UNIFORM AND L-CONVERGENCE OF LOGARITHMIC MEANS OF CUBICAL PARTIAL SUMS OF DOUBLE WALSH-FOURIER SERIES

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Abstract. The two-dimensional (Nörlund) logarithmic means of cubical partial sums of the double Fourier series of the integrable function f is:

$$\frac{1}{l_n}\sum_{i=1}^{n-1}\frac{S_{i,i}(f)}{n-i}, \quad \text{where } l_n := \sum_{k=1}^{n-1}\frac{1}{k}.$$

In this paper we discuss some convergence and divergence properties of this logarithmic means of the two-dimensional Walsh-Fourier series of functions in the uniform, and in the L Lebesgue norm. We give necessary and sufficient conditions for the convergence regarding the modulus of continuity of the function, and also the function space.

1. Introduction

In the literature, it is known the notion of the Riesz's logarithmic means of a Fourier series. The n-th mean of the Fourier series of the integrable function f is defined by

$$\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{k}.$$

This Riesz's logarithmic means with respect to the trigonometric system has been studied by a lot of authors. We mention for instance the papers of Szász, and Yabuta [11, 13]. This mean with respect to the Walsh, Vilenkin system is discussed by Simon, and Gát [10, 1].

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The Nörlund means for the Fourier series of f are defined by

$$\frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k}S_k(f),$$

where $Q_n := \sum_{k=1}^{n-1} q_k$. If $q_k = \frac{1}{k}$, then we get the (Nörlund) logarithmic means:

$$\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{n-k}.$$

In this paper we call it - it will not cause any misunderstood - logarithmic means. Although, it is a kind of "reverse" Riesz's logarithmic means. Móricz [7] investigates the approximation properties of some special Nörlund means of Walsh-Fourier series of L^p functions in norm. The case, when $q_k = \frac{1}{k}$ is excluded, since the methods of Móricz are not applicable for logarithmic means. In [6] we proved some convergence and divergence properties of the logarithmic means of functions in the class of continuous functions, and in the Lebesgue

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space L. Among others, we proved that the maximal norm convergence function space of this logarithmic meams is $L\log^+ L$. With respect to the two-dimensional logarithmic means

$$\frac{1}{l_n}\sum_{i=1}^{n-1}\frac{S_{i,i}(f)}{(n-i)}.$$

there is no known corresponding result yet. On the other hand, with respect to approximation properties of some other means of double Walsh-Fourier series see for instance the paper [4].

In this paper we discuss some convergence and divergence properties of logarithmic means of cubical partial sums of the two-dimensional Walsh-Fourier series of functions in the uniform, and in the L Lebesgue norm. We give necessary and sufficient conditions for the convergence regarding the modulus of continuity of the function, and also the function space.

Let $r_0(x)$ be a function defined by

$$r_0(x)=\begin{cases} 1, \text{ if } x\in[0,1/2) \\ -1, \text{ if } x\in[1/2,1) \end{cases}, \qquad r_0(x+1)=r_0(x).$$

The Rademacher system is defined by

$$r_n(x)=r_0(2^n x), \quad n\geq 1 \quad \text{and} \quad x\in[0,1).$$

Let $w_0, w_1, \ldots$ represent the Walsh functions, i.e. $w_0(x)=1$ and if $k=2^{n_1}+\cdots+2^{n_s}$ is a positive integer with $n_1>n_2>\cdots>n_s\geq 0$, then

$$w_k(x)=r_{n_1}(x)\cdots r_{n_s}(x).$$

The idea of using products of Rademacher's functions to define the Walsh system originated from Paley [8].

The Walsh-Dirichlet kernel is defined by

$$D_n(x)=\sum_{k=0}^{n-1}w_k(x).$$

Recall that

$$D_{2^n}(x)=\begin{cases} 2^n, \text{if } x\in[0,1/2^n), \\ 0, \text{ if } x\in[1/2^n,1). \end{cases} \tag{1}$$

We consider the double system $\{w_n(x)\times w_m(y): n,m=0,1,2,\ldots\}$ on the unit square $I^2=[0,1)\times[0,1)$.

As usual, denote by $L(I^2)$ the set of all measurable functions defined on I^2, for which

$$\|f\|_1=\int_0^1\int_0^1|f(x,y)|\,dxdy<\infty$$

and, by $C(I^2)$ the space of continuous functions on I^2, with the supremum norm

$$\|f\|_C=\sup_{x,y\in I}|f(x,y)| \qquad \left(f\in C\left(I^2\right)\right).$$

The positive logarithm $\log^+$ is defined as

$$\log^+(x) := \begin{cases} \log(x), & \text{if } x > 1, \\ 0, & \text{otherwise }. \end{cases}$$

Let a be a positive real. We say that the function $f \in L\left(I^2\right)$ belongs to the logarithm space $L(\log^+ L)^a(I^2)$ if the integral

$$\|f\|_{L(\log^+ L)^a} := \int\limits_0^1 \int\limits_0^1 |f(x,y)| \left(\log^+ |f(x,y)|\right)^a dxdy$$

is finite. Let $X = X(I^2)$ denote either the space $L(I^2)$, or the space of continuous functions, that is, $C(I^2)$. The corresponding norm is denoted by $\|.\|_X$. The total modulus of continuity, when $X = C$, and the total integrated modulus of continuity, where $X = L$ are defined by

$$\omega\left(\delta, f\right)_X = \sup\left\{\|f(x+u, y+v) - f(x,y)\|_X : u^2 + v^2 \leq \delta^2\right\}.$$

The partial modulus of continuity, when $X = C$, and the partial integrated modulus of continuity - in this case $X = L$ - are defined by

$$\omega_1\left(\delta, f\right)_X = \sup\left\{\|f(x+u, y) - f(x,y)\|_X : |u| \leq \delta\right\},$$
$$\omega_2\left(\delta, f\right)_X = \sup\left\{\|f(x, y+v) - f(x,y)\|_X : |v| \leq \delta\right\}.$$

