frac{\alpha(n_1)}{\beta(n_1)}, \alpha(n_1)\beta(n_1)],$$

that is for all $n_1 \in L_{k,1}$

$$\frac{\alpha(n_1)}{\beta(n_1)} \leq 2^{b_2+ka_2-2} \quad \text{and} \quad 2^{b_2+(k+1)a_2} \leq \alpha(n_1)\beta(n_1).$$

Discuss the second one first. It is sufficient to prove

$$2^{b_2+(k+1)a_2} \leq \alpha(2^{b_1+ka_1-2})\beta(2^{b_1-2})$$

(remark that both α and β are increasing).

$$\begin{aligned}
&\alpha(2^{b_1+ka_1-2}) \\
&\geq \alpha(2^{b_1}\zeta^{\lfloor \log_\zeta(2^{ka_1-2})\rfloor}) \\
&\geq \gamma_1^{\lfloor \log_\zeta(2^{ka_1-2})\rfloor}\alpha(2^{b_1}) \\
&\geq \gamma_1^{(ka_1-2)\log_\zeta(2)-1}2^{b_2-1} \\
&= 2^{ka_1\frac{1}{\log_{\gamma_1}(\zeta)}}2^{b_2}\frac{1}{2}\gamma_1^{-2\log_\zeta(2)-1}.
\end{aligned}$$

This $\gamma_2 \geq \gamma_1$, and the definition of a_2 gives

$$\alpha(2^{b_1+ka_1-2})\beta(2^{b_1-1}) \geq 2^{b_2+ka_2}2^{a_2}\tilde{C}\frac{1}{2\gamma_1^{2\log_\zeta(2)+1}} \geq 2^{b_2+(k+1)a_2},$$

valid if

$$\tilde{C} \geq 2\gamma_1^{2\log_\zeta(2)+1}.$$

Discuss the first one. It is sufficient to prove

$$2^{b_2+ka_2-2} \geq \frac{\alpha(2^{b_1+(k+1)a_1})}{\beta(2^{b_1-2})}.$$

Let $j = \lceil (k+1)a_1 \log_\zeta(2) \rceil$. Then

$$\begin{aligned}
&\alpha(2^{b_1+(k+1)a_1}) \\
&\leq \alpha(2^{b_1}\zeta^j) \\
&\leq \alpha(2^{b_1})\gamma_2^j \\
&\leq 2^{b_2}\gamma_2^j \\
&\leq 2^{b_2}\gamma_2^{(k+1)a_1\log_\zeta(2)}\gamma_2 \\
&= 2^{b_2+(k+1)a_2}\gamma_2.
\end{aligned}$$

We need to prove

$$\beta(2^{b_1-2}) \geq 2^{a_2}4\gamma_2.$$

Since

$$\beta(2^{b_1-2}) \geq \tilde{C}2^{a_2},$$

hence the inequality holds when $\tilde{C} \geq 4\gamma_2$ is supposed. That is, let $\tilde{C} \geq 4\gamma_2, 2\gamma_1^{2\log_\varsigma(2)+1}$ be any fixed real number. Then the relation $L_k \subset \mathbb{N}_{\alpha,\beta,1}$ is fulfilled for each $k \in \mathbb{N}$.

Lemma 4.3. *Let $a, b \in \mathbb{N}^2$ as above. We prove*

$$\sup_{n\in\mathbb{N}_{\alpha,\beta,1}} \sigma_n f_{b,a}(y) \geq 2^{-9}$$

for almost every $y \in T^2$.

Proof. It is easy to show that for $-2/n \leq u \leq 2/n$ $(u \neq 0, n \in \mathbb{N} \setminus \{0\})$

$$\begin{aligned} K_n(u) &= \frac{1}{2(n+1)}\left(\frac{\sin(\frac{u}{2}(n+1))}{\sin(\frac{u}{2})}\right)^2 \\ &\geq \frac{1}{2(n+1)}\frac{4}{u^2}2^{-4}(n+1)^2u^2/4 \\ &= 2^{-5}(n+1), \end{aligned}$$

since for $0 < |x| \leq 2$ we have $|\sin(x)| \geq 0.25|x|$ and $(n+1)|u/2| \leq 2$.

Let $y \in T^2$. Then there exists a unique $t_m \in T^2$, $k \in \mathbb{N}$ such that $y \in \Omega^k_{b,a}(t_m)$ and hence a unique $t \in J^k_{b,a}(t_m)$ with

$$y \in \bigcup \mathfrak{I}_{b+ka,a}(t).$$

Then we have $y \in I_{b_1+ka_1+j}(t_1) \times I_{b_2+(k+1)a_2-j}(t_2)$ for a $j \in \{0, 1, \ldots, \wedge a\}$. Then by the nonnegativity of the function $f_{b,a}$ and the Fejér kernels it is not difficult to give a lower bound for $\sup_{n\in\mathbb{N}_{\alpha,\beta,1}} \sigma_n f_{b,a}(y)$:

