UNIFORM AND L-CONVERGENCE OF LOGARITHMIC MEANS OF WALSH-FOURIER SERIES

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Abstract. The (Nörlund) logarithmic means of the Fourier series of the integrable function f is:

$$\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{n-k}, \quad \text{where } l_n := \sum_{k=1}^{n-1}\frac{1}{k}.$$

In this paper we discuss some convergence and divergence properties of this logarithmic means of the Walsh-Fourier series of functions in the uniform, and in the L^1 Lebesgue norm. Among others, as an application of our divergence results we give a negative answer for a question of Móricz concerning the convergence of logarithmic means in norm.

In the literature it is known the notion of the Riesz's logarithmic means of a Fourier series. The nth mean of the Fourier series of the integrable function f is defined by

$$\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{k}.$$

This Riesz's logarithmic means with respect to the trigonometric system has been studied by a lot of authors. We mention for instance the papers of Szász, and Yabuta [11, 13]. This mean with respect to the Walsh, Vilenkin system is discussed by Simon, and Gát [10, 3]. For results of this kind with respect to the Walsh-Kaczmarz system, and some generalization of the Walsh system see [7, 1]

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The Nörlund means for the Fourier series of f are defined by

$$\frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k} S_k(f),$$

where $Q_n := \sum_{k=1}^{n-1} q_k$. If $q_k = \frac{1}{k}$, then we get the (Nörlund) logarithmic means:

$$\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{n-k}.$$

In this paper we call it - it will not cause any misunderstood - as logarithmic means. Although, it is a kind of "reverse" Riesz's logarithmic means. Móricz [6] investigates the approximation properties of some special Nörlund means of Walsh-Fourier series of L^p functions in norm. The case, when $q_k = \frac{1}{k}$ is excluded, since the methods of Móricz are not applicable for logarithmic means. The aim of this paper is to prove some convergence and divergence properties of the logarithmic means of functions in the class of continuous functions, and in

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the Lebesgue space L^1. We mean convergence and divergence in norm. Among others, we give a negative answer for the question of Móricz [6].

Let $\mathbb{N}$ denote the set of nonnegative integers and $I=[0,1)$ the unit interval. By a dyadic intervals in I we mean one of the form $[l2^{-k},(l+1)\,2^{-k})$ for some $k\in\mathbb{N}, 0\leq l<2^k$. For a given $k\in\mathbb{N}$ and $x\in I$, $I_k(x)$ denote the dyadic interval of length 2^{-k} which contains the point x. Set $I_k(0)=I_k$ and $I_0(x)=I$.

Let $r_0(x)$ be the function defined by

$$r_0(x)=\begin{cases}1, & \text{if } x\in I_1\\ -1, & \text{if } x\in I\backslash I_1\end{cases}, \quad r_0(x+1)=r_0(x).$$

The Rademacher system is defined by

$$r_n(x)=r_0(2^n x), \quad n\geq 1 \quad \text{and} \quad x\in I.$$

Let $w_0,w_1,...$ represent the Walsh functions, i.e. $w_0(x)=1$ and if $k=2^{n_1}+\cdots+2^{n_s}$ is a positive integer with $n_1>n_2>\cdots>n_s\geq 0$ then

$$w_k(x)=r_{n_1}(x)\cdots r_{n_s}(x).$$

The idea of using products of Rademacher's functions to define the Walsh system originated from Paley [8].

The Walsh-Dirichlet kernel is defined by

$$D_n(x)=\sum_{k=0}^{n-1}w_k(x).$$

Recall that

$$D_{2^n}(x)=\begin{cases}2^n, & \text{if } x\in I_n,\\ 0, & \text{if } x\notin I_n\end{cases}, \quad (n\in\mathbb{N}). \tag{1}$$

Suppose that f is a Lebesgue integrable $(f\in L(I))$ function on I and 1-periodic. Then its Walsh-Fourier series is defined by

$$\sum_{k=0}^{\infty}\hat{f}(k)\,w_k(x),$$

where

$$\hat{f}(k)=\int_0^1 f(t)\,w_k(t)\,dt$$

is called the k-th Walsh-Fourier coefficient of function f.

