ALMOST EVERYWHERE CONVERGENCE OF A SUBSEQUENCE OF THE LOGARITHMIC MEANS OF QUADRATICAL PARTIAL SUMS OF DOUBLE WALSH-FOURIER SERIES

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Abstract. The main aim of this paper is to prove that the maximal operator of the logarithmic means of quadratical partial sums of double Walsh-Fourier series is of weak type $(1,1)$ provided that the supremum in the maximal operator is taken over special indicies. The set of Walsh polynomials is dense in $L_1(I\times I)$, so by the well-known density argument we have that $t_{2^n}f\left(x^1,x^2\right)\to f\left(x^1,x^2\right)$ a. e. for all integrable two-variable function f.

1. Introduction

The partial sums $S_n(f)$ of the Walsh-Fourier series of a function $f\in L(I)$, $\quad I=[0,1]$ converges in measure on I ([8],Ch. 5). The condition $f\in L\ln L(I\times I)$ provides convergence in measure on $I\times I$ of the rectangular partial sums $S_{n,m}(f)$ of double Fourier-Walsh series ([13], Ch. 3.) The first example of a function from classes wider than $L\ln L(I\times I)$ with $S_{n,n}(f)$ divergent in measure on $I\times I$ was obtained in [3]. Moreover, in each Orlicz space wider than $L\ln L(I\times I)$ the set of functions which quadratic Walsh-Fourier sums converge in measure on $I\times I$ is of first Baire category [11]. In [2] we proved that similar proposition is true also for logarithmic means of quadratical partial sums

$$t_nf\left(x^1,x^2\right):=\frac{1}{l_n}\sum_{i=1}^{n-1}\frac{S_{i,i}(f,x,y)}{n-i}$$

of double Walsh-Fourier series. We proved that for any Orlicz space, which is not a subspace of $L\ln L(I\times I)$, the set of the functions that these means converges in measure is of first Baire category. From this result follows that in classes wider than $L\ln L(I\times I)$ there exists functions f for which logarithmic means $t_n(f)$ of quadratical partial sums of double Walsh-Fourier series diverges in measure.

Besides, it is surprising that the two cases (the logarithmic means of quadratical and the two-dimensional partial sums) are not different in this point of view. Namely, for instance in the case of $(C,1)$ means we have a quite different situation. That is, it is well-known [13] that the Marcinkiewicz means $\sigma_n(f)=\frac{1}{n}\sum\limits_{j=1}^{n}S_{j,j}(f)$, that is the $(C,1)$ means of quadratical partial sums of double trigonometric Fourier series of a function $f\in L$ converges in L-norm and a.e. to f. Analogical questions with respect to the Walsh, Vilenkin systems are discussed by Weisz [12], Goginava [5] and Gát [1].

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Thus, in question of convergence in measure logarithmic means of quadratical partial sums of double Walsh-Fourier series differs from Marcinkiewicz means and like the usual quadratical partial sums of double Walsh-Fourier series. In spite of this in [7] it is proved the difference between Nörlund logarithmic summability and the usual convergence for Walsh-Fourier series.

The main aim of this paper is to prove that the maximal operator of the logarithmic means of quadratical partial sums of double Walsh-Fourier series is of weak type $(1,1)$ provided that the supremum in the maximal operator is taken over special indicies. The set of Walsh polynomials is dense in $L_1(I\times I)$, so by the well-known density argument we have that $t_{2^n}f(x^1,x^2)\to f(x^1,x^2)$ a. e. for all integrable two-variable function f.

2. Definitions and Notation

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N}:=\mathbf{P}\cup\{0\}$. Denote Z_2 the discrete cyclic group of order 2, that is $Z_2=\{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such a way that the measure of a singleton is $1/2$. Let I be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of I are of the form $x=(x_0,x_1,...,x_k,...)$ with $x_k\in\{0,1\}\ (k\in\mathbf{N})$. The group operation on I is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group I is called the Walsh group. A base for the neighborhoods of I can be given in the following way:

$$I_0(x):=I,\ \ I_n(x):=\{y\in I:\ y=(x_0,...,x_{n-1},y_n,y_{n+1},...)\},$$
$$(x\in I,n\in\mathbf{N}).$$

These sets are called the dyadic intervals. Let $0=(0:i\in\mathbf{N})\in I$ denote the null element of I, $I_n:=I_n(0)\ (n\in\mathbf{N})$. Set $\overline{I}_n:=I\backslash I_n$.