We also use the notion of the mixed modulus of continuity, where $X = C$, and the mixed integrated modulus of continuity (in this case $X = L$). They are defined as follows

$$\begin{aligned}\omega_{1,2}\left(\delta_1, \delta_2, f\right)_X = \sup\{&\|f(x \oplus u, y \oplus v) - f(x \oplus u, y) \\ &-f(x, y \oplus v) + f(x,y)\|_X : |u| \leq \delta_1, |v| \leq \delta_2\}, \quad f \in X\left(I^2\right).\end{aligned}$$

The rectangular partial sums of double Fourier series with respect to the Walsh system are defined by

$$S_{M,N}\left(f, x, y\right) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}\left(m, n\right) w_m(x) w_n(y).$$

The logarithmic means of cubical partial sums of double Walsh-Fourier series is defined as follows

$$t_n\left(f, x, y\right) = \frac{1}{l_n} \sum_{i=1}^{n-1} \frac{S_{i,i}\left(f, x, y\right)}{n-i},$$

where

$$l_n = \sum_{k=1}^{n-1} \frac{1}{k}.$$

It is evident that

$$t_n\left(f, x, y\right) - f\left(x, y\right) = \int\limits_0^1 [f(x \oplus t, y \oplus s) - f(x,y)] F_n\left(t, s\right) dtds,$$

where

$$F_n\left(t, s\right) = \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{D_k\left(t\right) D_k\left(s\right)}{n-k}$$

and $\oplus$ denotes the dyadic addition [9, 5].

Denote by

$$F_n^{(1)}(t)=\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{D_k(t)}{n-k}$$

the nth one dimensional logarithmic kernel. The following well-known inequality is proved by Getsadze [2].

Theorem A. Let either $X=C$, or $X=L$. Then the inequality

$$\left\|S_{n,m}(f)-f\right\|_X\leq c\left\{\omega_1\left(\frac{1}{n},f\right)_X\log(n+1)+\omega_2\left(\frac{1}{m},f\right)_X\log(m+1)\right.$$
$$\left.+\omega_{1,2}\left(\frac{1}{n},\frac{1}{m},f\right)_X\log(n+1)\log(m+1)\right\}$$

holds.

Let either $X=C$, or $X=L$. It is evident that the condition

$$\omega(\delta,f)_X=o\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right),$$

provides the convergence of $\left\|S_{n,n}(f)-f\right\|_X$.

Since

$$\left\|t_n(f)-f\right\|_X\leq\frac{1}{l_n}\sum_{i=1}^{n-1}\frac{\left\|S_{i,i}(f)-f\right\|_X}{(n-i)},$$

then from Theorem A we conclude that the following is true.

Theorem 1. *Let either $X=C$, or $X=L$. Let $f\in X(I^2)$, and*

$$\omega(\delta,f)_X=o\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right).$$

Then we have

$$\left\|t_n(f)-f\right\|_X\to 0\quad as\ \ n\to\infty.$$

In this paper we investigate sharpness of this results. In particular, we prove

Theorem 2. *There exists function $f\in C(I^2)$ such that*

$$\omega(\delta,f)_C=O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right)$$

and $t_n(f,0,0)$ diverges.

Theorem 3. *There exists function $g\in L(I^2)$ for which*

$$\omega(\delta,g)_L=O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right)$$

and $t_n(g)$ does not converge to g in L-norm.

What can be said, if we do not provide any condition of the modulus of continuity of the function? In [12] Tkebuchava proved

Theorem B. If $f \in L\left(\log^+ L\right)^2$, then

$$\left\|S_{m,n}(f) - f\right\|_L \to 0 \quad (m,n \to \infty).$$

From this follows that

$$\left\|t_n(f) - f\right\|_L \leq \frac{1}{l_n} \sum_{i=1}^{n-1} \frac{\left\|S_{i,i}(f) - f\right\|_L}{(n-i)} \to 0 \quad (n \to \infty)$$

for each $f \in L\left(\log^+ L\right)^2$.

So, the norm convergence holds for every function in the space $L\left(\log^+ L\right)^2$. Can this result be improved? The answer is no. Not only the result of Tkebuchava can not be improved in the point of view of the function space, but we prove even more. Namely, we prove that the maximal convergence space with respect to the $L(I^2)$-norm convergence is $L(\log^+ L)^2$. That is, let $\delta : [0,+\infty) \to [0,+\infty)$ be measurable, and $\delta(+\infty) = 0$. Then the following is true.

Theorem 4. *There exists a two-variable function* $h \in L\left(\log^+ L\right)^2 \delta(L)(I^2)$ *such that* $t_n(h)$ *does not converge to* h *in* $L(I^2)$*-norm.*

2. Auxiliary results

Lemma 1. [6] *The inequality*

$$\left\|F_{p_A}^{(1)}\right\|_L \geq c \log p_A$$

holds, where $p_A = 2^{2A} + 2^{2A-2} + \cdots + 2^2 + 2^0$.

Lemma 2. *The estimation*

$$\left\|F_{p_A}\right\|_L \geq c\left(\log p_A\right)^2$$

holds.

