

ALMOST EVERYWHERE CONVERGENCE OF (C, α) -MEANS OF QUADRATICAL PARTIAL SUMS OF DOUBLE VILENKIN-FOURIER SERIES

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ABSTRACT. In this paper we prove that the maximal operator of the (C, α) -means of quadratical partial sums of double Vilenkin-Fourier series is of weak type $(1,1)$. Moreover, the (C, α) -means $t_n^\alpha f$ of the function $f \in L^1$ converge a. e. to f as $n \rightarrow \infty$.

1. INTRODUCTION

In 1939 for the two-dimensional trigonometric Fourier partial sums $S_{j,j}f$ Marcinkiewicz [8] proved that for all $f \in L \log L([0, 2\pi]^2)$ the a.e. relation

$$t_n^1 f = \frac{1}{n} \sum_{j=0}^n S_{j,j} f \rightarrow f$$

holds as $n \rightarrow \infty$. Zhizhiashvili [13] improved this result and proved for $f \in L([0, 2\pi]^2)$ the (C, α) means

$$t_n^\alpha f = \frac{1}{A_n^\alpha} \sum_{j=0}^n A_{n-j}^{\alpha-1} S_{j,j} f$$

converges to f a.e. for any $\alpha > 0$. Dyachenko [3] proved this result for dimensions greater than 2. In papers [12, 6] written by Weisz and Goginava one can find that the $(C, 1)$ means $t_n^1 f$ of the double Walsh-Fourier series of a function $f \in L([0, 1]^2)$ converges to f a.e. Recently, Gát proved [4] this result with respect to two-dimensional Vilenkin systems. The d -dimensional Walsh-Fourier case is discussed in [7]. The aim of this paper is to generalize the result of Zhizhiashvili [13] concerning the (C, α) means with respect to two-dimensional (bounded) Vilenkin systems.

First, we give a brief introduction to the theory of Vilenkin systems. These orthonormal systems were introduced by N. Ja. Vilenkin in 1947 (see e.g. [11, 1]) as follows.

Let $m := (m_k, k \in \mathbb{N})$ ($\mathbb{N} := \{0, 1, \dots\}$, $\mathbb{P} := \mathbb{N} \setminus \{0\}$) be a sequence of integers each of them not less than 2. Let Z_{m_k} denote the discrete cyclic group of order m_k . That is, Z_{m_k} can be represented by the set $\{0, 1, \dots, m_k - 1\}$, with the group operation mod m_k addition. Since the group is discrete, then every subset is open. The normalized Haar measure on Z_{m_k} , μ_k is defined by $\mu_k(\{j\}) := 1/m_k$ ($j \in \{0, 1, \dots, m_k - 1\}$). Let

$$G_m := \prod_{k=0}^{\infty} Z_{m_k}.$$

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Then every $x \in G_m$ can be represented by a sequence $x = (x_i, i \in \mathbb{N})$, where $x_i \in Z_{m_i}$ ($i \in \mathbb{N}$). The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ), which is the normalized Haar measure, and the topology are the product measure and topology. Consequently, G_m is a compact Abelian group. If $\sup_{n \in \mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. In this paper we discuss bounded Vilenkin groups, only. The Vilenkin group is metrizable in the following way:

$$d(x, y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{M_{i+1}} \quad (x, y \in G_m).$$

The topology induced by this metric, the product topology, and the topology given by intervals defined below, are the same. A base for the neighborhoods of G_m can be given by the intervals:

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \text{ for } i < n\}$$

for $x \in G_m, n \in \mathbb{P}$. Let $0 = (0, i \in \mathbb{N}) \in G_m$ denote the nullelement of G_m and $I_n(0) := I_n, \bar{I}_n = G_m \setminus I_n$.

Furthermore, let $L^p(G_m)$ ($1 \leq p \leq \infty$) denote the usual Lebesgue spaces ($\|\cdot\|_p$ the corresponding norms) on G_m , \mathcal{G}_n the σ -algebra generated by the sets $I_n(x)$ ($x \in G_m$), and E_n the conditional expectation operator with respect to \mathcal{G}_n ($n \in \mathbb{N}$).

Let be $1 \leq p \leq +\infty$ be a real. We say that operator T is of type (p, p) if there exists an absolute constant $C > 0$ for which $\|Tf\|_p \leq C\|f\|_p$ for all $f \in L^p$. T is said to be of weak type $(1, 1)$ if there exist an absolute constant $C > 0$ such that $\|Tf\|_{weak-L^1} \leq C\|f\|_1$ for all $f \in L^1(G_m)$, where $\|f\|_{weak-L^1} = \sup_{\lambda > 0} \lambda \mu(|f| > \lambda)$. It is known that the operator which maps a function f to the maximal function $f^* := \sup |E_n f|$ is of weak type $(1, 1)$, and of type (p, p) for all $1 < p \leq \infty$ (see e.g. [2]).

Let $M_0 := 1, M_{n+1} := m_n M_n$ ($n \in \mathbb{N}$) be the so-called generalized powers. Then each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0, 1, \dots, m_i - 1\}, i \in \mathbb{N}),$$

where only a finite number of n_i 's differ from zero. For $1 \leq n \in \mathbb{N}$ we denote by $|n| := \max\{k \in \mathbb{N} : M_k \leq n\}$ the order of the natural number n . In other words, $M_{|n|} \leq n < M_{|n|+1}$. The generalized Rademacher functions are defined as

$$r_n(x) := \exp(2\pi i \frac{x_n}{m_n}) \quad (x \in G_m, n \in \mathbb{N}, i := \sqrt{-1}).$$

The n^{th} Vilenkin function is

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \quad (n \in \mathbb{N}).$$

The system $\psi := (\psi_n : n \in \mathbb{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m , and all the characters of G_m are of this form. Define the m -adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j \pmod{m_j}) M_j \quad (k, n \in \mathbb{N}).$$

Then, $\psi_{k \oplus n} = \psi_k \psi_n$, $\psi_n(x + y) = \psi_n(x) \psi_n(y)$, $\psi_n(-x) = \bar{\psi}_n(x)$, $|\psi_n| = 1$ ($k, n \in \mathbb{N}, x, y \in G_m$).