For $k\in\mathbf{N}$ and $x\in I$ denote

$$r_k(x):=(-1)^{x_k}\quad (x\in I,k\in\mathbf{N})$$

the k-th Rademacher function. If $n\in\mathbf{N}$, then $n=\sum\limits_{i=0}^{\infty}n_i2^i$, where $n_i\in\{0,1\}\ (i\in\mathbf{N})$, i. e. n is expressed in the number system of base 2. Denote $|n|:=\max\{j\in\mathbf{N}:n_j\neq 0\}$, that is, $2^{|n|}\leq n<2^{|n|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_n(x):=\prod_{k=0}^{\infty}(r_k(x))^{n_k}=r_{|n|}(x)(-1)^{\sum\limits_{k=0}^{|n|-1}n_kx_k}\quad (x\in I,n\in\mathbf{P}).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x)=\sum_{k=0}^{n-1}w_k(x).$$

Recall that

$$D_{2^n}(x)=\begin{cases}2^n,\text{if } x\in I_n,\\ 0,\ \ \text{if } x\in\overline{I}_n.\end{cases}\tag{1}$$

The rectangular partial sums of the 2-dimensional Walsh-Fourier series are defined as follows:

$$S_{M,N}(f;x^1,x^2):=\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\widehat{f}(i,j)\,w_i\left(x^1\right)w_j\left(x^2\right),$$

where the number

$$\widehat{f}(i,j)=\int_{I\times I}f\left(x^1,x^2\right)w_i\left(x^1\right)w_j\left(x^2\right)d\mu\left(x^1,x^2\right)$$

is said to be the (i,j)th Walsh-Fourier coefficient of the function f.

The norm of the space $L_p(I\times I)$ is defined by

$$\|f\|_p:=\left(\int_{I\times I}\left|f\left(x^1,x^2\right)\right|^p d\mu\left(x^1,x^2\right)\right)^{1/p}\qquad(1\leq p<\infty),$$

and $\|f\|_\infty:=$ ess sup$|f(x^1,x^2)|$. The space weak-$L_1(I\times I)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak}-L_1(I\times I)}:=\sup_{\lambda>0}\lambda\mu\left(|f|>\lambda\right)<+\infty.$$

The logarithmic means of cubical partial sums of the double Walsh-Fourier series are defined as follows

$$t_nf\left(x^1,x^2\right)=\frac{1}{l_n}\sum_{i=1}^{n-1}\frac{S_{i,i}\left(f,x^1,x^2\right)}{n-i},$$

where

$$l_n=\sum_{k=1}^{n-1}\frac{1}{k}.$$

Denote

$$F_n(x)=\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{D_k(x)}{n-k},$$

$$F_n\left(x^1,x^2\right)=\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{D_k\left(x^1\right)D_k\left(x^2\right)}{n-k},$$

$$K_n(x)=\frac{1}{n}\sum_{k=1}^{n}D_k(x),$$

$$K_n\left(x^1,x^2\right)=\frac{1}{n}\sum_{k=1}^{n}D_k\left(x^1\right)D_k\left(x^2\right).$$

For the function f we consider the maximal operators

$$t_\#f=\sup_{n\in\mathbf{N}}|t_{2^n}f|.$$

3. Formulation of the Main Results

Theorem 1. *Let $f \in L_1(I \times I)$. Then*

$$\mu\{t_{\#} f > \lambda\} \leq \frac{c}{\lambda} \|f\|_1 .$$

Corollary 1. *Let $f \in L_1(I \times I)$. Then*

$$t_{2^n} f\left(x^1, x^2\right) \to f\left(x^1, x^2\right) \quad a.e. \ \ as \ \ n \to \infty.$$