Denote the n-th partial sum of the Walsh-Fourier series of the function f by $S_n(f,x)$. Namely

$$S_n(f,x)=\sum_{k=0}^{n-1}\hat{f}(k)\,w_k(x).$$

The logarithmic means of the Walsh-Fourier series is defined as follows

$$t_n(f,x)=\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f,x)}{n-k},$$

where

$$l_n=\sum_{k=1}^{n-1}\frac{1}{k}.$$

It is evident that

$$t_n(f,x)-f(x)=\int_0^1[f(x\oplus t)-f(x)]F_n(t)\,dt,$$

where

$$F_n(t)=\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{D_k(t)}{n-k}$$

and $\oplus$ denotes dyadic addition [9, 5].

Denote by $C(I)$ the space of continuous function on I with period 1 .

Let $f\in C(I)$. The expression

$$\omega(\delta,f)_C=\sup_{|h|\le\delta}\|f(\cdot+h)-f(\cdot)\|_C$$

is called modulus of continuity of the function f, while the integral modulus of continuity defined by

$$\omega(\delta,f)_L=\sup_{|h|\le\delta}\|f(\cdot+h)-f(\cdot)\|_L.$$

It is well-known that the following are true [9, 4, 5].

Theorem A. Let $f\in C(I)$ and

$$\omega(\delta,f)_C=o\left(\frac{1}{\log(1/\delta)}\right),$$

then

$$\|S_n(f)-f\|_C\to 0\quad\text{as}\quad n\to\infty.$$

Theorem B. Let $f\in L(I)$ and

$$\omega(\delta,f)_L=o\left(\frac{1}{\log(1/\delta)}\right),$$

then

$$\|S_n(f)-f\|_L\to 0\quad\text{as}\quad n\to\infty.$$

Theorem C. Let $f\in L\log^+L(I)$. Then

$$\|S_n(f)-f\|_L\to 0\quad\text{as}\quad n\to\infty.$$

By

$$\|t_n(f)-f\|_C \le \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{\|S_k(f)-f\|_C}{n-k},$$

$$\|t_n(f)-f\|_L \le \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{\|S_k(f)-f\|_L}{n-k}$$

and by the fact that the logarithmic summability method is regular from Theorems A, B and C we obtain

Theorem 1. *Let $f\in C(I)$ and*

$$\omega(\delta,f)_C = o\left(\frac{1}{\log(1/\delta)}\right),$$

then

$$\|t_n(f)-f\|_C \to 0 \quad \text{as} \quad n\to\infty.$$

Theorem 2. *Let $f\in L(I)$ and*

$$\omega(\delta,f)_L = o\left(\frac{1}{\log(1/\delta)}\right)$$

then

$$\|t_n(f)-f\|_L \to 0 \quad \text{as} \quad n\to\infty.$$

Theorem 3. *Let $f\in L\log^+ L(I)$. Then*

$$\|t_n(f)-f\|_L \to 0 \quad \textit{as} \quad n\to\infty.$$

In this paper we investigate the sharpness of these results. In particular, we prove

Theorem 4. *There exists function $f\in C(I)$ such that*

$$\omega(\delta,f)_C = O\left(\frac{1}{\log(1/\delta)}\right)$$

and $t_n(f,0)$ diverges.

Theorem 5. *There exists a function $g\in L(I)$ such that*

$$\omega(\delta,g)_L = O\left(\frac{1}{\log(1/\delta)}\right)$$

and $t_n(g)$ does not converge to g in L-norm.

Using Ulyanov's [12] embedding theorem from the Theorem 5 we have

Theorem 6. *There exists function $g\in L(\log L)^{1-\varepsilon}(I)$ ($\varepsilon>0$) for which $t_n(g)$ does not converge to g in L-norm.*

We prove even more. Namely, we prove that the maximal convergence space with respect to the L-norm convergence is $L(\log L)$. That is, let $\delta:[0,+\infty)\to[0,+\infty)$ be measurable, and $\delta(+\infty)=0$. Then the following is true

Theorem 7. *There exists function $h \in L\left(\log^{+} L\delta\left(L\right)\right)\left(I\right)$ for which $t_n\left(h\right)$ does not converge to h in L-norm.*

In [6, Theorem 1] Móricz proved the following theorem for Nörlund means with some nonincreasing sequence $(q_k : k \geq 1)$: Let $f \in L^p, 1 \leq p \leq \infty$, $n = 2^m + k$, $1 \leq k \leq 2^m$, then

$$\left\| \frac{1}{Q_n} \sum_{k=1}^{n-1} q_{n-k} S_k(f) - f \right\|_p \leq \frac{c}{Q_n} \sum_{j=0}^{m-1} (Q_{n-2^j+1} - Q_{n-2^{j+1}+1}) \omega_p(f, 2^{-j}) + O(\omega_p(f, 2^{-m})).$$