Set $A_n^\alpha := \frac{(1+\alpha)\dots(n+\alpha)}{n!}$ for any $n \in \mathbb{N}, \alpha \in \mathbb{R}$. It is known that $A_n^\alpha \sim n^\alpha$. Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the (C, α) means, kernels, and the Fejér means and kernels with respect to the Vilenkin system ψ as follows

$$\begin{aligned} \hat{f}(n) &:= \int_{G_m} f \bar{\psi}_n d\mu, \\ S_n f &:= \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \\ D_n &:= \sum_{k=0}^{n-1} \psi_k, \\ \sigma_n^\alpha f &= \frac{1}{A_n^\alpha} \sum_{k=0}^n A_{n-k}^{\alpha-1} S_k f, \\ K_n^\alpha &:= \frac{1}{A_n^\alpha} \sum_{k=0}^n A_{n-k}^{\alpha-1} D_k, \\ \sigma_n f &:= \sigma_n^1 f, \quad K_n := K_n^1 \quad (f \in L^1(G_m)). \end{aligned}$$

It is well-known that

$$S_n f(y) = \int_{G_m} f(x) D_n(y-x) d\mu(x), \quad (n \in \mathbb{N}, y \in G_m, f \in L^1(G_m)).$$

It is also well-known [1] that

$$(1) \quad \begin{aligned} D_{M_n}(x) &= \begin{cases} M_n & , \text{if } x \in I_n := I_n(0) \\ 0 & , \text{if } x \notin I_n \end{cases} \\ S_{M_n} f(x) &= M_n \int_{I_n(x)} f d\mu = E_n f(x) \quad (f \in L^1(G_m), n \in \mathbb{N}). \end{aligned}$$

Next, we introduce some notation with respect to the theory of two-dimensional Vilenkin systems. Let \tilde{m} be a sequence like m . The relation between the sequence (\tilde{m}_n) and (\tilde{M}_n) is the same as between sequence (m_n) and (M_n) . The group $G_m \times G_{\tilde{m}}$ is called a two-dimensional Vilenkin group. The normalized Haar measure is denoted by μ , just as in the one-dimensional case. It will not cause any misunderstanding. In this paper we also suppose that $m = \tilde{m}$. That is, $G_m \times G_{\tilde{m}} = G_m^2$.

The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, the Dirichlet kernels, the (C, α) means, the kernels of the (C, α) means with respect to the

two-dimensional Vilenkin system are defined as follows:

$$\begin{aligned}\hat{f}(n_1, n_2) &:= \int_{G_m^2} f(x, y) \bar{\psi}_{n_1}(y) \bar{\psi}_{n_2}(y) d\mu(x, y), \\ S_{n_1, n_2} f(u, v) &:= \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \hat{f}(k_1, k_2) \psi_{k_1}(u) \psi_{k_2}(v), \\ D_{n_1, n_2}(x, y) &:= D_{n_1}(x) D_{n_2}(y), \\ t_n^\alpha f &:= \frac{1}{A_n^\alpha} \sum_{j=0}^n A_{n-j}^{\alpha-1} S_{j, j} f, \\ T_n^\alpha &:= \frac{1}{A_n^\alpha} \sum_{j=0}^n A_{n-j}^{\alpha-1} D_{j, j}.\end{aligned}$$

It is also well-known that

$$t_n^\alpha f(u, v) = \int_{G_m^2} f(x, y) T_n^\alpha(u-x, v-y) d\mu(x, y).$$

For the two-dimensional variable $(x, y) \in G_m^2$ we use the notations

$$\begin{aligned}\psi_n^1(x, y) &= \psi_n(x), & D_n^1(x, y) &= D_n(x), & K_n^{\alpha, 1}(x, y) &= K_n^\alpha(x), \\ \psi_n^2(x, y) &= \psi_n(y), & D_n^2(x, y) &= D_n(y), & K_n^{\alpha, 2}(x, y) &= K_n^\alpha(y)\end{aligned}$$

for any $\alpha > 0$ and $n \in \mathbb{N}$.

Set the maximal operator $t_*^\alpha f := \sup_{n \in \mathbb{N}} |t_n^\alpha f|$ for any $f \in L^1(G_m^2)$ and $\alpha > 0$.

2. MAIN RESULTS

Theorem 1. *Let $f \in L^1(G_m^2)$ and $\alpha > 0$. Then*

$$\|t_*^\alpha f\|_{weak-L^1} \leq C \|f\|_1.$$

Corollary 1. *Let $f \in L^1(G_m^2)$ and $\alpha > 0$. Then*

$$t_n^\alpha(f) \rightarrow f \quad a.e. \quad as \quad n \rightarrow \infty.$$

3. AUXILIARY RESULTS

Lemma 1. *Let $0 \leq j < n_s M_s$ and $0 \leq n_s < m_s$. Then*

$$D_{n_s M_s - j} = D_{n_s M_s} - \psi_{n_s M_s - 1} \bar{D}_j.$$

Proof. It is clear that

$$D_{n_s M_s} = D_{n_s M_s - j} + \sum_{k=n_s M_s - j}^{n_s M_s - 1} \psi_k = D_{n_s M_s - j} + \sum_{k=0}^{j-1} \psi_{n_s M_s - k - 1}.$$

By

$$\begin{aligned}
\psi_{n_s M_s - k - 1}(x) &= \psi_{(n_s - 1)M_s + (m_{s-1} - 1)M_{s-1} + \dots + (m_0 - 1)M_0 - k}(x) \\
&= \psi_{(n_s - k_s - 1)M_s + (m_{s-1} - k_{s-1} - 1)M_{s-1} + \dots + (m_0 - k_0 - 1)M_0}(x) \\
&= \psi_{(n_s - 1)M_s + (m_{s-1} - 1)M_{s-1} + \dots + (m_0 - 1)M_0}(x) \bar{\psi}_k(x) \\
&= \psi_{n_s M_s - 1}(x) \bar{\psi}_k(x)
\end{aligned}$$

Lemma 1 is proved.