4. Auxiliary Propositions

Lemma 1. *(Calderon-Zygmund decomposition* [10]*) Let $f \in L_1(I \times I)$, $\lambda > \|f\|_1$. Then there exists $(u^{(i,1)}, u^{(i,2)}) \in I \times I$, $k_i \in \mathbf{N}\,(i = 1, 2, \ldots,)$ and a decomposition*

$$f = f_0 + \sum_{i=1}^{\infty} f_i,$$

where

1)

$$\|f_0\|_1 \leq c\lambda, \quad \|f_0\|_1 \leq c\|f\|_1 ;$$

2)

$$\operatorname{supp} f_i \subset I_{k_i}\left(u^{i,1}\right) \times I_{k_i}\left(u^{i,2}\right), \qquad \int_{I \times I} f_i = 0, \ i = 1, 2, ...;$$

3)

$$\mu\left(\bigcup_{i=1}^{\infty}\left(I_{k_i}\left(u^{i,1}\right) \times I_{k_i}\left(u^{i,2}\right)\right)\right) \leq c\|f\|_1/\lambda.$$

Lemma 2. [6]*Let $A \geq k, A, k \in \mathbf{N}$. Then*

$$\int_{\overline{I}_k} \sup_{n \geq 2^A} |K_n(x)|\, d\mu(x) \leq c\frac{A-k+1}{2^{A-k}}.$$

Lemma 3. [1]*Let $k \in \mathbf{N}$. Then*

$$\int_{\overline{I_k \times I_k}} \sup_{n \geq 2^k} \left|K_n\left(x^1, x^2\right)\right| d\mu\left(x^1, x^2\right) \leq c < \infty.$$

5. Proof of the Main results

Proof of Theorem 1. Since

$$D_{2^n - j} = D_{2^n} - w_{2^n - 1} D_j,$$

we can write

$$
\begin{aligned}
(2)\qquad F_{2^n}\left(x^1,x^2\right) &= \frac{1}{l_{2^n}}\sum_{j=1}^{2^n-1}\frac{D_{2^n-j}\left(x^1\right)D_{2^n-j}\left(x^2\right)}{j} \\
&= D_{2^n}\left(x^1\right)D_{2^n}\left(x^2\right)-\frac{D_{2^n}\left(x^1\right)w_{2^n-1}\left(x^2\right)}{l_{2^n}}\sum_{j=1}^{2^n-1}\frac{D_j\left(x^2\right)}{j} \\
&\quad -\frac{D_{2^n}\left(x^2\right)w_{2^n-1}\left(x^1\right)}{l_{2^n}}\sum_{j=1}^{2^n-1}\frac{D_j\left(x^1\right)}{j} \\
&\quad +\frac{w_{2^n-1}\left(x^1\right)w_{2^n-1}\left(x^2\right)}{l_{2^n}}\sum_{j=1}^{2^n-1}\frac{D_j\left(x^1\right)D_j\left(x^2\right)}{j} \\
&= F_n^{(1)}\left(x^1,x^2\right)-F_n^{(2)}\left(x^1,x^2\right)-F_n^{(3)}\left(x^1,x^2\right)+F_n^{(4)}\left(x^1,x^2\right).
\end{aligned}
$$

Denote

$$t_n^{(i)}f:=f*F_n^{(i)},\quad i=1,2,3,4.$$

Since the operator

$$\sup_{n\in\mathbf{N}}2^{2n}\left|\int\limits_{I_n(x^1)\times I_n(x^2)}f\left(u^1,u^2\right)d\mu\left(u^1,u^2\right)\right|$$

is of weak type $(1,1)$ and

$$t_{\#}^{(1)}f:=\sup_{n\in\mathbf{N}}\left|t_n^{(1)}f\right|=\sup_{n\in\mathbf{N}}2^{2n}\left|\int\limits_{I_n(x^1)\times I_n(x^2)}f\left(u^1,u^2\right)d\mu\left(u^1,u^2\right)\right|,$$

we obtain that

$$(3)\qquad \left\|t_{\#}^{(1)}f\right\|_{weak-L_1(I\times I)}\leq c\left\|f\right\|_1.$$

We prove that

$$(4)\qquad \int\limits_{I_N\times I_N}\sup_{n\geq N}\left|F_n^{(4)}\left(x^1,x^2\right)d\mu\left(x^1,x^2\right)\right|\leq c<\infty.$$

Using Abel's transformation we can write that

$$\sum_{j=1}^{2^n-1}\frac{D_j\left(x^1\right)D_j\left(x^2\right)}{j}=\sum_{j=1}^{2^n-2}\frac{K_j\left(x^1,x^2\right)}{j+1}+K_{2^n-1}\left(x^1,x^2\right).$$