Proof. We write

$$G_{p_A}(x,y) = \sum_{k=1}^{p_A-1} \frac{D_{p_A-k}(x) D_{p_A-k}(y)}{k}$$

$$= \sum_{k=1}^{p_{A-1}-1} \frac{D_{p_A-k}(x) D_{p_A-k}(y)}{k} + \sum_{k=p_{A-1}}^{p_A-1} \frac{D_{p_A-k}(x) D_{p_A-k}(y)}{k}$$

$$= I + II. \tag{2}$$

Since [6]

$$D_{2^{2A}-k}(x) = D_{2^{2A}}(x) - w_{2^{2A}-1}(x) D_k(x)$$

we have

$$II = \sum_{k=0}^{2^{2A}-1} \frac{D_{2^{2A}-k}(x) D_{2^{2A}-k}(y)}{k + p_{A-1}}$$

$$= D_{2^{2A}}(x) D_{2^{2A}}(y) \sum_{k=0}^{2^{2A}-1} \frac{1}{k + p_{A-1}}$$

$$-D_{2^{2A}}(x)\, w_{2^{2A}-1}(y) \sum_{k=1}^{2^{2A}-1} \frac{D_k(y)}{k+p_{A-1}}$$

$$-D_{2^{2A}}(y)\, w_{2^{2A}-1}(x) \sum_{k=1}^{2^{2A}-1} \frac{D_k(x)}{k+p_{A-1}}$$

$$+w_{2^{2A}-1}(x)\, w_{2^{2A}-1}(y) \sum_{k=1}^{2^{2A}-1} \frac{D_k(x)\, D_k(y)}{k+p_{A-1}}$$

$$= II_1 + II_2 + II_3 + II_4. \tag{3}$$

By (1) we get

$$\|II_1\|_1 = O(1). \tag{4}$$

Denote by K_k the kth Fejér kernel function, that is, $K_k = \frac{1}{k}\sum_{j=1}^{k} D_j$. Using the Abel transform we have

$$\|II_2 + II_3\|_1 \le 2 \int_0^1 \left| \sum_{k=0}^{2^{2A}-1} \frac{D_k(y)}{k+p_{A-1}} \right| dy$$

$$\le 2 \sum_{k=1}^{2^{2A}-2} \left(\frac{1}{k+p_{A-1}} - \frac{1}{k+p_{A-1}+1} \right) k \int_0^1 |K_k(y)|\, dy$$

$$+2 \frac{2^{2A}-1}{p_{A-1}-1} \int_0^1 |K_{2^{2A}-1}(y)|\, dy = O(1). \tag{5}$$

Since [3]

$$\frac{1}{n} \int_0^1 \left| \sum_{j=1}^{n} D_j(x)\, D_j(y) \right| dx dy \le c < \infty$$

the estimation of II_4 is analogous to estimation of II_2 and we have

$$\|II_4\|_1 = O(1). \tag{6}$$

Next, we investigate I. Since

$$D_{p_A - k}(x) = D_{2^{2A}}(x) + r_{2A}(x)\, D_{p_{A-1}-k}(x),$$

then we have

$$I = D_{2^{2A}}(x)\, D_{2^{2A}}(y)\, l_{p_{A-1}} + D_{2^{2A}}(x)\, r_{2A}(y)\, G^{(1)}_{p_{A-1}}(y)$$

$$+D_{2^{2A}}(y)\, r_{2A}(x)\, G^{(1)}_{p_{A-1}}(x) + r_{2A}(x)\, r_{2A}(y)\, G_{p_{A-1}}(x,y)$$

$$= I_1 + I_2 + I_3 + I_4, \tag{7}$$

where

$$G^{(1)}_n(u) = \sum_{k=1}^{n-1} \frac{D_{n-k}(u)}{k}.$$

It is evident that

$$\|I\|_1$$

$$= \left(\int_0^{2^{-2A}} \int_0^{2^{-2A}} + \int_0^{2^{-2A}} \int_{2^{-2A}}^{1} + \int_{2^{-2A}}^{1} \int_0^{2^{-2A}} + \int_{2^{-2A}}^{1} \int_{2^{-2A}}^{1} \right) \left(I_1 + I_2 + I_3 + I_4 \right) dxdy$$

$$= B + C + D + E. \tag{8}$$

Since

$$G_n\left(0,0\right) = \sum_{k=1}^{n-1} \frac{\left(n-k\right)^2}{k} = O\left(n^2 \log n\right)$$

and

$$G_n^{(1)}\left(0\right) = \sum_{k=1}^{n-1} \frac{n-k}{k} = O\left(n \log n\right),$$

then for B we obtain

$$\begin{aligned} B = & \int_0^{2^{-2A}} \int_0^{2^{-2A}} \Big| D_{2^{2A}}\left(x\right) D_{2^{2A}}\left(y\right) l_{p_{A-1}} + D_{2^{2A}}\left(x\right) r_{2A}\left(y\right) G_{p_{A-1}}^{(1)}\left(y\right) \\ & + D_{2^{2A}}\left(y\right) r_{2A}\left(x\right) G_{p_{A-1}}^{(1)}\left(x\right) + r_{2A}\left(x\right) r_{2A}\left(y\right) G_{p_{A-1}}\left(x,y\right) \Big| dxdy \\ & \leq l_{p_{A-1}} + \frac{1}{2^{2A}} \left| G_{p_{A-1}}^{(1)}\left(0\right) \right| + \frac{1}{2^{2A}} \left| G_{p_{A-1}}^{(1)}\left(0\right) \right| + \frac{1}{2^{4A}} \left| G_{p_{A-1}}\left(0,0\right) \right| \\ & = O\left(A\right). \end{aligned} \tag{9}$$