The case when we have $q_k = k^{-1}$ differs from the types discussed by Móricz in [6]. His method is not applicable for logarithmic means. So, it is a question that his inequality above holds in this case also, or not. Theorems 5,6, and 7 show that the answer is no. The theorem of Móricz does not hold for $p = 1$, and $q_k = k^{-1}$. This means a negative answer for his question [6, Problem 1, p. 386]

In order to prove these theorems we need the following lemmas

Lemma 1. *Let $1 \leq j \leq 2^k - 1$. Then*

$$D_{2^k-j}\left(u\right) = D_{2^k}\left(u\right) - w_{2^k-1}\left(u\right) D_j\left(u\right).$$

Proof. Since $\left(2^k - 1\right) \oplus \nu = 2^k - 1 - \nu$ for $\nu = 0, 1, ..., 2^k - 1$ and $w_{n \oplus m}\left(x\right) = w_n\left(x\right) w_m\left(x\right)$ we get

$$D_{2^k-j}\left(u\right) = D_{2^k}\left(u\right) - \sum_{\nu=2^k-j}^{2^k-1} w_\nu\left(u\right) =$$

$$= D_{2^k}\left(u\right) - w_{2^k-1}\left(u\right) \sum_{\nu=2^k-j}^{2^k-1} w_{\left(2^k-1\right)\oplus\nu}\left(u\right)$$

$$= D_{2^k}\left(u\right) - w_{2^k-1}\left(u\right) \sum_{\nu=2^k-j}^{2^k-1} w_{2^k-1-\nu}\left(u\right)$$

$$= D_{2^k}\left(u\right) - w_{2^k-1}\left(u\right) \sum_{\nu=0}^{j-1} w_\nu\left(u\right)$$

$$= D_{2^k}\left(u\right) - w_{2^k-1}\left(u\right) D_j\left(u\right).$$

The proof of Lemma 1 is complete. □

Lemma 2. *Let $p_A = 2^{2A} + \cdots + 2^2 + 2^0$. Then*

$$\left\| F_{p_A} \right\|_L \geq c \log p_A.$$

Proof . Set

$$G_n\left(t\right) = \sum_{k=1}^{n-1} \frac{D_{n-k}\left(t\right)}{k}.$$

Then we have

$$G_{p_A}\left(x\right) = \sum_{k=1}^{2^{2A-2}+\cdots+2^0-1} \frac{1}{k} D_{2^{2A}+2^{2A-2}+\cdots+2^0-k}\left(x\right) +$$

$$+\sum_{k=2^{2A-2}+\dots+2^0}^{2^{2A}+2^{2A-2}+\dots+2^0-1}\frac{1}{k}D_{2^{2A}+2^{2A-2}+\dots+2^0-k}(x)=:B_1+B_2.$$

First discuss B_1. Since $k<2^{2A-2}+\cdots+2^0$, then

$$D_{2^{2A}+2^{2A-2}+\dots+2^0-k}(x)=D_{2^{2A}}(x)+r_{2A}D_{2^{2A-2}+\dots+2^0-k}(x).$$

This gives

$$B_1=D_{2^{2A}}(x)\left(A2\log_e(2)+c_A^1\right)+r_{2A}(x)\,G_{2^{2A-2}+\dots+2^0},$$

where $c_A^1/A\to 0(A\to+\infty)$, that is, $c_A^1=o(A)$. Discuss B_2. Define k' as $k=2^{2A-2}+\cdots+2^0+k'$. This means $k'\in[0,2^{2A})$. From Lemma 1 we write

$$D_{2^{2A}+2^{2A-2}+\dots+2^0-k}(x)=D_{2^{2A}-k'}(x)=D_{2^{2A}}(x)-\omega_{2^{2A}-1}(x)\,D_{k'}(x).$$