Then we have

$$\int\limits_{I_N\times I_N}\sup_{n\geq N}\left|F_n^{(4)}\left(x^1,x^2\right)d\mu\left(x^1,x^2\right)\right| \tag{5}$$
$$\leq \int\limits_{I_N\times I_N}\sup_{n\geq N}\frac{1}{l_{2^n}}\sum_{j=1}^{2^n-2}\frac{\left|K_j\left(x^1,x^2\right)\right|}{j+1}d\mu\left(x^1,x^2\right)$$
$$+\int\limits_{I_N\times I_N}\sup_{n\geq N}\left|K_{2^n-1}\left(x^1,x^2\right)\right|d\mu\left(x^1,x^2\right)$$
$$= I+II.$$

Since [4]

$$\sup_n\int\limits_{I\times I}\left|K_n\left(x^1,x^2\right)\right|d\mu\left(x^1,x^2\right)<\infty,$$

from Lemma 3 we get

$$II\leq c<\infty, \tag{6}$$

and also

$$I \leq \int\limits_{I_N\times I_N}\sup_{n\geq N}\frac{1}{l_{2^n}}\sum_{j=1}^{2^N-1}\frac{\left|K_j\left(x^1,x^2\right)\right|}{j+1}d\mu\left(x^1,x^2\right) \tag{7}$$
$$+\int\limits_{I_N\times I_N}\sup_{n\geq N}\frac{1}{l_{2^n}}\sum_{j=2^N}^{2^n-2}\frac{\left|K_j\left(x^1,x^2\right)\right|}{j+1}d\mu\left(x^1,x^2\right)$$
$$\leq \frac{1}{l_N}\sum_{j=1}^{2^N-1}\frac{1}{j}\int\limits_{I\times I}\left|K_j\left(x^1,x^2\right)\right|d\mu\left(x^1,x^2\right)$$
$$+\sup_{n\geq N}\frac{1}{l_{2^n}}\sum_{j=2^N}^{2^n-2}\frac{1}{j+1}\int\limits_{I_N\times I_N}\sup_{j\geq 2^N}\left|K_j\left(x^1,x^2\right)\right|d\mu\left(x^1,x^2\right)$$
$$\leq c<\infty.$$

Combining (5) - (7) we obtain the proof of (4).

Hence, we can write that (see Gát [1])

$$\left\|t_{\#}^{(4)}f\right\|_{weak-L_1(I\times I)}\leq c\left\|f\right\|_1. \tag{8}$$

Finally, we prove that

$$\left\|t_{\#}^{(2)}f\right\|_{weak-L_1(I\times I)}\leq c\left\|f\right\|_1. \tag{9}$$

Since

$$\sum_{j=1}^{2^n-1}\frac{D_j(u)}{j}=\sum_{j=1}^{2^n-2}\frac{K_j(u)}{j+1}+K_{2^n-1}(u),$$

we have

$$t_n^{(2)}f\left(y^1,y^2\right)$$

$$\begin{aligned}
(10)\qquad &= \int\limits_{I\times I} f\left(x^1,x^2\right)\frac{D_{2^n}\left(x^1+y^1\right)w_{2^n-1}\left(x^2+y^2\right)}{l_{2^n}}\sum_{j=1}^{2^n-1}\frac{D_j\left(x^2+y^2\right)}{j}d\mu\left(x^1,x^2\right)\\
&= \int\limits_{I\times I} f\left(x^1,x^2\right)\frac{D_{2^n}\left(x^1+y^1\right)w_{2^n-1}\left(x^2+y^2\right)}{l_{2^n}}\sum_{j=1}^{2^n-2}\frac{K_j\left(x^2+y^2\right)}{j+1}d\mu\left(x^1,x^2\right)\\
&\quad+\int\limits_{I\times I} f\left(x^1,x^2\right)\frac{D_{2^n}\left(x^1+y^1\right)w_{2^n-1}\left(x^2+y^2\right)}{l_{2^n}}K_{2^n-1}\left(x^2+y^2\right)d\mu\left(x^1,x^2\right)\\
&= t_n^{(2,1)}f\left(y^1,y^2\right)+t_n^{(2,2)}f\left(y^1,y^2\right).
\end{aligned}$$