It is evident that

$$\begin{aligned} C = & \int_0^{2^{-2A}} \int_{2^{-2A}}^{1} \left| D_{2^{2A}}\left(x\right) r_{2A}\left(y\right) G_{p_{A-1}}^{(1)}\left(y\right) + r_{2A}\left(x\right) r_{2A}\left(y\right) G_{p_{A-1}}\left(x,y\right) \right| dxdy \\ & \geq \int_{2^{-2A}}^{1} \left| G_{p_{A-1}}^{(1)}\left(y\right) \right| dy - \int_0^{2^{-2A}} \int_{2^{-2A}}^{1} \left| G_{p_{A-1}}\left(x,y\right) \right| dxdy \\ & = \left\| G_{p_{A-1}}^{(2)} \right\|_1 - \int_0^{2^{-2A}} \left| G_{p_{A-1}}^{(1)}\left(y\right) \right| dy - \int_0^{2^{-2A}} \int_{2^{-2A}}^{1} \left| G_{p_{A-1}}\left(x,y\right) \right| dxdy \\ & = \left\| G_{p_{A-1}}^{(1)} \right\|_1 - \frac{1}{2^{2A}} \left| G_{p_{A-1}}^{(1)}\left(0\right) \right| \\ & - \frac{1}{2^{2A}} \int_{2^{-2A}}^{1} \left| \sum_{k=1}^{p_{A-1}-1} \frac{p_{A-1}-k}{k} D_{p_{A-1}-k}\left(y\right) \right| dy \\ & \geq \left\| G_{p_{A-1}}^{(1)} \right\|_1 - \frac{1}{2^{2A}} \left| G_{p_{A-1}}^{(1)}\left(0\right) \right| - \frac{p_{A-1}}{2^{2A}} \left\| G_{p_{A-1}}^{(1)} \right\|_1 \\ & - \frac{p_{A-1}}{2^{2A}} \left\| K_{p_{A-1}-1} \right\|_1, \end{aligned} \tag{10}$$

analogously we have

$$D \geq \left\| G_{p_{A-1}}^{(1)} \right\|_1 - \frac{1}{2^{2A}} \left| G_{p_{A-1}}^{(1)}\left(0\right) \right| - \frac{p_{A-1}}{2^{2A}} \left\| G_{p_{A-1}}^{(1)} \right\|_1$$

$$-\frac{p_{A-1}}{2^{2A}}\left\|K_{p_{A-1}-1}\right\|_1. \tag{11}$$

For E we write

$$E=\int\limits_{2^{-2A}}^{1}\int\limits_{2^{-2A}}^{1}\left|G_{p_{A-1}}(x,y)\right|dxdy$$

$$=\left\|G_{p_{A-1}}\right\|_1-\int\limits_{0}^{2^{-2A}}\int\limits_{2^{-2A}}^{1}\left|G_{p_{A-1}}(x,y)\right|dxdy$$

$$-\int\limits_{2^{-2A}}^{1}\int\limits_{0}^{2^{-2A}}\left|G_{p_{A-1}}(x,y)\right|dxdy-\int\limits_{0}^{2^{-2A}}\int\limits_{0}^{2^{-2A}}\left|G_{p_{A-1}}(x,y)\right|dxdy$$

$$\geq\left\|G_{p_{A-1}}\right\|_1-\frac{2p_{A-1}}{2^{2A}}\left\|G^{(1)}_{p_{A-1}}\right\|_1-\frac{2p_{A-1}}{2^{2A}}\left\|K_{p_{A-1}-1}\right\|_1-\frac{1}{2^{4A}}\left|G_{p_{A-1}}(0,0)\right|. \tag{12}$$

Combining (8)-(12) we have

$$\|I\|_1\geq 2\left\|G^{(1)}_{p_{A-1}}\right\|_1-\frac{2}{2^{2A}}\left|G^{(1)}_{p_{A-1}}(0)\right|-\frac{4p_{A-1}}{2^{2A}}\left\|G^{(1)}_{p_{A-1}}\right\|_1$$

$$-\frac{4p_{A-1}}{2^{2A}}\left\|K_{p_{A-1}-1}\right\|_1+\left\|G_{p_{A-1}}\right\|_1-O(A)$$

$$\geq 2\left\|G^{(1)}_{p_{A-1}}\right\|_1+\left\|G_{p_{A-1}}\right\|_1-\frac{4p_{A-1}}{2^{2A}}\left\|G^{(1)}_{p_{A-1}}\right\|_1-O(A).$$

Since

$$\frac{p_{A-1}}{2^{2A}}<\frac{1}{3}$$

from Lemma 1 we get

$$\|I\|_1\geq\frac{2}{3}\left\|G^{(1)}_{p_{A-1}}\right\|_1+\left\|G_{p_{A-1}}\right\|_1-O(A)$$

$$\geq c(2A-2)^2+\left\|G_{p_{A-1}}\right\|_1$$

$$\geq\cdots\geq c\sum_{j=2}^{A}(2j-2)^2\geq cA^3\geq c(\log p_A)^3.$$

Lemma 2 is proved.

3. PROOFS OF THEOREMS

Proof of Theorem 2. We choose a monotonically increasing sequence of positive integers $\{n_k : k\geq 1\}$ such that

$$n_{k-1}^2\leq n_k, \tag{13}$$

$$\sum_{l=1}^{k-1}\frac{2^{2n_l}}{n_l^2}<\frac{2^{2n_k}}{n_k^2}. \tag{14}$$

First, set

$$\psi_{n_k}(x)=\begin{cases} 2^{2n_k+2}x, \text{ if } 0\leq x<2^{-2n_k-2} \\ -2^{2n_k+2}\left(x-2^{-2n_k-1}\right), \text{ if } 2^{-2n_k-2}\leq x\leq 2^{-2n_k-1} \\ 0, \quad \text{otherwise,} \end{cases}$$

$$\varphi_{n_k}(x)=\sum_{j=0}^{2^{2n_k+1}-1}\psi_{n_k}\left(x-\frac{j}{2^{2n_k+1}}\right), \quad \varphi_{n_k}(x+1)=\varphi_{n_k}(x).$$