Consequently,

$$\begin{aligned}B_2&=\sum_{k'=0}^{2^{2A}-1}\frac{1}{2^{2A-2}+\cdots+2^0+k'}\left(D_{2^{2A}}(x)-\omega_{2^{2A}-1}(x)\,D_{k'}(x)\right)\\&=D_{2^{2A}}(x)\,c_A^2-\omega_{2^{2A}-1}(x)\sum_{k'=0}^{2^{2A}-1}\frac{1}{2^{2A-2}+\cdots+2^0+k'}D_{k'}(x)=B_{2,1}+B_{2,2},\end{aligned}$$

where $0\le c_A^2\le c$. We give an upper bound for the integral of $|B_{2,2}|$. By the means of the Abel transform we have:

$$\begin{aligned}|B_{2,2}|&\le\sum_{k'=0}^{2^{2A}-1}\left(\frac{1}{2^{2A-2}+\cdots+2^0+k'}-\frac{1}{2^{2A-2}+\cdots+2^0+k'+1}\right)|k'K_{k'}(x)|\\&+\frac{1}{2^{2A}+2^{2A-2}+\cdots+2^0}(2^{2A}-1)|K_{2^{2A}-1}(x)|.\end{aligned}$$

(K_k is the kth one-dimensional Fejér kernel). This implies

$$\|B_{2,2}\|_L\le c\sum_{k'=0}^{2^{2A}-1}\frac{1}{(2^{2A-2}+\cdots+2^0+k')^2}k'+\|K_{2^{2A}-1}\|_L\le c.$$

This gives

$$\|B_2\|_1\le c_A^2+c\le c.$$

So, we have

$$\|G_{2^{2A}+2^{2A-2}+\dots+2^0}\|_1\ge\|2A\log_e(2)D_{2^{2A}}+r_{2A}G_{2^{2A-2}+\dots+2^0}\|_1-o(A).\tag{2}$$

We discuss the right hand side of this inequality, more exactly we give a lower bound for it. That is, for the integral on the set $I\setminus I_{2A}$, and then on the set I_{2A} .

$$\begin{aligned}&\int_{I\setminus I_{2A}}|2A\log_e(2)D_{2^{2A}}(x)+r_{2A}(x)\,G_{2^{2A-2}+\dots+2^0}(x)|dx\\&=\int_{I\setminus I_{2A}}|G_{2^{2A-2}+\dots+2^0}(x)|dx\\&=\|G_{2^{2A-2}+\dots+2^0}\|_L-\int_{I_{2A}}|G_{2^{2A-2}+\dots+2^0}(x)|dx\\&=\|G_{2^{2A-2}+\dots+2^0}\|_L-\frac{1}{2^{2A}}G_{2^{2A-2}+\dots+2^0}(0).\end{aligned}$$

Besides,

$$\int_{I_{2A}} \left|2A\log_e(2)D_{2^{2A}}(x)+r_{2A}G_{2^{2A-2}+\cdots+2^0}(x)\right|dx$$
$$\geq 2A\log_e(2)-\frac{1}{2^{2A}}G_{2^{2A-2}+\cdots+2^0}(0).$$

This means

$$\|G_{2^{2A}+2^{2A-2}+\cdots+2^0}\|_L \geq 2A\log_e(2)+\|G_{2^{2A-2}+\cdots+2^0}\|_L$$
$$-2\frac{1}{2^{2A}}G_{2^{2A-2}+\cdots+2^0}(0)-o(A).$$

What can be said about function G_n at the point zero?

$$G_n(0)=\sum_{k=1}^{n-1}\frac{n-k}{k}=n\sum_{k=1}^{n-1}\frac{1}{k}-n+1=n\log_e(n)+O(n).$$

So,

$$2\cdot\frac{1}{2^{2A}}\cdot\frac{1}{2A}G_{2^{2A-2}+\cdots+2^0}(0)$$
$$=2\cdot\frac{1}{2^{2A}}\cdot\frac{1}{2A}(2^{2A-2}+\cdots+2^0)\log_e(2^{2A-2}+\cdots+2^0)+o(1)$$
$$\leq 2\cdot\frac{1}{2^{2A}}\cdot\frac{1}{2A}\cdot\frac{1}{3}2^{2A}(2A)+o(1)\leq\frac{2}{3}+o(1).$$

Finally,

$$\frac{1}{2A}\|G_{2^{2A}+2^{2A-2}+\cdots+2^0}\|_L$$
$$\geq\log_e(2)+\frac{1}{2A}\|G_{2^{2A-2}+\cdots+2^0}\|_L-\frac{2}{3}-o(1)$$
$$\geq\frac{1}{40}+\frac{1}{2A}\|G_{2^{2A-2}+\cdots+2^0}\|_L-o(1).$$