Denote (use the notation of Lemma 1)

$$g(t):=\sum_{i=1}^{\infty}\frac{|f_i(t)|}{k_i},L(t):=\sum_{i=1}^{\infty}\frac{|K_i(t)|}{i+1}.$$

Let

$$(11)\qquad \left(y^1,y^2\right)\in\overline{\bigcup_{i=1}^{\infty}(I_{k_i}(u^{i,1})\times I_{k_i}(u^{i,2}))}.$$

Since $\int f_i=0$ we have

$$(12)\qquad t_n^{(2,1)}f_i\left(y^1,y^2\right)=0\quad\text{for}\quad n\leq k_i.$$

Let $y^1\in\overline{I_{k_i}(u^{i,1})}$. Then from (1) we can write that $t_n^{(2,1)}f_i(y^1,y^2)=0$ for $n>k_i$. Hence $t_n^{(2,1)}f_i(y^1,y^2)\neq 0$ implies that $y^1\in I_{k_i}(u^{i,1})$. Consequently, from (11) we can suppose that

$$y^2\in\bigcap_{i=1}^{\infty}\overline{I_{k_i}(u^{i,2})}.$$

Then we write

$$\begin{aligned}
(13)\qquad D\ :&=\mu\left\{\left(y^1,y^2\right)\in I\times\left(\bigcap_{i=1}^{\infty}\overline{I_{k_i}(u^{i,2})}\right):t_{\#}^{(2,1)}f\left(y^1,y^2\right)>c\lambda\right\}\\
&\leq\int\limits_{\bigcap_{i=1}^{\infty}\overline{I_{k_i}(u^{i,2})}}\mu\left\{y^1\in I:t_{\#}^{(2,1)}\left(\sum_{i=1}^{\infty}f_i\right)\left(y^1,y^2\right)>c\lambda\right\}d\mu\left(y^2\right).
\end{aligned}$$

From (12), we have

$$
\begin{aligned}
&\left|t_n^{(2,1)}\left(\sum_{i=1}^{\infty} f_i\right)\left(y^1,y^2\right)\right| \\
\leq & \sum_{i=1}^{\infty}\left|\int\limits_{I_{k_i}(u^{i,1})\times I_{k_i}(u^{i,2})} f_i\left(x^1,x^2\right)\frac{D_{2^n}\left(x^1+y^1\right)w_{2^n-1}\left(x^2+y^2\right)}{l_{2^n}}\sum_{j=1}^{2^n-2}\frac{K_j\left(x^2+y^2\right)}{j+1}d\mu\left(x^1,x^2\right)\right| \\
\leq & \int\limits_I\left(\int\limits_I\sum_{i=1}^{\infty}\frac{\left|f_i\left(x^1,x^2\right)\right|}{k_i}\sum_{j=1}^{2^n-2}\frac{\left|K_j\left(x^2+y^2\right)\right|}{j+1}d\mu\left(x^2\right)\right)D_{2^n}\left(x^1+y^1\right)d\mu\left(x^1\right) \\
= & \int\limits_I\left(\int\limits_I g\left(x^1,x^2\right)L\left(x^2+y^2\right)d\mu\left(x^2\right)\right)D_{2^n}\left(x^1+y^1\right)d\mu\left(x^1\right).
\end{aligned}
$$

The one-dimensional operator $\sup_{n\in\mathbf{N}}|S_{2^n}f|$ is of weak type $(1,1)$. We apply this fact for the one-dimensional function $h(x^1):=\int_I g(x^1,x^2)L(x^2+y^2)d\mu(x^2)$ for every fixed $y^2\in I$. Consequently, from (13) and by the above we can write

$$
\begin{aligned}
(14)\qquad D\leq & \int\limits_{\overline{\bigcap_{i=1}^{\infty} I_{k_i}(u^{i,2})}}\mu\Bigg\{y^1\in I:\sup_n\int\limits_I\left(\int\limits_I g\left(x^1,x^2\right)L\left(x^2+y^2\right)d\mu\left(x^2\right)\right) \\
& \cdot D_{2^n}\left(x^1+y^1\right)d\mu\left(x^1\right)>c\lambda\Bigg\}d\mu(y^2)
\end{aligned}
$$