Lemma 2. *Let $\alpha \in (0, 1)$ and $n := n^{(A)} = n_A M_A + \dots + n_0 M_0$. Then in the one-dimensional case*

$$|K_n^\alpha| \leq \frac{c(\alpha)}{n^\alpha} \sum_{i=0}^A \left\{ \sum_{p=1}^i M_p^{\alpha-1} \sum_{j=M_{p-1}}^{M_p-1} |K_j| + M_i^\alpha |K_{M_i-1}| + M_i^\alpha D_{M_i} \right\}.$$

Proof. It is evident that

$$(2) \quad \sum_{j=0}^n A_{n-j}^{\alpha-1} D_j = \sum_{j=0}^{n_A M_A - 1} A_{n-j}^{\alpha-1} D_j + \sum_{j=n_A M_A}^n A_{n-j}^{\alpha-1} D_j = I + II.$$

Since [1] for $r \in \{0, \dots, m_A - 1\}$

$$D_{j+rM_A} = \left(\sum_{q=0}^{r-1} \psi_{M_A}^q \right) D_{M_A} + \psi_{M_A}^r D_j,$$

then for I we write

$$\begin{aligned}
I &= \sum_{r=0}^{n_A-1} \sum_{j=0}^{M_A-1} A_{n-j-rM_A}^{\alpha-1} \left(\sum_{q=0}^{r-1} \psi_{M_A}^q \right) D_{M_A} \\
&\quad + \sum_{r=0}^{n_A-1} \left(\sum_{j=0}^{M_A-1} A_{(n_A-r-1)M_A+n^{(A-1)+j}}^{\alpha-1} D_{M_A-j} \right) \psi_{M_A}^r = I_1 + I_2.
\end{aligned}$$

It is evident that

$$(3) \quad |I_1| \leq c(\alpha) M_A^\alpha D_{M_A}.$$

Using Lemma 1 for I_2 we obtain ($D_0 = 0$)

$$(4) \quad |I_2| \leq c(\alpha) \left\{ M_A^\alpha D_{M_A} + \sum_{r=0}^{n_A-1} \left| \sum_{j=1}^{M_A-1} A_{(n_A-r-1)M_A+n^{(A-1)+j}}^{\alpha-1} \bar{D}_j \right| \right\}.$$

Since

$$D_{j+n_A M_A} = D_{n_A M_A} + \psi_{n_A M_A} D_j$$

we write

$$(5) \quad |II| \leq c(\alpha) \left\{ M_A^\alpha D_{M_A} + n^{(A-1)} |K_{n^{(A-1)}}^\alpha| \right\}$$

Combining (2)-(5) we obtain

$$n |K_n^\alpha| \leq c(\alpha) \left\{ M_A^\alpha D_{M_A} + \sum_{r=0}^{n_A-1} \left| \sum_{j=1}^{M_A-1} A_{(n_A-r-1)M_A+n^{(A-1)+j}}^{\alpha-1} \bar{D}_j \right| + n^{(A-1)} |K_{n^{(A-1)}}^\alpha| \right\}.$$

Iterating this inequality we obtain that

$$n |K_n^\alpha| \leq c(\alpha) \left\{ \sum_{i=0}^A M_i^\alpha D_{M_i} + \sum_{i=0}^A \sum_{r=0}^{n_i-1} \left| \sum_{j=1}^{M_i-1} A_{(n_i-r-1)M_i+n^{(i-1)+j}}^{\alpha-1} \bar{D}_j \right| \right\}.$$

Applying Abel's transformation we write

$$\begin{aligned} \sum_{j=1}^{M_i-1} A_{(n_i-r-1)M_i+n^{(i-1)+j}}^{\alpha-1} \bar{D}_j &= - \sum_{j=1}^{M_i-2} A_{(n_i-r-1)M_i+n^{(i-1)+j}}^{\alpha-2} j \bar{K}_j \\ &\quad + A_{(n_i-r)M_i+n^{(i-1)-1}}^{\alpha-1} (M_i - 1) K_{M_i-1}, \end{aligned}$$

consequently,

$$n |K_n^\alpha| \leq c(\alpha) \left\{ \sum_{i=0}^A M_i^\alpha D_{M_i} + \sum_{i=0}^A \sum_{p=1}^i M_p^{\alpha-1} \sum_{j=M_{p-1}}^{M_p-1} |K_j| + \sum_{i=0}^A M_i^\alpha |K_{M_i-1}| \right\}.$$

Lemma 2 is proved.

Lemma 3. *Let $A \geq k$. Then*

$$\int_{\bar{I}_k} \sup_{n \geq A} |K_{M_n}| \leq c \frac{M_k}{M_A}.$$

Proof. Since [9]

$$|K_{M_n}(x)| \leq c M_s \sum_{x_s=1}^{m_s-1} \mathbf{1}_{I_n(0)+e_s x_s}(x), \quad x \in I_s \setminus I_{s+1}, \quad s = 0, \dots, n-1,$$

where $\mathbf{1}_E$ is characteristic function of set E and $e_s := (0, \dots, 0, 1, 0, \dots) \in G$, the s -th coordinate of which is 1 and the rest are zeros, we obtain that

$$\begin{aligned} \int_{\bar{I}_k} \sup_{n \geq A} |K_{M_n}| &\leq \sum_{n=A}^{\infty} \int_{\bar{I}_k} |K_{M_n}| = \sum_{n=A}^{\infty} \sum_{s=0}^{k-1} \int_{I_{s+1} \setminus I_s} |K_{M_n}| \\ &\leq c \sum_{n=A}^{\infty} \sum_{s=0}^{k-1} M_s \sum_{x_s=1}^{m_s-1} \int_{I_{s+1} \setminus I_s} \mathbf{1}_{I_n(0)+e_s x_s} \leq c \sum_{n=A}^{\infty} \frac{1}{M_n} \sum_{s=0}^{k-1} M_s \leq c \frac{M_k}{M_A}. \end{aligned}$$

Lemma 3 is proved.