By some easy assumptions we also have

$$\|F_{p_A}\|_L\geq c\log p_A.$$

The proof of the Lemma 2 is complete. □

Proof of Theorem 4 . We choose a monotonically increasing sequence of positive integers $\{n_k : k\geq 1\}$ such that

$$n_k^2\leq n_{k+1}, \tag{3}$$

$$\sum_{l=1}^{k-1}\frac{2^{n_l}}{n_l}<\frac{2^{n_k}}{n_k}. \tag{4}$$

First, set

$$\psi_{n_k}(x)=\begin{cases}2^{2n_k+2}x, & \text{if } 0\leq x<2^{-2n_k-2}\\ -2^{2n_k+2}\left(x-2^{-2n_k-1}\right), & \text{if } 2^{-2n_k-2}\leq x\leq 2^{-2n_k-1}\\ 0, & \text{otherwise,}\end{cases}$$

$$\varphi_{n_k}(x)=\sum_{j=0}^{2^{2n_k+1}-1}\psi_{n_k}\left(x-\frac{j}{2^{2n_k+1}}\right),\quad \varphi_{n_k}(x+1)=\varphi_{n_k}(x).$$

We construct a function f defined as follows. Set

$$f(x) = \sum_{l=1}^{\infty} \frac{f_{n_k}(x)}{n_k},$$

where

$$f_{n_k}(x) = \varphi_{n_k}(x) \operatorname{sign} F_{p_{n_k}}(x)$$

and

$$p_{n_k} = 2^{2n_k} + 2^{2n_k-2} + \cdots + 2^2 + 2^0.$$

First we prove that

$$\omega(\delta, f)_C = O\left(\frac{1}{\log(1/\delta)}\right).$$

For every $\delta > 0$, small enough there exists a positive integers k such that

$$2^{-2n_k} \le \delta < 2^{-2n_{k-1}}.$$

Since $|\varphi_{n_l}(x+\delta) - \varphi_{n_l}(x)| = O(\delta 2^{2n_l})$ for $l = 1, 2, ..., k-1,$ from (3) and (4) we get

$$|f(x+\delta) - f(x)| \le \sum_{l=1}^{k-1} \frac{1}{n_l} |f_{n_l}(x+\delta) - f_{n_l}(x)| + 2\sum_{l=k}^{\infty} \frac{1}{n_l} =$$

$$= O\left(\delta \sum_{l=1}^{k-1} \frac{2^{2n_l}}{n_l}\right) + O\left(\frac{1}{n_k}\right) = O\left(\frac{1}{\log(1/\delta)}\right).$$

Next, we shall prove that $t_{p_{n_k}}(f, 0)$ diverges.

It is clear that

$$\left|t_{p_{n_k}}(f,0) - f(0)\right| = \left|t_{p_{n_k}}(f,0)\right| =$$

$$= \left|\int_0^1 f(t) F_{p_{n_k}}(t)\, dt\right| \ge \frac{c}{n_k}\left|\int_0^1 f_{n_k}(t) F_{p_{n_k}}(t)\, dt\right| -$$

$$-\sum_{i=0}^{k-1} \frac{c}{n_i}\left|\int_0^1 f_{n_i}(t) F_{p_{n_k}}(t)\, dt\right| - \sum_{i=k+1}^{\infty} \frac{c}{n_i}\left|\int_0^1 f_{n_i}(t) F_{p_{n_k}}(t)\, dt\right| =$$

$$= I - II - III. \tag{5}$$

From Lemma 2 we get

$$I = \frac{c}{n_k}\int_0^1 \varphi_{n_k}(t) \left|F_{p_{n_k}}(t)\right| dt \ge \frac{c}{n_k}\left\|F_{p_{n_k}}\right\|_1 \ge c > 0. \tag{6}$$