We construct a function f defined as follows. Set

$$f(x,y)=\sum_{k=1}^{\infty}\frac{f_{n_k}(x,y)}{n_k^2},$$

where

$$f_{n_k}(x,y)=\varphi_{n_k}(x)\,\varphi_{n_k}(y)\,\mathrm{sign}F_{p_{n_k}}(x,y)$$

and

$$p_{n_k}=2^{2n_k}+2^{2n_k-2}+\cdots+2^2+2^0.$$

First we prove that

$$\omega(\delta,f)_C=O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right). \tag{15}$$

For every $\delta>0$, small enough there exists a positive integer k such that

$$2^{-2n_k}\leq\delta<2^{-2n_{k-1}}.$$

Since $|\varphi_{n_l}(x+\delta)-\varphi_{n_l}(x)|=O\left(\delta 2^{2n_l}\right)$ for $l=1,2,...,k-1,$ from (13) and (14) we get

$$|f(x+\delta,y)-f(x,y)|\leq\sum_{l=1}^{k-1}\frac{1}{n_l^2}\left|f_{n_l}(x+\delta,y)-f_{n_l}(x,y)\right|+2\sum_{l=k}^{\infty}\frac{1}{n_l^2}=$$

$$=O\left(\delta\sum_{l=1}^{k-1}\frac{2^{2n_l}}{n_l^2}\right)+O\left(\frac{1}{n_k^2}\right)=O\left(\delta\frac{2^{2n_{k-1}}}{n_{k-1}^2}\right)+O\left(\frac{1}{n_k^2}\right)$$

$$=O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right).$$

Consequently

$$\omega_1(\delta,f)_C=O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right). \tag{16}$$

Analogously we obtain

$$\omega_2(\delta,f)_C=O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right). \tag{17}$$

Since

$$\omega(\delta,f)_C\leq\omega_1(\delta,f)_C+\omega_2(\delta,f)_C$$

from (16) and (17) we get (15).

Next, we prove that $t_{p_{n_k}}(f,0,0)$ diverges.

It is clear that

$$\left|t_{p_{n_k}}(f,0,0)-f(0,0)\right|=\left|t_{p_{n_k}}(f,0,0)\right|=$$

$$=\left|\int_0^1\int_0^1 f(t,s)\,F_{p_{n_k}}(t,s)\,dtds\right|\geq\frac{1}{n_k^2}\left|\int_0^1\int_0^1 f_{n_k}(t,s)\,F_{p_{n_k}}(t,s)\,dtds\right|-$$

$$-\sum_{i=1}^{k-1}\frac{1}{n_i^2}\left|\int_0^1\int_0^1 f_{n_i}(t,s)\,F_{p_{n_k}}(t,s)\,dtds\right|-\sum_{i=k+1}^{\infty}\frac{1}{n_i^2}\left|\int_0^1\int_0^1 f_{n_i}(t,s)\,F_{p_{n_k}}(t,s)\,dtds\right|=$$

$$=I-II-III. \tag{18}$$

From Lemma 2 we get

$$I=\frac{1}{n_k^2}\left|\int_0^1\int_0^1 f_{n_k}(t,s)\,F_{p_{n_k}}(t,s)\,dtds\right|=$$

$$=\frac{1}{n_k^2}\sum_{j=0}^{2^{2n_k+1}-1}\sum_{i=0}^{2^{2n_k+1}-1}\int\limits_{j2^{-2n_k-1}}^{(j+1)2^{-2n_k-1}}\int\limits_{i2^{-2n_k-1}}^{(i+1)2^{-2n_k-1}}\varphi_{n_k}(t)\,\varphi_{n_k}(s)\left|F_{p_{n_k}}(t,s)\right|dtds=$$

$$=\frac{1}{n_k^2}\sum_{j=0}^{2^{2n_k+1}-1}\sum_{i=0}^{2^{2n_k+1}-1}\left|F_{p_{n_k}}\left(\frac{j}{2^{2n_k+1}},\frac{i}{2^{2n_k+1}}\right)\right|\int\limits_{j2^{-2n_k-1}}^{(j+1)2^{-2n_k-1}}\int\limits_{i2^{-2n_k-1}}^{(i+1)2^{-2n_k-1}}\varphi_{n_k}(t)\,\varphi_{n_k}(s)\,dtds=$$

$$=\frac{1}{4n_k^2}\sum_{j=0}^{2^{2n_k+1}-1}\sum_{i=0}^{2^{2n_k+1}-1}\left|F_{p_{n_k}}\left(\frac{j}{2^{2n_k+1}},\frac{i}{2^{2n_k+1}}\right)\right|\int\limits_{j2^{-2n_k-1}}^{(j+1)2^{-2n_k-1}}\int\limits_{i2^{-2n_k-1}}^{(i+1)2^{-2n_k-1}}1\,dtds=$$

$$=\frac{1}{4n_k^2}\sum_{j=0}^{2^{2n_k+1}-1}\sum_{i=0}^{2^{2n_k+1}-1}\int\limits_{j2^{-2n_k-1}}^{(j+1)2^{-2n_k-1}}\int\limits_{i2^{-2n_k-1}}^{(i+1)2^{-2n_k-1}}\left|F_{p_{n_k}}(t,s)\right|dtds=\frac{1}{4n_k^2}\left\|F_{p_{n_k}}\right\|_1\geq c>0. \tag{19}$$