Lemma 4. *Let $A \geq k$. Then*

$$\int_{\bar{I}_k} \sup_{n \geq M_A} |K_n| \leq c \frac{M_k (A - k + 1)}{M_A}.$$

Proof. Since

$$|K_n| \leq c \sum_{j=0}^A \frac{M_j}{M_A} |K_{M_j}|, \quad \text{for } M_A \leq n < M_{A+1},$$

By Lemma 3 and from the fact that [1]

$$(6) \quad \sup_{n \geq 1} \int_{G_m} |K_n| < \infty,$$

we obtain

$$\begin{aligned} \int_{\bar{I}_k} \sup_{n \geq M_A} |K_n| &\leq \sum_{v=A}^{\infty} \sum_{j=0}^v \frac{M_j}{M_v} \int_{\bar{I}_k} |K_{M_j}| \\ &= \sum_{v=A}^{\infty} \sum_{j=0}^k \frac{M_j}{M_v} \int_{\bar{I}_k} |K_{M_j}| + \sum_{v=A}^{\infty} \sum_{j=k+1}^v \frac{M_j}{M_v} \int_{\bar{I}_k} |K_{M_j}| \\ &\leq c \left\{ \frac{M_k}{M_A} + \sum_{v=A}^{\infty} \frac{M_k(v-k+1)}{M_v} \right\} \leq c \frac{M_k(A-k+1)}{M_A}. \end{aligned}$$

Lemma 4 is proved.

Lemma 5. *Let $\alpha \in (0, 1)$ and $A \geq k$. Then*

$$\int_{\bar{I}_k} \sup_{n \geq M_A} |K_n^\alpha| \leq c(\alpha) \frac{A-k+1}{(M_A/M_k)^\alpha}.$$

Proof. From Lemma 2 we get

$$\begin{aligned} \int_{\bar{I}_k} \sup_{n \geq M_A} |K_n^\alpha| &\leq c(\alpha) \int_{\bar{I}_k} \sup_{N \geq A} \frac{1}{M_N^\alpha} \sum_{i=0}^N \sum_{p=1}^i M_p^{\alpha-1} \sum_{j=M_{p-1}}^{M_p-1} |K_j| \\ &\quad + c(\alpha) \int_{\bar{I}_k} \sup_{N \geq A} \frac{1}{M_N^\alpha} \sum_{i=0}^N M_i^\alpha |K_{M_{i-1}}| \\ (7) \quad &\quad + c(\alpha) \int_{\bar{I}_k} \sup_{N \geq A} \frac{1}{M_N^\alpha} \sum_{i=0}^N M_i^\alpha D_{M_i} = I + II + III. \end{aligned}$$

from (1), (6) and Lemma 4 we obtain that

$$\begin{aligned} (8) \quad III &\leq c(\alpha) \sum_{N=A}^{\infty} \frac{1}{M_N^\alpha} \sum_{i=0}^k M_i^\alpha \int_{\bar{I}_k} D_{M_i} \leq c(\alpha) \frac{M_k^\alpha}{M_A^\alpha}, \\ II &\leq c(\alpha) \sum_{N=A}^{\infty} \frac{1}{M_N^\alpha} \sum_{i=0}^N M_i^\alpha \int_{\bar{I}_k} |K_{M_{i-1}}| \\ &= c(\alpha) \sum_{N=A}^{\infty} \frac{1}{M_N^\alpha} \sum_{i=0}^k M_i^\alpha \int_{\bar{I}_k} |K_{M_{i-1}}| \end{aligned}$$

$$\begin{aligned}
(9) \quad & +c(\alpha) \sum_{N=A}^{\infty} \frac{1}{M_N^\alpha} \sum_{i=k+1}^N M_i^\alpha \int_{\bar{I}_k} |K_{M_i-1}| \\
& \leq c(\alpha) \frac{M_k^\alpha}{M_A^\alpha} + c(\alpha) \sum_{N=A}^{\infty} \frac{1}{M_N^\alpha} \sum_{i=k+1}^N M_i^\alpha \frac{(i-k) M_k}{M_i} \leq c(\alpha) \frac{M_k^\alpha}{M_A^\alpha}, \\
& I \leq c(\alpha) \int_{\bar{I}_k} \sup_{N \geq A} \frac{1}{M_N^\alpha} \sum_{i=0}^k \sum_{p=1}^i M_p^{\alpha-1} \sum_{j=M_{p-1}}^{M_p-1} |K_j| \\
& + c(\alpha) \int_{\bar{I}_k} \sup_{N \geq A} \frac{1}{M_N^\alpha} \sum_{i=k+1}^N \sum_{p=1}^k M_p^{\alpha-1} \sum_{j=M_{p-1}}^{M_p-1} |K_j| \\
& + c(\alpha) \int_{\bar{I}_k} \sup_{N \geq A} \frac{1}{M_N^\alpha} \sum_{i=k+1}^N \sum_{p=k+1}^i M_p^{\alpha-1} \sum_{j=M_{p-1}}^{M_p-1} |K_j| \\
(10) \quad & = I_1 + I_2 + I_3.
\end{aligned}$$

$$(11) \quad I_1 \leq c(\alpha) \frac{M_k^\alpha}{M_A^\alpha},$$

$$(12) \quad I_2 \leq c(\alpha) \frac{M_k^\alpha}{M_A^\alpha} (A - k + 1),$$

$$\begin{aligned}
(13) \quad & I_3 \leq c(\alpha) \sum_{N=A}^{\infty} \frac{1}{M_N^\alpha} \sum_{i=k+1}^N \sum_{p=k+1}^i M_p^\alpha \int_{\bar{I}_k} \sup_{l \geq M_{p-1}} |K_l| \\
& \leq c(\alpha) \sum_{N=A}^{\infty} \frac{1}{M_N^\alpha} \sum_{i=k+1}^N \sum_{p=k+1}^i M_p^\alpha \frac{p-k}{(M_p/M_k)} \\
& \leq c(\alpha) \sum_{N=A}^{\infty} \frac{N-k+1}{(M_N/M_k)^\alpha} \leq c(\alpha) \frac{A-k+1}{(M_A/M_k)^\alpha}.
\end{aligned}$$

Combining (7)-(13) we complete the proof of Lemma 5.