$$
\begin{aligned}
&\leq\frac{c}{\lambda}\int\limits_{\overline{\bigcap_{i=1}^{\infty} I_{k_i}(u^{i,2})}}\left[\int\limits_I\left(\int\limits_I g\left(x^1,x^2\right)L\left(x^2+y^2\right)d\mu\left(x^2\right)\right)d\mu\left(x^1\right)\right]d\mu\left(y^2\right) \\
&=\frac{c}{\lambda}\int\limits_{\overline{\bigcap_{i=1}^{\infty} I_{k_i}(u^{i,2})}}\left[\int\limits_I\left(\int\limits_I g\left(x^1,x^2\right)d\mu\left(x^1\right)\right)L\left(x^2+y^2\right)d\mu\left(x^2\right)\right]d\mu\left(y^2\right) \\
&\leq\frac{c}{\lambda}\sum_{i=1}^{\infty}\frac{1}{k_i}\int\limits_{\overline{I_{k_i}(u^{i,2})}}\left[\int\limits_{I_{k_i}(u^{i,2})}\left(\int\limits_{I_{k_i}(u^{i,1})}\left|f_i\left(x^1,x^2\right)\right|d\mu\left(x^1\right)\right)L\left(x^2+y^2\right)d\mu\left(x^2\right)\right]d\mu\left(y^2\right) \\
&=\frac{c}{\lambda}\sum_{i=1}^{\infty}\frac{1}{k_i}\int\limits_{\overline{I_{k_i}(u^{i,2})}}\Bigg[\int\limits_{I_{k_i}(u^{i,2})}\left(\int\limits_{I_{k_i}(u^{i,1})}\left|f_i\left(x^1,x^2\right)\right|d\mu\left(x^1\right)\right) \\
&\qquad\cdot\sum_{j=1}^{2^{k_i}-1}\frac{\left|K_j\left(x^2+y^2\right)\right|}{j+1}d\mu\left(x^2\right)\Bigg]d\mu\left(y^2\right)
\end{aligned}
$$

$$+\frac{c}{\lambda}\sum_{i=1}^{\infty}\frac{1}{k_i}\int\limits_{\overline{I_{k_i}(u^{i,2})}}\left[\int\limits_{I_{k_i}(u^{i,2})}\left(\int\limits_{I_{k_i}(u^{i,1})}\left|f_i\left(x^1,x^2\right)\right|d\mu\left(x^1\right)\right)\right.$$
$$\left.\cdot\sum_{j=2^{k_i}}^{\infty}\frac{\left|K_j\left(x^2+y^2\right)\right|}{j+1}d\mu\left(x^2\right)\right]d\mu\left(y^2\right)$$
$$=S+M.$$

Since [10]

$$\int\limits_I\left|K_j(x)\right|d\mu(x)\le c<\infty,$$

we have

$$\begin{aligned}S &\le \frac{c}{\lambda}\sum_{i=1}^{\infty}\frac{1}{k_i}\int\limits_{I_{k_i}(u^{i,2})}\left[\int\limits_{I_{k_i}(u^{i,1})}\left|f_i\left(x^1,x^2\right)\right|d\mu\left(x^1\right)\right.\\ &\quad\left.\int\limits_{\overline{I_{k_i}(u^{i,2})}}\sum_{j=1}^{2^{k_i}-1}\frac{\left|K_j\left(x^2+y^2\right)\right|}{j+1}d\mu\left(y^2\right)\right]d\mu\left(x^2\right)\\ &\le \frac{c}{\lambda}\sum_{i=1}^{\infty}\|f_i\|_1\le\frac{c}{\lambda}\|f\|_1\end{aligned}\tag{15}$$

