ON ALMOST EVERYWHERE CONVERGENCE AND DIVERGENCE OF MARCINKIEWICZ-LIKE MEANS OF INTEGRABLE FUNCTIONS WITH RESPECT TO THE TWO-DIMENSIONAL WALSH SYSTEM

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ABSTRACT. Let $|n|$ be the lower integer part of the binary logarithm of the positive integer n and $\alpha : \mathbb{N}^2 \to \mathbb{N}^2$. In this paper we generalize the notion of the two dimensional Marcinkiewicz means of Fourier series of two-variable integrable functions as $t_n^\alpha f := \frac{1}{n}\sum_{k=0}^{n-1} S_{\alpha(|n|,k)} f$ and give a kind of necessary and sufficient condition for functions in order to have the almost everywhere relation $t_n^\alpha f \to f$ for all $f \in L^1([0,1)^2)$ with respect to the Walsh-Paley system. The original version of the Marcinkiewicz means are defined by $\alpha(|n|,k) = (k,k)$ and discussed a lot of authors. See for instance [13, 8, 6, 3, 11].

First, we give a brief introduction to the theory of the Walsh-Fourier series.

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$, and $I := [0,1)$. For any set E let E^2 the cartesian product $E \times E$. Thus $\mathbb{N}^2$ is the set of integral lattice points in the first quadrant and I^2 is the unit square. Let $E^1 = E$ and fix $j = 1$ or 2. Denote the j -dimensional Lebesgue measure of any set $E \subset I^j$ by $\mathrm{mes}(E)$. Denote the $L^p(I^j)$ norm of any function f by $\|f\|_p$ $(1 \le p \le \infty)$.

Denote the dyadic expansion of $n \in \mathbb{N}$ and $x \in I$ by $n = \sum_{j=0}^\infty n_j 2^j$ and $x = \sum_{j=0}^\infty x_j 2^{-j-1}$ (in the case of $x = \frac{k}{2^m}$ $k, m \in \mathbb{N}$ choose the expansion which terminates in zeros). n_i, x_i are the i-th coordinates of n, x, respectively. Set $e_i := 1/2^{i+1} \in I$, the i th coordinate of e_i is 1, the rest are zeros $(i \in \mathbb{N})$. Define the dyadic addition $+$ as

$$x + y = \sum_{j=0}^\infty |x_j - y_j| 2^{-j-1}.$$

The sets $I_n(x) := \{y \in I : y_0 = x_0, ..., y_{n-1} = x_{n-1}\}$ for $x \in I$, $I_n := I_n(0)$ for $n \in \mathbb{P}$ and $I_0(x) := I$ are the dyadic intervals of I. The set of the dyadic intervals on I is denoted by $\mathcal{I} := \{I_n(x) : x \in I, n \in \mathbb{N}\}$. Denote by $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)\,(x \in I)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n\,(n \in \mathbb{N})$. C denotes a constant which may be different from line to line.

For $t = (t^1, t^2) \in I^2$, $b = (b^1, b^2) \in \mathbb{N}^2$ set the two-dimensional dyadic rectangle, i.e. two-dimensional dyadic interval

$$I_b(t) := I_{b^1}(t^1) \times I_{b^2}(t^2).$$

For $n = (n^1, n^2) \in \mathbb{N}^2$ denote by $E_n = E_{n^1,n^2}$ the two-dimensional expectation operator with respect to the σ algebra $\mathcal{A}_n = \mathcal{A}_{n^1,n^2}$ generated by the two-dimensional rectangles $I_{n^1}(x^1) \times I_{n^2}(x^2)\,(x = (x^1, x^2) \in I^2)$. For $n \in \mathbb{P}$ denote by $|n| := \max(j \in \mathbb{N} : n_j \ne 0)$, that

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is, $2^{|n|} \leq n < 2^{|n|+1}$. The Rademacher functions on I are defined as:

$$r_n(x) := (-1)^{x_n} \quad (x \in I,\, n \in \mathbb{N}).$$

The Walsh-Paley system (on I) is defined as the sequence of the Walsh-Paley functions:

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k}, \quad (x \in I,\, n \in \mathbb{N}).$$

That is, $\omega := (\omega_n, n \in \mathbb{N})$. (For details see Fine [1].)