Lemma 6. *Let $\alpha \in (0, 1)$ and $n = n_A M_A + \dots + n_0 M_0$. Then*

$$|T_n^\alpha| \leq c(\alpha) \sum_{i=1}^{10} B_i,$$

where

$$\begin{aligned}
B_1 &= \frac{1}{n^\alpha} \sum_{s=0}^{A+1} \sum_{r=0}^{s-1} M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} |T_j^1|, \\
B_2 &= \frac{1}{n^\alpha} \sum_{s=0}^A M_s^\alpha |T_{n_s M_s}^1|,
\end{aligned}$$

$$\begin{aligned}
B_3 &= \frac{1}{n^\alpha} \sum_{s=0}^{A+1} |D_{n_s M_s}^1| \sum_{r=0}^{s-1} M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} |K_j^{1,2}|, \\
B_4 &= \frac{1}{n^\alpha} \sum_{s=0}^A |D_{n_s M_s}^1| M_s^\alpha |K_{n_s M_s}^{1,2}|, \\
B_5 &= \frac{1}{n^\alpha} \sum_{s=0}^{A+1} |D_{n_s M_s}^2| \sum_{r=0}^{s-1} M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} |K_j^{1,1}|, \\
B_6 &= \frac{1}{n^\alpha} \sum_{s=0}^A |D_{n_s M_s}^2| M_s^\alpha |K_{n_s M_s}^{1,1}|, \\
B_7 &= \frac{1}{n^\alpha} \sum_{s=0}^A M_s^\alpha |D_{n_s M_s, n_s M_s}|, \\
B_8 &= \frac{1}{n^\alpha} \sum_{s=1}^A |D_{n_s M_s}^2| A_{n^{(s-1)}}^\alpha |K_{n^{(s-1)}}^{\alpha,1}|, \\
B_9 &= \frac{1}{n^\alpha} \sum_{s=1}^A |D_{n_s M_s}^1| A_{n^{(s-1)}}^\alpha |K_{n^{(s-1)}}^{\alpha,2}|, \\
B_{10} &\leq c(\alpha).
\end{aligned}$$

Proof. It is evident that

$$\begin{aligned}
A_n^\alpha T_n^\alpha &= \sum_{j=1}^{n_A M_A} A_{n-j}^{\alpha-1} D_{j,j} + \sum_{j=n_A M_A+1}^n A_{n-j}^{\alpha-1} D_{j,j} \\
&= \sum_{j=0}^{n_A M_A-1} A_{n^{(A-1)+j}}^{\alpha-1} D_{n_A M_A-j, n_A M_A-j} + \sum_{j=1}^{n^{(A-1)}} A_{n^{(A-1)-j}}^{\alpha-1} D_{j+n_A M_A, j+n_A M_A}.
\end{aligned}$$

Since (see Lemma 1)

$$\begin{aligned}
D_{n_A M_A-j, n_A M_A-j} &= D_{n_A M_A, n_A M_A} - D_{n_A M_A}^1 \psi_{n_A M_A-1}^2 \overline{D}_j^2 \\
&\quad - D_{n_A M_A}^2 \psi_{n_A M_A-1}^1 \overline{D}_j^1 + \psi_{n_A M_A-1}^1 \psi_{n_A M_A-1}^2 \overline{D}_{j,j}
\end{aligned}$$

and

$$\begin{aligned}
D_{j+n_A M_A, j+n_A M_A} &= D_{n_A M_A, n_A M_A} + D_{n_A M_A}^1 \psi_{n_A M_A}^2 D_j^2 \\
&\quad + D_{n_A M_A}^2 \psi_{n_A M_A}^1 D_j^1 + \psi_{n_A M_A}^1 \psi_{n_A M_A}^2 D_{j,j},
\end{aligned}$$

we get

$$\begin{aligned}
A_n^\alpha T_n^\alpha &= A_{n^{(A)-1}}^\alpha D_{n_A M_A, n_A M_A} \\
&\quad - \left(\sum_{j=0}^{n_A M_A-1} A_{n^{(A-1)+j}}^{\alpha-1} \overline{D}_j^2 \right) D_{n_A M_A}^1 \psi_{n_A M_A-1}^2 \\
&\quad - \left(\sum_{j=0}^{n_A M_A-1} A_{n^{(A-1)+j}}^{\alpha-1} \overline{D}_j^1 \right) D_{n_A M_A}^2 \psi_{n_A M_A-1}^1
\end{aligned}$$

$$\begin{aligned}
& + \left(\sum_{j=0}^{n_A M_A - 1} A_{n^{(A-1)+j}}^{\alpha-1} \overline{D}_{j,j} \right) \psi_{n_A M_A - 1}^1 \psi_{n_A M_A - 1}^2 \\
& + A_n^\alpha K_{n^{(A-1)}}^{\alpha,2} D_{n_A M_A}^1 \psi_{n_A M_A}^2 + A_n^\alpha K_{n^{(A-1)}}^{\alpha,1} D_{n_A M_A}^2 \psi_{n_A M_A}^1 \\
& + \psi_{n_A M_A}^1 \psi_{n_A M_A}^2 A_n^\alpha T_{n^{(A-1)}}^\alpha.
\end{aligned}$$