Since [4]

$$\|S_n(f) - f\|_C \le c\omega\left(\frac{1}{n}, f\right)_C \log(n+1)$$

and

$$\frac{\omega(\delta, f)}{\delta} \le 2\frac{\omega(\delta', f)}{\delta'}, \quad 0 < \delta' < \delta,$$

we write

$$\|t_n(f)-f\|_C \leq \frac{c}{\log n}\sum_{i=1}^{n-1}\frac{\|S_i(f)-f\|_C}{n-i}$$

$$\leq \frac{c}{\log n}\sum_{i=1}^{n-1}\frac{\omega\left(\frac{1}{i},f\right)_C \log(i+1)}{n-i} \leq$$

$$\leq c\sum_{i=1}^{n-1}\frac{i\omega\left(\frac{1}{i},f\right)_C}{i(n-i)} \leq c\omega\left(\frac{1}{n},f\right)_C \sum_{i=1}^{n-1}\frac{n}{i(n-i)}$$

$$= c\omega\left(\frac{1}{n},f\right)_C \sum_{i=1}^{n-1}\left(\frac{1}{i}+\frac{1}{(n-i)}\right) \leq c\omega\left(\frac{1}{n},f\right)_C \log(n+1). \tag{7}$$

It is easy to have

$$\omega\left(f_{n_i},\frac{1}{2^{2n_k}}\right)=O\left(\frac{2^{2n_i}}{2^{2n_k}}\right),\quad i=1,2...,k-1.$$

Then from (3), (4) and (7) we get

$$II \leq c\sum_{i=0}^{k-1}\frac{1}{n_i}\omega\left(f_{n_i},\frac{1}{2^{2n_k}}\right)n_k =$$

$$= O\left(\frac{n_k}{2^{2n_k}}\sum_{i=0}^{k-1}\frac{2^{2n_i}}{n_i}\right)=O\left(\frac{n_k}{2^{2n_k}}\frac{2^{2n_{k-1}}}{n_{k-1}}\right)=o(1) \quad \text{as } k\to\infty. \tag{8}$$

Since

$$\|F_n\|_1 = O\left(\frac{1}{\log n}\sum_{i=1}^{n-1}\frac{\|D_i\|_1}{n-i}\right)=$$

$$= O\left(\frac{1}{\log n}\sum_{i=1}^{n-1}\frac{\log(i+1)}{n-i}\right)=O(\log(n+1))$$

by (3) we have

$$III = O\left(\sum_{i=k+1}^{\infty}\frac{1}{n_i}\left\|F_{p_{n_k}}\right\|_1\right)=O\left(\frac{n_k}{n_{k+1}}\right)=o(1) \quad \text{as } k\to\infty. \tag{9}$$

After substituting (6), (8) and (9) in (5) we obtain

$$\overline{\lim_{k\to\infty}}\left|t_{p_{n_k}}(f_0,0)-f_0(0)\right|>0.$$

Theorem 4 is proved. □

Proof of Theorem 5. We choose a monotonically increasing sequence of positive integers $\{m_k : k\geq 1\}$ such that

$$2m_{k-1}\leq m_k, \tag{10}$$

$$\sum_{l=1}^{k-1} \frac{2^{2m_l}}{m_l} < \frac{2^{2m_k}}{m_k}. \tag{11}$$

We construct a function g defined as follows. Set

$$g(x) = \sum_{j=1}^{\infty} g_j(x),$$

where

$$g_j(x) = \frac{D_{2^{2m_j+1}}(x)}{m_j}.$$

First we prove that

$$\omega(\delta, g)_L = O\left(\frac{1}{\log(1/\delta)}\right). \tag{12}$$

For every $\delta > 0$ there exists a positive integer k such that

$$2^{-2m_k} \le \delta < 2^{-2m_{k-1}}.$$

Since $(l = 1, 2, ..., k-1, \delta > 0)$

$$\int_0^1 \left|D_{2^{2m_l+1}}(x+\delta) - D_{2^{2m_l+1}}(x)\right| dx$$

$$= \int_0^{2^{-2m_l-1}-\delta} \left|D_{2^{2m_l+1}}(x+\delta) - D_{2^{2m_l+1}}(x)\right| dx$$

$$+ \int_{2^{-2m_l-1}-\delta}^{2^{-2m_l-1}} \left|D_{2^{2m_l+1}}(x+\delta) - D_{2^{2m_l+1}}(x)\right| dx$$

$$+ \int_{2^{-2m_l-1}}^{1-\delta} \left|D_{2^{2m_l+1}}(x+\delta) - D_{2^{2m_l+1}}(x)\right| dx$$

$$+ \int_{1-\delta}^{1} \left|D_{2^{2m_l+1}}(x+\delta) - D_{2^{2m_l+1}}(x)\right| dx$$