Consider the Dirichlet and the Fejér kernel functions:

$$D_n := \sum_{k=0}^{n-1} \omega_k,\ \ K_n := \frac{1}{n}\sum_{k=0}^{n-1} D_k,\ \ D_0, K_0 := 0.$$

The Fourier coefficients, the n-th partial sum of the Fourier series, the n-th $(C,1)$ mean of $f \in L^1(I)$:

$$\hat{f}(n) := \int_I f(x)\omega_n(x)\,dx \ (n \in \mathbb{N}),$$

$$S_n f(y) := \sum_{k=0}^{n-1} \hat{f}(k)\omega_k(y) = \int_I f(x+y) D_n(x)\,dx = f * D_n(y),$$

$$\sigma_n f(y) := \frac{1}{n}\sum_{k=0}^{n-1} S_k f(y) = \int_I f(x+y) K_n(x)\,dx = f * K_n(y),\ (n \in \mathbb{P}, y \in I).$$

Moreover, for $n \in \mathbb{N}$ we have ([10, page 7])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{otherwise} \end{cases}$$

and for $n \in \mathbb{P}$ ([10, page 28])

$$D_n(x) = \omega_n(x) \sum_{i=0}^{|n|} n_i r_i(x) D_{2^i}(x). \tag{1}$$

Then, this gives $S_{2^n} f(y) = 2^n \int_{I_n(y)} f(x)dx = E_n f(y)\, (n \in \mathbb{N})$. We say that an operator $T : L^1(I^j) \to L^0(I^j)$ ($L^0(I^j)$ is the space of measurable functions on I^j) is of type (L^p, L^p) (for $1 \leq p \leq \infty$) if $\|Tf\|_p \leq C_p \|f\|_p$ with some constant C_p depending only on p for all $f \in L^p(I^j)$. We say that T is of weak type (L^1, L^1) if $\text{mes}\{|Tf| > \lambda\}| \leq C\|f\|_1/\lambda$ for all $f \in L^1(I^j)$ and $\lambda > 0$ $(j = 1, 2)$. The two-dimensional Walsh-Paley functions, Dirichlet, Fejér and Marcinkiewicz kernels are defined as follows:

$$\omega_m(x) = \omega_{m^1}(x^1)\omega_{m^2}(x^2), \quad D_m(x) = D_{m^1}(x^1) D_{m^2}(x^2) \quad (m \in \mathbb{N}^2),$$

$$K_m(x) = K_{m^1}(x^1) K_{m^2}(x^2), \quad M_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} D_{k,k}(x) \quad (m \in \mathbb{P}^2, n \in \mathbb{P}),$$

where $x \in I^2$. Moreover, the two-dimensional Fourier coefficients, the m-th $(m \in \mathbb{N}^2)$ rectangular partial sum of the Fourier series, the m-th $(m \in \mathbb{P}^2)$ $(C,1)$ mean and the n-th

($n \in \mathbb{P}$) Marcinkiewicz mean of $f \in L^1(I^2)$:

$$\hat{f}(m) := \int_{I^2} f(x)\omega_m(x)\,dx,$$

$$S_m f(y) := \sum_{k^1=0}^{m^1-1}\sum_{k^2=0}^{m^2-1} \hat{f}(k^1,k^2)\omega_{(k^1,k^2)}(y) = \int_{I^2} f(x+y)D_m(x)\,dx,$$

$$\sigma_m f(y) := \frac{1}{m^1 m^2}\sum_{k^1=0}^{m^1-1}\sum_{k^2=0}^{m^2-1} S_k f(y) = \int_{I^2} f(x+y)K_m(x)\,dx,$$

$$t_n f(y) := \frac{1}{n}\sum_{k=0}^{n-1} S_{k,k} f(y) = \int_{I^2} f(x+y)M_n(x)\,dx \quad (y \in I^2).$$

Many papers investigate the behavior of the convergence (and some the divergence) properties of the two dimensional Fejér means with respect to the trigonometric or the Walsh system. We mention the papers [7], [4] (trigonometric) and [9], [2] (Walsh-Paley system). This is another story and also very interesting to discuss the almost everywhere convergence of the Marcinkiewicz means $\frac{1}{n}\sum_{j=0}^{n-1} S_{j,j} f$ of integrable functions with respect to orthonormal systems. Although, this mean is defined for two-variable functions, in the view of almost everywhere convergence there are similarities with the one-dimensional case. On the one side, the maximal convergence space for two dimensional Fejér means (no restriction on the set of indices other than they have to converge to $+\infty$) is $L\log^+ L$ ([4, 2]), and on the other side, for the Marcinkiewicz means we have a.e. convergence for every integrable functions (for the trigonometric, Walsh Paley systems).

We mention that the first result is due to Marcinkiewicz [8]. But he proved „only" for functions in the space $L\log^+ L$ the a.e. relation $t_n f \to f$ with respect to the trigonometric system. The „L^1 result" for the trigonometric, Walsh-Paley, and the so called bounded Vilenkin systems see the papers of Zhizhiasvili [13] (trigonometric system), Weisz [11] (Walsh system), Goginava [6, 5] (Walsh system) and Gát [3] (Vilenkin systems). Some of these results (including the proofs) can also be found in [12].