Consequently,

$$\begin{aligned}
A_n^\alpha |T_n^\alpha| & \leq A_{n^{(A)}-1} |D_{n_A M_A, n_A M_A}| + \left| \sum_{j=0}^{n_A M_A - 1} A_{n^{(A-1)+j}}^{\alpha-1} \overline{D}_j^2 \right| |D_{n_A M_A}^1| \\
& + \left| \sum_{j=0}^{n_A M_A - 1} A_{n^{(A-1)+j}}^{\alpha-1} \overline{D}_j^1 \right| |D_{n_A M_A}^2| + \left| \sum_{j=0}^{n_A M_A - 1} A_{n^{(A-1)+j}}^{\alpha-1} \overline{D}_{j,j} \right| \\
& + A_n^\alpha |K_{n^{(A-1)}}^{\alpha,2}| |D_{n_A M_A}^1| + A_n^\alpha |K_{n^{(A-1)}}^{\alpha,1}| |D_{n_A M_A}^2| + A_n^\alpha |T_{n^{(A-1)}}^\alpha|.
\end{aligned}$$

Iterating this inequality we obtain that

$$\begin{aligned}
A_n^\alpha |T_n^\alpha| & \leq \sum_{s=1}^A \left\{ A_{n^{(s)}-1} |D_{n_s M_s, n_s M_s}| + \left| \sum_{j=0}^{n_s M_s - 1} A_{n^{(s-1)+j}}^{\alpha-1} \overline{D}_j^2 \right| |D_{n_s M_s}^1| \right. \\
& + \left| \sum_{j=0}^{n_s M_s - 1} A_{n^{(s-1)+j}}^{\alpha-1} \overline{D}_j^1 \right| |D_{n_s M_s}^2| + \left| \sum_{j=0}^{n_s M_s - 1} A_{n^{(s-1)+j}}^{\alpha-1} \overline{D}_{j,j} \right| \\
(14) \quad & \left. + A_{n^{(s-1)}}^\alpha |K_{n^{(s-1)}}^{\alpha,2}| |D_{n_s M_s}^1| + A_{n^{(s-1)}}^\alpha |K_{n^{(s-1)}}^{\alpha,1}| |D_{n_s M_s}^2| \right\} + A_{n^{(0)}}^\alpha |T_{n^{(0)}}^\alpha|.
\end{aligned}$$

Applying Abel's transformation we write

$$\begin{aligned}
& \left| \sum_{j=0}^{n_s M_s - 1} A_{n^{(s-1)+j}}^{\alpha-1} \overline{D}_{j,j} \right| \\
& \leq c(\alpha) \left\{ \sum_{j=0}^{n_s M_s - 2} (n^{(s-1)} + j)^{\alpha-2} j |T_j^1| + (n^{(s)})^{\alpha-1} (n^{(s)} - 1) |T_{n_s M_s - 1}^1| \right\} \\
& \leq c(\alpha) \left\{ \sum_{r=0}^s \sum_{j=M_r}^{M_{r+1}-1} (n^{(s-1)} + j)^{\alpha-2} j |T_j^1| + (n^{(s)})^\alpha |T_{n_s M_s - 1}^1| \right\} \\
(15) \quad & \leq c(\alpha) \left\{ \sum_{r=0}^s M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} |T_j^1| + (n^{(s)})^\alpha |T_{n_s M_s - 1}^1| \right\}.
\end{aligned}$$

Analogously,

$$\begin{aligned}
& \left| \sum_{j=0}^{n_s M_s - 1} A_{n^{(s-1)+j}}^{\alpha-1} \overline{D}_j^1 \right| \\
(16) \quad & \leq c(\alpha) \left\{ \sum_{r=0}^s M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} |K_j^{1,l}| + (n^{(s)})^\alpha |T_{n_s M_s - 1}^1| \right\}, \quad l = 1, 2.
\end{aligned}$$

Combining (14)-(16) we complete the proof of Lemma 6.

Corollary 2. *Let $\alpha \in (0, 1)$. Then*

$$\sup_{n \geq 1} \int_{G_m^2} |T_n^\alpha| < \infty.$$

Proof. Since [1],[4]

$$(17) \quad \sup_{n \geq 1} \int_{G_m^2} |T_n^1| < \infty$$

and

$$\sup_{n \geq 1} \int_{G_m} |K_n^\alpha| < \infty, \quad \alpha > 0$$

from Lemma 6 we obtain the proof of Corollary 2.

Lemma 7. *Let $\alpha \in (0, 1)$. Then*

$$\int_{I_k \times I_k} \sup_{n \geq M_k} |T_n^\alpha| \leq c(\alpha) < \infty.$$

Proof. Since in [4] one can find

$$\int_{I_k \times I_k} \sup_{n \geq M_k} |T_n^1| \leq c < \infty,$$

then from (17) we can write

$$\begin{aligned} \int_{I_k \times I_k} \sup_{n \geq M_k} B_1 &\leq c(\alpha) \int_{I_k \times I_k} \sup_{A \geq k} \frac{1}{M_A^\alpha} \sum_{s=0}^{A+1} \sum_{r=0}^{s-1} M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} |T_j^1| \\ &\leq c(\alpha) \int_{I_k \times I_k} \sup_{A \geq k} \frac{1}{M_A^\alpha} \sum_{s=0}^k \sum_{r=0}^{s-1} M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} |T_j^1| \\ &+ c(\alpha) \int_{I_k \times I_k} \sup_{A \geq k} \frac{1}{M_A^\alpha} \sum_{s=k+1}^{A+1} \sum_{r=0}^{k-1} M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} |T_j^1| \\ &+ c(\alpha) \int_{I_k \times I_k} \sup_{A \geq k} \frac{1}{M_A^\alpha} \sum_{s=k+1}^{A+1} \sum_{r=k}^{s-1} M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} |T_j^1| \\ &\leq c(\alpha) \frac{1}{M_k^\alpha} \sum_{s=0}^k \sum_{r=0}^{s-1} M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} \int_{I_k \times I_k} |T_j^1| \\ &+ c(\alpha) \frac{1}{M_k^\alpha} \sum_{r=0}^{k-1} M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} \int_{I_k \times I_k} |T_j^1| \end{aligned}$$

$$(18) \quad +c(\alpha) \int_{I_k \times I_k} \sup_{A \geq k} \frac{1}{M_A^\alpha} \sum_{s=k+1}^{A+1} \sum_{r=k}^{s-1} M_r^\alpha \sup_{l \geq M_k} |T_l^1| \leq c(\alpha) < \infty.$$