$$\begin{aligned} &\sup_{n\in L_k} \sigma_n f_{b,a}(y) \\ &\geq \int_{T^2} f_{b,a}(x)K_{2^{b_1+ka_1+j-2}}(y_1-x_1)K_{2^{b_2+(k+1)a_2-j-2}}(y_2-x_2)dx \\ &\geq \int_{I_{b_1+ka_1+\wedge a}(t_1)\times I_{b_2+(k+1)a_2}(t_2)} f_{b,a}(x)K_{2^{b_1+ka_1+j-2}}(y_1-x_1)K_{2^{b_2+(k+1)a_2-j-2}}(y_2-x_2)dx \\ &= 2^{\wedge a}\int_{I_{b_1+ka_1+\wedge a}(t_1)\times I_{b_2+(k+1)a_2}(t_2)} K_{2^{b_1+ka_1+j-2}}(y_1-x_1)K_{2^{b_2+(k+1)a_2-j-2}}(y_2-x_2)dx \\ &\geq 2^{\wedge a}\frac{4\pi^2}{2^{b_1+b_2+ka_1+\wedge a+(k+1)a_2}}2^{-10}2^{b_1+ka_1+j-2+b_2+(k+1)a_2-j-2} \\ &= \pi^2 2^{-12} \geq 2^{-9}. \end{aligned}$$

This completes the proof of Lemma 4.3. □

Finally we prove Theorem 1.4, that is we give the construction of a counterexample for the function f. Let $\omega : \mathbb{N} \to \mathbb{N}$ be a strictly monotone increasing sequence, such that

$$\delta(t) \leq \frac{1}{4^n}$$

for all $t \geq \omega_n, n \in \mathbb{N}$. This can be done, since $\lim_{+\infty} \delta = 0$. Let $\tau : \mathbb{N} \to \mathbb{N}$ be also a strictly monotone sequence (consequently $\tau(n) \geq n$), and $a_{n,1} := 2^{\tau_n}$. Let $a_{n,2}, b_{n,1}$ and $b_{n,2}$

be defined as above (e.g. $a_{n,2} := a_{n,1}/\log_{\gamma_2}(\zeta)$). Also suppose that $2^{a_{n,1}}, 2^{a_{n,2}} \geq \omega_n$. Then take

$$f := \sum_{n=0}^{\infty} 2^n f_n := \sum_{n=0}^{\infty} 2^n f_{b_n,a_n}.$$

By the construction of the functions $f_{b,a}$ it is easy to have

$$\mathrm{mes}\left(x \in T^2 : f_{b,a} \neq 0\right) = \frac{1}{2^{\wedge a}(1+\wedge a/2)}.$$

Thus, taking

$$\begin{aligned} H_{-1} &:= \left\{x \in T^2 : f_n(x) = 0 \, \forall n \in \mathbb{N}\right\}, \\ H_n &:= \left\{x \in T^2 : f_n(x) \neq 0, f_{n+1+j}(x) = 0 \, (j = 0, 1, \dots)\right\}, \ (n \in \mathbb{N}). \end{aligned}$$

$$\begin{aligned} &\mathrm{mes}\,(\limsup \{f_k \neq 0\}) \\ &= \mathrm{mes}\left(\bigcap_{n=1}^{\infty} \bigcup_{k \geq n} \{f_k \neq 0\}\right) \\ &\leq \lim_{n\to\infty} \sum_{k \geq n} \mathrm{mes}\,(\{f_k \neq 0\}) \\ &\leq \lim_{n\to\infty} \sum_{k \geq n} \frac{1}{2^{\wedge a_k}} \\ &\leq \lim_{n\to\infty} \sum_{k \geq n} \frac{1}{2^{\min(2^k, 2^k/\log_{\gamma_2}(\zeta))}} = 0. \end{aligned}$$

This implies the a.e. equality $\bigcup_{n=-1}^{\infty} H_n = T^2$. This will play an important role a couple of lines below in the proof of $f \in L^1 \log^+ L\delta(L)$. Let $x \in H_n$. Then

$$\begin{aligned} 2^n 2^{\min(a_{n,1}, a_{n,2})} &= 2^n f_n(x) \\ &\leq f(x) \\ &= \sum_{k=0}^{n} 2^k f_k(x) \\ &\leq \sum_{k=0}^{n} 2^k 2^{\min(a_{k,1}, a_{k,2})} \\ &\leq C 2^n 2^{\min(a_{n,1}, a_{n,2})} = C 2^n f_n(x). \end{aligned}$$

Then $\delta(f(x)) \leq \frac{1}{4^n}$ on the set $x \in H_n$. By Lemma 4.2 we easily obtain

$$\begin{aligned}
&\int_{T^2} f(x)\log^+(f(x))\delta(f(x))dx \\
&= \sum_{n=0}^{\infty}\int_{H_n} f(x)\log^+(f(x))\delta(f(x))dx \\
&\leq \sum_{n=0}^{\infty}\int_{H_n} C2^n f_n(x)\log^+(C2^n f_n(x))\frac{1}{4^n}dx \\
&\leq C\sum_{n=0}^{\infty}\frac{1}{2^n}\int_{T^2} f_n(x)\log^+(f(x))dx \\
&\leq C.
\end{aligned}$$

Now, by Lemma 4.3 we show that f is "a real counterexample" function. By the nonnegativity of the functions f_n and the Fejér kernel functions we have

$$\sup_{n\in\mathbb{N}_{\alpha,\beta,1}} \sigma_n f(y) \geq \sup_{n\in\mathbb{N}_{\alpha,\beta,1}} \sigma_n 2^k f_{b_k,a_k}(y) \geq 2^{-9}2^k$$

for almost all $y \in T^2$ and for all $k \in \mathbb{N}$. This implies that we have

$$\sup_{n\in\mathbb{N}_{\alpha,\beta,1}} \sigma_n f(y) = +\infty$$

almost everywhere. This completes the proof of Theorem 1.4.

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