After then, we turn our attention to the generalization of Marcinkiewicz means. Let $\alpha = (\alpha_1, \alpha_2) : \mathbb{N}^2 \to \mathbb{N}^2$ be a function. Define the following Marcinkiewicz-like kernels and means:

$$M_n^\alpha(x) := \frac{1}{n}\sum_{k=0}^{n-1} D_{\alpha_1(|n|,k)}(x^1)D_{\alpha_2(|n|,k)}(x^2), \quad t_n^\alpha f := f * M_n^\alpha \quad (f \in L^1(I^2), n \in \mathbb{P}).$$

The main aim of this paper is to describe the functions α for we have the a.e. convergence relation $t_n^\alpha f \to f$ for each integrable two-variable function f.

The following properties will play a prominent role in this investigation project. ($\#B$ denotes the cardinality of set B.) Roughly speaking they will be necessary and sufficient condition.

$$\#\{l \in \mathbb{N} : \alpha_j(|n|,l) = \alpha_j(|n|,k), l < n\} \le C \quad (k < n, n \in \mathbb{P}, j = 1,2) \tag{2}$$

$$\max\{\alpha_j(|n|,k) : k < n\} \le Cn \quad (n \in \mathbb{P}, j = 1,2). \tag{3}$$

More precisely, we prove the „theorem of convergence"

Theorem 1. *Let α satisfy (2) and (3). Then we have $t_n^\alpha f \to f$ for each $f \in L^1(I^2)$.*

Condition (2) is clearly a necessary one in the following sense. Let $\alpha_1(|n|,k)=0$, $\alpha_2(|n|,k)=k$ for every $n,k\in\mathbb{N}$. Then (3) is satisfied and (2) is not. It is very simple to give a function $f\in L^1(I^2)$ such as $t_n^\alpha f\to f$ fails to hold a.e. To construct an α with (2) which fails to satisfy (3) and a $f\in L^1(I^2)$ such that $t_n^\alpha f$ does not converge to f a.e. is more complicated.

The „theorem of divergence" aims to show that (3) is also a necessary condition in certain sense. That is, we prove

Theorem 2. *Let $\gamma:\mathbb{N}\to\mathbb{N}$ be any function with property $\gamma(+\infty)=+\infty$. Then there exists a function α satisfying (2),*

$$\max\{\alpha_1(|n|,k):k<n\}\le Cn,\quad \max\{\alpha_2(|n|,k):k<n\}\le Cn\gamma(n)\quad (n\in\mathbb{P})$$

and $f\in L^1(I^2)$ such that $\limsup_{n\in\mathbb{N}}|t_n^\alpha f|=+\infty$ almost everywhere.

Of course it would have been possible to write the conditions as $\alpha_1\le Cn\gamma(n)$ and $\alpha_2\le Cn$. We give a corollary of Theorem 1.

Corollary 3. *Let (a_n) be a lacunary sequence of reals, i.e. $a_{n+1}\ge a_nq$ for some $q>1$ $(n\in\mathbb{N})$ and α satisfy conditions (2) and $\alpha_j(n,k)\le Ca_n$ $(k<a_n,\ j=1,2)$ (modified version of condition (3)). Then for every integrable function $f\in L^1(I^2)$ we have*

$$\frac{1}{a_n}\sum_{k=0}^{a_n-1}S_{\alpha_1(n,k),\alpha_2(n,k)}f(x)\to f(x)$$

for a.e. $x\in I^2$.

Proof. The proof of this corollary runs as follows. Let $b_n=\lceil\log_2 a_n\rceil$ and

$$\tilde\alpha_j(b_n,k)=\begin{cases}\alpha(n,k), & \text{if } 0\le k<a_n,\\ k, & \text{if } a_n\le k<2^{b_n}\end{cases}\quad (j=1,2).$$

Then, $\tilde\alpha$ satisfies conditions (2) (trivially) and (3) since for $k<a_n$, $\tilde\alpha_j(b_n,k)=\alpha_j(n,k)\le Ca_n\le C2^{b_n}$. By Theorem 8 (see in this paper below) it follows that for the maximal operator $t_*^{\tilde\alpha}f:=\sup|t_n^{\tilde\alpha}f|$ we have $\operatorname{mes}\left\{t_*^{\tilde\alpha}f\ge\lambda\right\}\le C\|f\|_1/\lambda$ for all $f\in L^1(I^2)$ and $\lambda>0$. Since

$$\frac{1}{a_n}\sum_{k=0}^{a_n-1}S_{\alpha_1(n,k),\alpha_2(n,k)}f=\frac{2^{b_n}}{a_n}\frac{1}{2^{b_n}}\sum_{k=0}^{2^{b_n}-1}S_{\tilde\alpha_1(b_n,k),\tilde\alpha_2(b_n,k)}f-\frac{2^{b_n}}{a_n}\frac{1}{2^{b_n}}\sum_{k=a_n}^{2^{b_n}-1}S_{k,k}f,$$