Analogously, we obtain that

$$(19) \quad \int_{\bar{I}_k \times \bar{I}_k} \sup_{n \geq M_k} B_2 < \infty.$$

We have

$$(20) \quad \int_{\bar{I}_k \times \bar{I}_k} \sup_{n \geq M_k} B_3 = \int_{\bar{I}_k \times \bar{I}_k} \sup_{n \geq M_k} B_3 + \int_{\bar{I}_k \times I_k} \sup_{n \geq M_k} B_3 + \int_{I_k \times \bar{I}_k} \sup_{n \geq M_k} B_3.$$

It is evident that

$$(21) \quad \begin{aligned} & \int_{\bar{I}_k \times I_k} \sup_{n \geq M_k} B_3 \leq c(\alpha) \int_{\bar{I}_k \times I_k} \sup_{A \geq k} \frac{1}{M_A^\alpha} \sum_{s=0}^{A+1} \left(\sum_{n_s=0}^{m_s-1} |D_{n_s M_s}^1| \right) \sum_{r=0}^{s-1} M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} |K_j^{1,2}| \\ & \leq c(\alpha) \int_{\bar{I}_k \times I_k} \sup_{A \geq k} \frac{1}{M_A^\alpha} \sum_{s=0}^k \left(\sum_{n_s=0}^{m_s-1} |D_{n_s M_s}^1| \right) \sum_{r=0}^{s-1} M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} |K_j^{1,2}| \leq c(\alpha) < \infty. \end{aligned}$$

Analogously,

$$(22) \quad \int_{\bar{I}_k \times \bar{I}_k} \sup_{n \geq M_k} B_3 < \infty.$$

We write

$$(23) \quad \begin{aligned} & \int_{I_k \times \bar{I}_k} \sup_{n \geq M_k} B_3 \leq c(\alpha) \int_{I_k \times \bar{I}_k} \sup_{A \geq k} \frac{1}{M_A^\alpha} \sum_{s=0}^{A+1} \left(\sum_{n_s=0}^{m_s-1} |D_{n_s M_s}^1| \right) \sum_{r=0}^{s-1} M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} |K_j^{1,2}| \\ & \leq c(\alpha) \int_{I_k \times \bar{I}_k} \sup_{A \geq k} \frac{1}{M_A^\alpha} \sum_{s=0}^k \left(\sum_{n_s=0}^{m_s-1} |D_{n_s M_s}^1| \right) \sum_{r=0}^{s-1} M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} |K_j^{1,2}| \\ & + c(\alpha) \int_{I_k \times \bar{I}_k} \sup_{A \geq k} \frac{1}{M_A^\alpha} \sum_{s=k+1}^{A+1} \left(\sum_{n_s=0}^{m_s-1} |D_{n_s M_s}^1| \right) \sum_{r=0}^{k-1} M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} |K_j^{1,2}| \\ & + c(\alpha) \int_{I_k \times \bar{I}_k} \sup_{A \geq k} \frac{1}{M_A^\alpha} \sum_{s=k+1}^{A+1} \left(\sum_{n_s=0}^{m_s-1} |D_{n_s M_s}^1| \right) \sum_{r=k}^{s-1} M_r^{\alpha-1} \sum_{j=M_r}^{M_{r+1}-1} |K_j^{1,2}| \\ & =: M + N + R. \end{aligned}$$

It is evident that

$$(24) \quad M \leq c(\alpha) < \infty,$$

$$(25) \quad N \leq c(\alpha) \sum_{A=k}^{\infty} \frac{M_k^\alpha}{M_A^\alpha} \sum_{s=k+1}^{A+1} \sum_{n_s=0}^{m_s-1} \int_{I_k} |D_{n_s M_s}^1| \leq c(\alpha) \sum_{A=k}^{\infty} \frac{A-k+1}{(M_A/M_k)^\alpha} \leq c(\alpha) < \infty,$$

Using Lemma 4 and (1) for R we get

$$(26) \quad \begin{aligned} R &\leq c(\alpha) \sum_{A=k}^{\infty} \frac{1}{M_A^\alpha} \sum_{s=k+1}^{A+1} \sum_{n_s=0}^{m_s-1} \sum_{r=k}^{s-1} M_r^\alpha \int_{I_k \times \bar{I}_k} |D_{n_s M_s}^1| \sup_{l \geq M_r} |K_l^{1,2}| \\ &\leq c(\alpha) \sum_{A=k}^{\infty} \frac{1}{M_A^\alpha} \sum_{s=k+1}^{A+1} \sum_{r=k}^{s-1} M_r^\alpha \frac{r-k+1}{(M_r/M_k)} \\ &\leq c(\alpha) \sum_{A=k}^{\infty} \frac{M_k^\alpha}{M_A^\alpha} \sum_{s=k+1}^{A+1} \sum_{r=k}^{s-1} \frac{r-k+1}{(M_r/M_k)^{1-\alpha}} \\ &\leq c(\alpha) \sum_{A=k}^{\infty} \frac{M_k^\alpha}{M_A^\alpha} (A-k+1) \leq c(\alpha) < \infty. \end{aligned}$$

Combining (23)-(26) we obtain that

$$(27) \quad \int_{I_k \times \bar{I}_k} \sup_{n \geq M_k} B_3 < \infty.$$

After substituting (21), (22) and (27) in (20) we have

$$(28) \quad \int_{\bar{I}_k \times \bar{I}_k} \sup_{n \geq M_k} B_3 < \infty.$$

Analogously, we obtain that

$$(29) \quad \int_{\bar{I}_k \times I_k} \sup_{n \geq M_k} B_j < \infty, \quad j = 4, 5, 6, 7.$$