$$= \int_{2^{-2m_l-1}-\delta}^{2^{-2m_l-1}} D_{2^{2m_l+1}}(x) + \int_{1-\delta}^{1} D_{2^{2m_l+1}}(x+\delta)\, dx = 22^{2m_l+1}\delta, \tag{13}$$

from (10) and (11) we get

$$\int_I |g(x+\delta) - g(x)|\, dx$$

$$\le \sum_{l=1}^{k-1} \frac{1}{m_l} \int_I \left|D_{2^{2m_l+1}}(x+\delta) - D_{2^{2m_l+1}}(x)\right| dx$$

$$+2\sum_{l=k}^{\infty}\frac{1}{m_l}=O\left(\delta\sum_{l=1}^{k-1}\frac{2^{2m_l}}{m_l}\right)+O\left(\frac{1}{m_k}\right)$$

$$=O\left(\delta\frac{2^{2m_{k-1}}}{m_{k-1}}\right)+O\left(\frac{1}{m_k}\right)=O\left(\frac{1}{\log(1/\delta)}\right)$$

which proves (12).

It is evident that

$$\left\|t_{p_{m_k}}(g)-g\right\|_L\geq\left\|t_{p_{m_k}}\left(\sum_{i=k}^{\infty}g_i\right)\right\|_L-\sum_{i=k}^{\infty}\|g_i\|_L-$$

$$-\left\|t_{p_{m_k}}\left(\sum_{i=1}^{k-1}g_i\right)-\sum_{i=1}^{k-1}g_i\right\|_L=$$

$$=I-II-III. \tag{14}$$

The estimation of $\|t_n(g)-g\|_L$ is analogous to estimation of $\|t_n(g)-g\|_C$ (see (7)) and from (10), (11), (13) we have

$$III\leq\sum_{i=1}^{k-1}\left\|t_{p_{m_k}}(g_i)-g_i\right\|_L=O\left(\sum_{i=1}^{k-1}\omega\left(g_i,\frac{1}{2^{2m_k}}\right)_L m_k\right)=$$

$$=O\left(\frac{m_k}{2^{2m_k}}\sum_{i=1}^{k-1}\frac{2^{2m_i}}{m_i}\right)=O\left(\frac{m_k}{2^{2m_k}}\frac{2^{2m_{k-1}}}{m_{k-1}}\right)=o(1)\qquad\text{as}\quad k\to\infty. \tag{15}$$

From (10) we get

$$II\leq\sum_{i=k}^{\infty}\frac{1}{m_i}=O\left(\frac{1}{m_k}\right)=o(1)\quad\text{as}\quad k\to\infty. \tag{16}$$

It is easy to have

$$t_{p_{m_k}}(g_i)=\frac{1}{m_i}F_{p_{m_k}}(x),\quad i=k,k+1,...$$

Consequently, by Lemma 2 we get

$$\left\|t_{p_{m_k}}\left(\sum_{i=k}^{\infty}g_i\right)\right\|_L=\sum_{i=k}^{\infty}\frac{1}{m_i}\left\|F_{p_{m_k}}\right\|_L\geq$$

$$\geq c\frac{1}{m_k}\left\|F_{p_{m_k}}\right\|\geq c>0. \tag{17}$$

Combining (14)-(17) we conclude that

$$\overline{\lim_{k\to\infty}}\left\|t_{p_{m_k}}(g)-g\right\|_L>0.$$

□

Proof of Theorem 7. It is well-known that operator S_n is of weak type (L^1,L^1), and of type (L^p,L^p) for all $1<p<\infty$, uniformly in n. This implies the inequality

$$\|S_n(f)\|_L \leq c\|f\|_{L\log^+ L} + c \tag{18}$$

for all $f \in L\log^+ L$. (See for instance the book of Bennett and Sharpley [2]). Since

$$\|t_n(f)\|_L \leq \frac{c}{\log(n+1)} \sum_{k=1}^{n-1} \frac{\|S_k(f)\|_L}{n-k}$$

from (18) we get

$$\|t_n(f)\|_L \leq c\|f\|_{L\log^+ L} + c. \tag{19}$$

Later, we will need it. Let (A_j) be a sequence of natural numbers, and $(\lambda_j) \in l^1$ a sequence of positive reals. Set

$$h_j := D_{2^{2A_j+1}}, \quad p_{A_j} := 2^{2A_j} + 2^{2A_j-2} + \cdots + 2^0 \quad (j \in \mathbb{N}).$$