Using Lemma 2 for M we have

$$\begin{aligned}M &\le \frac{c}{\lambda}\sum_{i=1}^{\infty}\frac{1}{k_i}\int\limits_{I_{k_i}(u^{i,2})}\left[\int\limits_{I_{k_i}(u^{i,1})}\left|f_i\left(x^1,x^2\right)\right|d\mu\left(x^1\right)\right.\\ &\quad\left.\int\limits_{\overline{I_{k_i}(u^{i,2})}}\sum_{j=2^{k_i}}^{\infty}\frac{\left|K_j\left(x^2+y^2\right)\right|}{j+1}d\mu\left(y^2\right)\right]d\mu\left(x^2\right)\\ &\le \frac{c}{\lambda}\sum_{i=1}^{\infty}\frac{1}{k_i}\int\limits_{I_{k_i}(u^{i,2})}\left[\int\limits_{I_{k_i}(u^{i,1})}\left|f_i\left(x^1,x^2\right)\right|d\mu\left(x^1\right)\right.\\ &\quad\left.\sum_{r=k_i}^{\infty}\sum_{j=2^r}^{2^{r+1}-1}\frac{1}{j}\int\limits_{\overline{I_{k_i}(u^{i,2})}}\left|K_j\left(x^2+y^2\right)\right|d\mu\left(y^2\right)\right]d\mu\left(x^2\right)\\ &\le \frac{c}{\lambda}\sum_{i=1}^{\infty}\left(\sum_{r=k_i}^{\infty}\frac{r-k_j+1}{2^{r-k_i}}\right)\int\limits_{I_{k_i}(u^{i,2})}\int\limits_{I_{k_i}(u^{i,1})}\left|f_i\left(x^1,x^2\right)\right|d\mu\left(x^1,x^2\right)\end{aligned}\tag{16}$$

$$\leq \frac{c}{\lambda} \sum_{i=1}^{\infty} \|f_i\|_1 \leq \frac{c}{\lambda} \|f\|_1 .$$

Combining (14)-(16) we obtain

$$\mu \left\{ \left(y^1, y^2\right) \in \overline{\bigcup_{i=1}^{\infty} \left(I_{k_i}\left(u^{i,1}\right) \times I_{k_i}\left(u^{i,2}\right)\right)} : t_{\#}^{(2,1)} f\left(y^1, y^2\right) > c\lambda \right\} \leq \frac{c}{\lambda} \|f\|_1 . \tag{17}$$

From Lemma 1, we get

$$\begin{aligned} &\mu \left\{ \left(y^1, y^2\right) \in \bigcup_{i=1}^{\infty} \left(I_{k_i}\left(u^{i,1}\right) \times I_{k_i}\left(u^{i,2}\right)\right) : t_{\#}^{(2,1)} f\left(y^1, y^2\right) > c\lambda \right\} \\ \leq\ & \mu \left(\bigcup_{i=1}^{\infty} \left(I_{k_i}\left(u^{i,1}\right) \times I_{k_i}\left(u^{i,2}\right)\right) \right) \leq \frac{c}{\lambda} \|f\|_1 , \end{aligned} \tag{18}$$

and consequently from (17) and (18) we have

$$\mu \left\{ \left(y^1, y^2\right) \in I \times I : t_{\#}^{(2,1)} f\left(y^1, y^2\right) > c\lambda \right\} \leq \frac{c}{\lambda} \|f\|_1 . \tag{19}$$

Analogously, we can prove that

$$\mu \left\{ \left(y^1, y^2\right) \in I \times I : t_{\#}^{(2,2)} f\left(y^1, y^2\right) > c\lambda \right\} \leq \frac{c}{\lambda} \|f\|_1 . \tag{20}$$

Combining (10), (19) and (20) we obtain

$$\mu \left\{ \left(y^1, y^2\right) \in I \times I : t_{\#}^{(2)} f\left(y^1, y^2\right) > c\lambda \right\} \leq \frac{c}{\lambda} \|f\|_1 . \tag{21}$$

The estimation of $\mu \left\{ \left(y^1, y^2\right) \in I \times I : t_{\#}^{(3)} f\left(y^1, y^2\right) > c\lambda \right\}$ is analogous to the estimation of $\mu \left\{ \left(y^1, y^2\right) \in I \times I : t_{\#}^{(2)} f\left(y^1, y^2\right) > c\lambda \right\}$ and we have

$$\mu \left\{ \left(y^1, y^2\right) \in I \times I : t_{\#}^{(3)} f\left(y^1, y^2\right) > c\lambda \right\} \leq \frac{c}{\lambda} \|f\|_1 . \tag{22}$$

Combining (2), (3), (8), (21) and (22) we complete the proof of Theorem 1.

By making use of the well-known density argument due to Marcinkiewicz and Zygmund [9] we can show that Corollary 1 follows from Theorem 1.

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