Since

$$\omega_{1,2}\left(\frac{1}{i},\frac{1}{j},f\right)_C\leq 2\omega_1\left(\frac{1}{i},f\right)_C,$$

$$\omega_{1,2}\left(\frac{1}{i},\frac{1}{j},f\right)_C\leq 2\omega_2\left(\frac{1}{j},f\right)_C$$

and

$$\frac{\omega(\delta,f)_C}{\delta}\leq\frac{\omega(\delta',f)_C}{\delta'},\quad 0<\delta'\leq\delta$$

from Theorem A we have

$$\|t_n(f)-f\|_C\leq\frac{c}{\log n}\sum_{j=1}^{n}\frac{\|S_{j,j}(f)-f\|_C}{n-j}$$

$$\leq\frac{c}{\log n}\sum_{j=1}^{n}\left[\frac{\omega_1(1/j,f)_C}{n-j}\log(j+1)+\frac{\omega_2(1/j,f)_C}{n-j}\log(j+1)\right.$$

$$+\frac{\sqrt{\omega_1\left(1/j,f\right)_C}\sqrt{\omega_2\left(1/j,f\right)_C}}{n-j}\log^2\left(j+1\right)$$

$$\leq c\log n\left[\omega_1\left(1/n,f\right)_C+\omega_2\left(1/n,f\right)_C+\sqrt{\omega_1\left(1/n,f\right)_C}\sqrt{\omega_2\left(1/n,f\right)_C}\right]$$

$$\times\sum_{j=1}^{n}\frac{n}{j\left(n-j\right)}$$

$$\leq c\log^2 n\left[\omega_1\left(1/n,f\right)_C+\omega_2\left(1/n,f\right)_C+\sqrt{\omega_1\left(1/n,f\right)_C}\sqrt{\omega_2\left(1/n,f\right)_C}\right]. \tag{20}$$

It is evident that

$$\omega_1\left(f_{n_i},\frac{1}{2^{2n_k}}\right)_C=\omega_2\left(f_{n_i},\frac{1}{2^{2n_k}}\right)_C=O\left(\frac{2^{2n_i}}{2^{2n_k}}\right),\quad i=1,2...,k-1,$$

Then from (20) we get

$$\left|t_{p_{n_k}}\left(f_{n_i},0,0\right)\right|\leq\left\|t_{p_{n_k}}\left(f_{n_i}\right)-f_{n_i}\right\|_C\leq c\frac{2^{2n_i}}{2^{2n_k}}\log^2 p_{n_k},$$

consequently from (13) and (14) we have

$$II=O\left(\frac{n_k^2}{2^{2n_k}}\sum_{i=1}^{k-1}\frac{2^{2n_i}}{n_i^2}\right)=O\left(\frac{n_k^2}{2^{2n_k}}\frac{2^{2n_{k-1}}}{n_{k-1}^2}\right)=o\left(1\right)\ \text{ as } k\to\infty. \tag{21}$$

Since

$$\left\|F_n\right\|_1=O\left(\frac{1}{\log n}\sum_{i=1}^{n-1}\frac{\left\|D_i\right\|_1^2}{n-i}\right)=$$

$$=O\left(\frac{1}{\log n}\sum_{i=1}^{n-1}\frac{\log^2\left(i+1\right)}{n-i}\right)=O\left(\log^2\left(n+1\right)\right),$$

by (13) we have

$$III=O\left(\sum_{i=k+1}^{\infty}\frac{1}{n_i^2}\left\|F_{p_{n_k}}\right\|_1\right)=O\left(\left(\frac{n_k}{n_{k+1}}\right)^2\right)=o\left(1\right)\ \ as\ k\to\infty \tag{22}$$

After substituting (19),(21) and (22) in (18) we obtain

$$\overline{\lim_{k\to\infty}}\left|t_{p_{n_k}}\left(f,0,0\right)-f\left(0,0\right)\right|>0.$$

Theorem 2 is proved. □

Proof of Theorem 3. We choose a monotonically increasing sequence of positive integers $\{m_k:k\geq 1\}$ such that

$$2m_{k-1}\leq m_k, \tag{23}$$

$$\sum_{l=1}^{k-1}\frac{2^{2m_l}}{m_l^2}<\frac{2^{2m_k}}{m_k^2}. \tag{24}$$

We construct a function g defined as follows. Set

$$g(x,y)=\sum_{j=1}^{\infty}\frac{g_j(x,y)}{m_j^2},$$

where

$$g_j(x,y)=D_{2^{2m_j+1}}(x)\,D_{2^{2m_j+1}}(y).$$

First we prove that

$$\omega(\delta,g)_L=O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right). \tag{25}$$

For every $\delta>0$ there exists a positive integer k such that

$$2^{-2m_k}\leq\delta<2^{-2m_{k-1}}.$$

Since $(l=1,2,...,k-1,\delta>0)$

$$\int_0^1\left|D_{2^{2m_l+1}}(x+\delta)-D_{2^{2m_l+1}}(x)\right|dx=22^{2m_l+1}\delta, \tag{26}$$

from (1), (23) and (24) we get

$$\int_0^1\int_0^1\left|g(x+\delta,y)-g(x,y)\right|dxdy$$

$$\leq\sum_{l=1}^{k-1}\frac{1}{m_l^2}\int_0^1\left|D_{2^{2m_l+1}}(x+\delta)-D_{2^{2m_l+1}}(x)\right|dx\int_0^1 D_{2^{2m_l+1}}(y)\,dy$$

$$+2\sum_{l=k}^{\infty}\frac{1}{m_l^2}=O\left(\delta\sum_{l=1}^{k-1}\frac{2^{2m_l}}{m_l^2}\right)+O\left(\frac{1}{m_k^2}\right)$$

$$=O\left(\delta\frac{2^{2m_{k-1}}}{m_{k-1}^2}\right)+O\left(\frac{1}{m_k^2}\right)=O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right).$$

Consequently

$$\omega_1(\delta,g)_L=O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right). \tag{27}$$

Analogously, we obtain

$$\omega_2(\delta,g)_L=O\left(\left(\frac{1}{\log(1/\delta)}\right)^2\right). \tag{28}$$

Since

$$\omega(\delta,g)_L\leq\omega_1(\delta,g)_L+\omega_2(\delta,g)_L,$$

then from (27) and (28) we get (25).