For B_9 we write

$$(30) \quad \int_{\bar{I}_k \times I_k} \sup_{n \geq M_k} B_9 = \int_{\bar{I}_k \times \bar{J}_k} \sup_{n \geq M_k} B_9 + \int_{\bar{I}_k \times I_k} \sup_{n \geq M_k} B_9 + \int_{I_k \times \bar{I}_k} \sup_{n \geq M_k} B_9.$$

Using Lemma 2 we have

$$\begin{aligned} \int_{\bar{I}_k \times I_k} \sup_{n \geq M_k} B_9 &\leq c(\alpha) \int_{\bar{I}_k \times I_k} \sup_{A \geq k} \frac{1}{M_A^\alpha} \sum_{s=1}^A \left(\sum_{n_s=0}^{m_s-1} |D_{n_s M_s}^1| \right) \sup_{1 \leq n \leq M_s} A_n^\alpha |K_n^{\alpha,2}| \\ &\leq c(\alpha) \frac{1}{M_k^\alpha} \sum_{s=1}^k \int_{G_m} \sup_{1 \leq n \leq M_s} A_n^\alpha |K_n^{\alpha,2}| \\ &\leq c(\alpha) \frac{1}{M_k^\alpha} \sum_{s=1}^k \sum_{i=0}^s \left\{ \sum_{p=1}^i M_p^{\alpha-1} \sum_{j=M_p-1}^{M_p-1} \int_{G_m} |K_j^{1,2}| \right\} \end{aligned}$$

$$(31) \quad \left. + M_i^\alpha \int_{G_m} |K_{M_i-1}^{1,2}| + M_i^\alpha \int_{G_m} |D_{M_i}| \right\} \leq c(\alpha) < \infty.$$

Analogously, we obtain

$$(32) \quad \int_{\bar{I}_k \times \bar{I}_k} \sup_{n \geq M_k} B_9 < \infty,$$

From Lemmas 2, 5 and by (1) we have

$$(33) \quad \begin{aligned} & \int_{I_k \times \bar{I}_k} \sup_{n \geq M_k} B_9 \leq c(\alpha) \int_{I_k \times \bar{I}_k} \sup_{A \geq k} \frac{1}{M_A^\alpha} \left(\sum_{s=k}^A \sum_{n_s=0}^{m_s-1} |D_{n_s M_s}^1| \right) \sum_{a=0}^A M_a^\alpha \sup_{|q|=a} |K_q^{\alpha,2}| \\ & \quad + c(\alpha) \int_{I_k \times \bar{I}_k} \sup_{A \geq k} \frac{1}{M_A^\alpha} \sum_{s=1}^{k-1} |D_{n_s M_s}^1| A_{n^{(s-1)}}^\alpha |K_{n^{(s-1)}}^{\alpha,2}| \\ & \leq c(\alpha) \sum_{A=k}^\infty \frac{1}{M_A^\alpha} \sum_{a=0}^{k-1} \int_{I_k \times \bar{I}_k} \sum_{s=k}^A \left(\sum_{n_s=0}^{m_s-1} |D_{n_s M_s}^1| \right) M_a^\alpha \sup_{|q|=a} |K_q^{\alpha,2}| \\ & \quad + c(\alpha) \sum_{A=k}^\infty \frac{1}{M_A^\alpha} \sum_{a=k}^A \int_{I_k \times \bar{I}_k} \sum_{s=k}^A \left(\sum_{n_s=0}^{m_s-1} |D_{n_s M_s}^1| \right) M_a^\alpha \sup_{|q|=a} |K_q^{\alpha,2}| \\ & \quad + c(\alpha) \int_{I_k \times \bar{I}_k} \frac{1}{M_k^\alpha} \sum_{s=1}^{k-1} \left(\sum_{n_s=0}^{m_s-1} |D_{n_s M_s}^1| \right) \sup_{1 \leq l \leq M_s} A_l^\alpha |K_l^{\alpha,2}| \\ & \leq c(\alpha) \left\{ \sum_{A=k}^\infty \frac{A-k+1}{M_A^\alpha} \sum_{a=0}^{k-1} M_a^\alpha + \sum_{A=k}^\infty \frac{A-k+1}{M_A^\alpha} \sum_{a=k}^A M_a^\alpha \frac{a-k+1}{(M_a/M_k)^\alpha} \right. \\ & \quad \left. + \frac{1}{M_k^\alpha} \sum_{s=1}^{k-1} \sum_{i=0}^s \left\{ \sum_{p=1}^i M_p^\alpha + M_i^\alpha \right\} \right\} \leq c(\alpha) < \infty. \end{aligned}$$

By virtue of (30)-(33) we have

$$(34) \quad \int_{I_k \times I_k} \sup_{n \geq M_k} B_9 \leq c(\alpha) < \infty.$$

Analogously, we get that

$$(35) \quad \int_{I_k \times I_k} \sup_{n \geq M_k} B_8 \leq c(\alpha) < \infty.$$

Combining (18), (19), (28), (29), (34) and (35), from Lemma 6 we complete the proof of Lemma 7.

Proof of Theorem 1. For $\alpha \geq 1$ Theorem 1 is proved in [4]. Hence it can be assumed that $0 < \alpha < 1$. As a consequence of Corollary 2 we have that the maximal operator

$t_*^\alpha f := \sup_{n \in \mathbb{N}} |t_n^\alpha f|$ is of type (∞, ∞) . Since this sublinear operator is quasi-local (this is what Lemma 4 means), then by standard argument (see e.g. the book [10]) it follows that it is of weak type $(1, 1)$. That is the proof of Theorem 1 is complete.

Proof of Corollary 1. The set of Vilenkin polynomials is dense in $L^1(G_m^2)$, so by the well-known density argument we have that $t_n^\alpha f \rightarrow f$ a.e. for all integrable two-variable function f . We remark that the Marcinkiewicz interpolation theorem (see e.g. [10]) also gives that the maximal operator t_*^α is of type (p, p) for all $1 < p \leq \infty$. The proof of Corollary 1 is complete.

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