Obviously, the function $h := \sum_{j=0}^{\infty} \lambda_j h_j$ is an element of L. It is easy to have

$$t_{p_{A_j}}(h_{j+i}, x) = F_{p_{A_j}}(x)$$

for all $i = 0, 1, \ldots$. Consequently, by Lemma 2 we have

$$\|t_{p_{A_j}}(\sum_{i=0}^{\infty} \lambda_{j+i} h_{j+i})\|_L = \sum_{i=0}^{\infty} \lambda_{j+i} \|F_{p_{A_j}}\|_L \geq cA_j \sum_{i=0}^{\infty} \lambda_{j+i} \geq cA_j\lambda_j.$$

By inequality (19) we also have the upper bound

$$\begin{aligned}
&\|t_{p_{A_j}}(\sum_{i=1}^{j} \lambda_{j-i} h_{j-i})\|_L \\
&\leq \sum_{i=1}^{j} \lambda_{j-i} \|t_{p_{A_j}} h_{j-i}\|_L \\
&\leq c\sum_{i=1}^{j} \lambda_{j-i}(\|h_{j-i}\|_{L\log^+ L} + 1) \\
&\leq c\sum_{i=1}^{j} \lambda_{j-i} A_{j-i}.
\end{aligned}$$

By the last two inequalities we get

$$\|t_{p_{A_j}} h\|_L \geq cA_j\lambda_j - c\sum_{i<j} \lambda_i A_i.$$

Let

$$\delta_j := \sup\left\{t > 0 : \delta\left(\frac{4^{j^2}}{t} 2^{2t}\right) > \frac{1}{8^{j^2}}\right\}.$$

If the set the supremum of which is taken is empty (that is, $\delta \leq 1/8^{j^2}$ for each positive t), then let $\delta_j := 1$ Since the function δ is vanishing at plus infinity, then the sequence (δ_j) is well-defined. We can define the sequence (A_j) and (λ_j) in the following way $(A_{-1} := 0)$:

$$A_j := \max\left\{6^{j^2}, A_{j-1} + 1, \delta_j + 1\right\}, \quad \lambda_j := \frac{4^{j^2}}{A_j}.$$

Then obviously $(\lambda_j) \in l^1$, and consequently $f \in L^1$. Besides,

$$\|t_{p_{A_j}} h\|_L \geq cA_j\lambda_j - c\sum_{i<j} \lambda_i A_i \geq c4^{j^2} - c\sum_{i<j} 4^{i^2} \geq c4^{j^2}.$$

This gives $\limsup_j \|t_{p_{A_j}} h\|_L = \infty$. The rest is to prove that $h \in L\log^+ L\delta(L)$. Since

$$\sum_{i\leq j} \lambda_i 2^{2A_i+1} = \sum_{i\leq j} \frac{4^{i^2}}{A_i} 2^{2A_i+1} \leq c\frac{4^{j^2}}{A_j} 2^{2A_j},$$

then we have

$$\log^+\left(\sum_{i\leq j} \lambda_i 2^{2A_i+1}\right) \leq \log^+\left(c\frac{4^{j^2}}{A_j} 2^{2A_j}\right) \leq cA_j.$$

We also conclude from $\sum_{i\leq j} \lambda_i 2^{2A_i+1} \geq \lambda_j 2^{2A_j+1} = \frac{4^{j^2}}{A_j} 2^{2A_j+1}$ that the inequality

$$\delta\left(\sum_{i\leq j} \lambda_i 2^{2A_i+1}\right) \leq \frac{1}{8^{j^2}}$$

holds. Finally, from the above we get

$$\begin{aligned}
&\|h\|_{L\log^+ L} \\
&= \sum_{j=0}^{\infty} \int_{I_{2A_j+1}\setminus I_{2A_j+2}} \sum_{i\leq j} \lambda_i 2^{2A_i+1} \log^+\left(\sum_{i\leq j} \lambda_i 2^{2A_i+1}\right) \delta\left(\sum_{i\leq j} \lambda_i 2^{2A_i+1}\right) dx \\
&= c\sum_{j=0}^{\infty} \frac{1}{2^{2A_j}} \cdot \frac{4^{j^2}}{A_j} 2^{2A_j} \cdot A_j \cdot \frac{1}{8^{j^2}} \\
&\leq c.
\end{aligned}$$

This completes the proof of Theorem 7. □

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