It is evident that

$$\left\|t_{p_{m_k}}(g)-g\right\|_L \geq \left\|t_{p_{m_k}}\left(\sum_{i=k}^{\infty}\frac{g_i}{m_i^2}\right)\right\|_L-\sum_{i=k}^{\infty}\frac{1}{m_i^2}\left\|g_i\right\|_L-$$

$$-\left\|t_{p_{m_k}}\left(\sum_{i=1}^{k-1}\frac{g_i}{m_i^2}\right)-\sum_{i=1}^{k-1}\frac{g_i}{m_i^2}\right\|_L=$$

$$=I-II-III. \tag{29}$$

Since

$$\omega_{1,2}\left(\frac{1}{i},\frac{1}{j},f\right)_L \leq 2\omega_1\left(\frac{1}{i},f\right)_L$$

and

$$\omega_{1,2}\left(\frac{1}{i},\frac{1}{j},f\right)_L \leq 2\omega_2\left(\frac{1}{j},f\right)_L,$$

then from Theorem A we get

$$\left\|S_{i,j}(f)-f\right\|_L=$$

$$=O\left(\omega_1\left(\frac{1}{i},f\right)_L\log(i+1)+\omega_2\left(\frac{1}{j},f\right)_L\log(j+1)\right.$$

$$\left.\sqrt{\omega_1\left(\frac{1}{i},f\right)_L\log(i+1)}\sqrt{\omega_2\left(\frac{1}{j},f\right)_L\log(j+1)}\right).$$

Since

$$\frac{\omega(\delta,f)_L}{\delta}\leq\frac{\omega(\delta',f)_L}{\delta'},\quad 0<\delta'\leq\delta$$

and

$$\omega_1(\delta,g_l)_L=\omega_2(\delta,g_l)_L=O\left(2^{2m_l}\delta\right)$$

we have

$$\left\|S_{i,i}(g_l)-g_l\right\|_L=O\left(\frac{1}{i}2^{2m_l}\log^2 p_{m_k}\right),$$

consequently,

$$\left\|t_{p_{m_k}}(g_l)-g_l\right\|_L$$

$$\leq\frac{c}{m_k}\sum_{i=1}^{p_{m_k}-1}\frac{\left\|S_{ii}(g_l)-g_l\right\|_L}{(p_{m_k}-i)}$$

$$=O\left(\frac{2^{2m_l}\log^2 p_{m_k}}{m_k}\sum_{i=1}^{p_{m_k}-1}\frac{1}{i}\frac{1}{(p_{m_k}-i)}\right)$$

$$=O\left(2^{2m_l}\frac{\log^2 p_{m_k}}{2^{2m_k}}\right),$$

$$III=O\left(\frac{\log^2 p_{m_k}}{2^{2m_k}}\sum_{l=1}^{k-1}\frac{2^{2m_l}}{m_l^2}\right)=O\left(\frac{m_k^2}{2^{2m_k}}\frac{2^{2m_{k-1}}}{m_{k-1}^2}\right)=o(1)\quad\text{as}\quad k\to\infty. \tag{30}$$

From (1) and (23) we get

$$II \le \sum_{i=k}^{\infty} \frac{1}{m_i^2} = O\left(\frac{1}{m_k^2}\right) = o(1) \quad \text{as} \quad k \to \infty \tag{31}$$

It is easy to have

$$t_{m_k}(g_i) = S_{2^{2m_i+1}, 2^{2m_i+1}}\left(F_{p_{m_k}}, x, y\right) = F_{p_{m_k}}(x,y), \quad i = k, k+1, ...$$

Consequently, by Lemma 2 we get

$$\left\| t_{p_{m_k}}\left(\sum_{i=k}^{\infty} \frac{g_i}{m_i^2}\right)\right\|_L = \sum_{i=k}^{\infty} \frac{1}{m_i^2} \left\|F_{p_{m_k}}\right\|_L \ge$$

$$\ge \frac{c}{m_k^2}\left\|F_{p_{m_k}}\right\|_L \ge c > 0. \tag{32}$$

Combining (29)-(32) we conclude that

$$\overline{\lim_{k\to\infty}} \left\| t_{p_{m_k}}(g) - g\right\|_L > 0.$$

This completes the proof of Theorem 3. □

Proof of Theorem 4. It is well-known the following inequality proved by Tkebuchava [12]

$$\|S_{m,n}(f)\|_L \le c\|f\|_{L(\log^+ L)^2} + c \tag{33}$$

for all $f \in L(\log^+ L)^2$. Since

$$\|t_n(f)\|_L \le \frac{c}{\log(n+1)} \sum_{k=1}^{n-1} \frac{\|S_{k,k}(f)\|_L}{n-k},$$

then from this inequality we get

$$\|t_n(f)\|_L \le c\|f\|_{L(\log^+ L)^2} + c \tag{34}$$

for the two-variable function f. Later, we will need it. Let (A_j) be a sequence of natural numbers (discussed later), and $(\lambda_j) \in l^1$ a sequence of positive reals. Set the two-variable function h_j as

$$h_j(x,y) := D_{2^{2A_j+1}}(x) D_{2^{2A_j+1}}(y).$$

The $L(I^2)$ norm of h_j is 1 ($j \in \mathbb{N}$). What can be said about $\|h_j\|_{L(\log^+ L)^2}$? It is bounded by cA_j^2. Obviously, the function $h := \sum_{j=0}^{\infty} \lambda_j h_j$ is an element of $L(I^2)$. Set

$$p_{A_j} = 2^{2A_j} + 2^{2A_j-2} + \cdots + 2^0 \quad (j \in \mathbb{N})$$

again. It is easy to have for any $(x,y) \in I^2$ that

$$t_{p_{A_j}}(h_{j+i}, x, y) = F_{p_{A_j}}(x,y)$$

for all $i = 0, 1, \ldots$. Consequently, by Lemma 2 we have

$$\|t_{p_{A_j}}(\sum_{i=0}^{\infty} \lambda_{j+i} h_{j+i})\|_L = \sum_{i=0}^{\infty} \lambda_{j+i} \|F_{p_{A_j}}\|_L \ge cA_j^2 \sum_{i=0}^{\infty} \lambda_{j+i} \ge cA_j^2 \lambda_j.$$

By inequality (34) we also have the upper bound

$$\begin{aligned}
&\|t_{p_{A_j}}(\sum_{i=1}^{j}\lambda_{j-i}h_{j-i})\|_L \\
&\le \sum_{i=1}^{j}\lambda_{j-i}\|t_{p_{A_j}}(h_{j-i})\|_L \\
&\le c\sum_{i=1}^{j}\lambda_{j-i}(\|h_{j-i}\|_{L(\log^+ L)^2}+1)^2 \\
&\le c\sum_{i=1}^{j}\lambda_{j-i}A_{j-i}^2.
\end{aligned}$$

By the last two inequalities we get

$$\|t_{p_{A_j}}(h)\|_L \ge cA_j^2\lambda_j - c\sum_{i<j}\lambda_i A_i^2.$$

Let

$$\delta_j := \sup\left\{t>0 : \delta\left(\frac{4^{j^2}}{t^2}2^{4t}\right) > \frac{1}{8^{j^2}}\right\}.$$

If the set the supremum of which is taken is empty (that is, $\delta \le 1/8^{j^2}$ for each positive t), then let $\delta_j := 1$. Since the function δ is vanishing at plus infinity, then the sequence (δ_j) is well-defined. We can define the sequence (A_j) and (λ_j) in the following way ($A_{-1} := 0$):

$$A_j := \max\left\{6^{j^2}, A_{j-1}+1, \delta_j+1\right\}, \quad \lambda_j := \frac{4^{j^2}}{A_j^2}.$$

Then obviously $(\lambda_j)\in l^1$, and consequently $h \in L(I^2)$. Besides,

$$\|t_{p_{A_j}}(h)\|_{L(I^2)} \ge cA_j^2\lambda_j - c\sum_{i<j}\lambda_i A_i^2 \ge c4^{j^2} - c\sum_{i<j}4^{i^2} \ge c4^{j^2}.$$

This gives $\overline{\lim}_j\|t_{p_{A_j}}h\|_{L(I^2)} = \infty$. The rest is to prove that $h \in L(\log^+ L)^2\delta(L)(I^2)$. Introduce the following notation.

$$J_{j,k} := \left[2^{-2A_{j+1}-1}, 2^{-2A_j-1}\right) \times \left[2^{-2A_{k+1}-1}, 2^{-2A_k-1}\right)$$

for any $j,k\in\mathbb{N}$. The construction of function h gives

$$\operatorname{supp} h \setminus \{0\} \subset \bigcup_{j,k\in\mathbb{N}} J_{j,k} = \left[0, 2^{-2A_0-1}\right) \times \left[0, 2^{-2A_0-1}\right).$$

Next, we discuss the value of h on the set $J_{j,k}$. Let $l := \min\{j,k\}$. Then for $(x,y)\in J_{j,k}$ we have

$$h(x,y) = \sum_{i\le l}\lambda_i h_i(x) = \sum_{i\le l}\lambda_i(2^{2A_i+1})^2 \le c\lambda_l 2^{4A_l+2}.$$

Since

$$\sum_{i\le l}\lambda_i 2^{4A_i+2} = \sum_{i\le l}\frac{4^{i^2}}{A_i^2}2^{4A_i+2} \le c\frac{4^{l^2}}{A_l^2}2^{4A_l},$$

then we have

$$\log^+\left(\sum_{i\le l}\lambda_i 2^{4A_i+2}\right)\le \log^+\left(c\frac{4^{l^2}}{A_l^2}2^{4A_l}\right)\le cA_l.$$

We also conclude from $\sum_{i\le l}\lambda_i 2^{4A_i+2}\ge \lambda_l 2^{4A_l+2}=\frac{4^{l^2}}{A_l^2}2^{4A_l+2}$ that the inequality

$$\delta\left(\sum_{i\le l}\lambda_i 2^{4A_i+2}\right)\le \frac{1}{8^{l^2}}$$

holds. Finally, from the above we get

$$\begin{aligned}
&\|h\|_{L(\log^+ L)^2\delta(L)}\\
&=\int_0^{2^{-2A_0-1}}\int_0^{2^{-2A_0-1}}|h(x,y)|\Big(\log^+(|h(x,y)|)\Big)^2\delta(|h(x,y)|)dx\,dy\\
&\le\sum_{j=0}^{\infty}\sum_{k=0}^{j}\int_{J_{j,k}}|h(x,y)|\Big(\log^+(|h(x,y)|)\Big)^2\delta(|h(x,y)|)dx\,dy\\
&+\sum_{k=0}^{\infty}\sum_{j=0}^{k}\int_{J_{j,k}}|h(x,y)|\Big(\log^+(|h(x,y)|)\Big)^2\delta(|h(x,y)|)dy\,dx\\
&=I+II.
\end{aligned}$$

Since to investigate I, and II is quite the same, then we discuss I, only. By the above written we have

$$\begin{aligned}
I&\le c\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{1}{2^{2A_j}}\frac{1}{2^{2A_k}}\frac{4^{k^2}}{A_k^2}2^{4A_k}A_k^2\frac{1}{8^{k^2}}\\
&=c\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{1}{2^{k^2}}4^{A_k-A_j}\\
&=c\sum_{k=0}^{\infty}\frac{1}{2^{k^2}}\sum_{j=k}^{\infty}4^{A_k-A_j}\le c.
\end{aligned}$$

This completes the proof of Theorem 4. □

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