and consequently, $|t_{a_n}^\alpha f|\le 2|t_{2^{b_n}}^{\tilde\alpha}f|+2|t_{2^{b_n}}^{\mathrm{id}}f|+2|t_{a_n}^{\mathrm{id}}f|$, then (id denotes the „identical function", i.e. $\mathrm{id}(n,k)=(k,k)$) $t_*^\alpha f\le Ct_*^{\tilde\alpha}f+Ct_*f$. The ordinary maximal Marcinkiewicz operator is of weak type (L^1,L^1) (see e.g. [3]) and this by standard argument [10] completes the proof of this corollary. □

Now, we turn our attention to the proof of the convergence theorem. Our first main aim is to prove that the operator $t_*^\alpha f:=\sup_{n\in\mathbb{P}}|t_n^\alpha f|$ is of weak type (L^1,L^1). In order to have this we need a sequence of lemmas. The first, which - may say - the very base of the proof of Theorem 1 is the most difficult one. However, the techniques of its proof will also be used in the proof of the forthcoming lemmas.

Denote for $k\in\mathbb{N}$ $J_k=I_k\setminus I_{k+1}$ and $n^s:=\sum_{k=s}^\infty n_k2^k$ $(n,s\in\mathbb{N})$. $n^0=n, n^{|n|+1}=0$.

Lemma 4. *Let $a \in \mathbb{N}$. Then*

$$\sum_{t^1=0}^{a}\sum_{t^2=t^1}^{\infty}\int_{J_{t^1}\times J_{t^2}}\sup_{A\geq a}\sup_{|n|=A}\frac{1}{2^A}\sum_{s=t^1}^{A}\left|\sum_{k=0}^{2^s-1}D_{\alpha_1(A,n^{s+1}+k)}(x^1)D_{\alpha_2(A,n^{s+1}+k)}(x^2)\right|dx \leq C.$$

Proof. First, for fixed $t=(t^1,t^2)$, s, A we discuss the integral

$$\int_{J_{t^1}\times J_{t^2}}\sup_{|n|=A}\left|\sum_{k=0}^{2^s-1}D_{\alpha_1(A,n^{s+1}+k)}(x^1)D_{\alpha_2(A,n^{s+1}+k)}(x^2)\right|dx.$$

Check the function $\sum_{k=0}^{2^s-1}D_{\alpha_1(A,n^{s+1}+k)}(x^1)D_{\alpha_2(A,n^{s+1}+k)}(x^2)$ on the set $J_{t^1}\times J_{t^2}$. Since we have $x^2\in J_{t^2}$, then by (1) we have $|D_j(x^2)|\leq 2^{t^2}$ for each $j\in\mathbb{N}$ and consequently $|D_{\alpha_2(A,n^{s+1}+k)}(x^2)|\leq 2^{t^2}$. On the other hand, again by (1) for $x^1\in J_{t^1}$ we have

$$D_{\alpha_1(A,n^{s+1}+k)}(x^1)=\omega_{[\alpha_1(A,n^{s+1}+k)]^{t^1+1}}(x^1)\left(\sum_{j=0}^{t^1-1}[\alpha_1(A,n^{s+1}+k)]_j2^j-[\alpha_1(A,n^{s+1}+k)]_{t^1}2^{t^1}\right)$$
$$=:\omega_{[\alpha_1(A,n^{s+1}+k)]^{t^1+1}}(x^1)[\alpha_1(A,n^{s+1}+k)]_{t^1}^{\sim}.$$

Apply the Cauchy-Buniakovski-Schwarz inequality:

$$\int_{J_{t^2}}\left[\int_{J_{t^1}}\sup_{|n|=A}\left|\sum_{k=0}^{2^s-1}D_{\alpha_1(A,n^{s+1}+k)}(x^1)D_{\alpha_2(A,n^{s+1}+k)}(x^2)\right|dx^1\right]dx^2$$
$$\leq\int_{J_{t^2}}2^{-t^1/2}\left[\int_{J_{t^1}}\sup_{|n|=A}\left|\sum_{k=0}^{2^s-1}D_{\alpha_1(A,n^{s+1}+k)}(x^1)D_{\alpha_2(A,n^{s+1}+k)}(x^2)\right|^2dx^1\right]^{1/2}dx^2$$
$$=\int_{J_{t^2}}2^{-t^1/2}\Bigg[\int_{J_{t^1}}\sup_{|n|=A}\sum_{k,l=0}^{2^s-1}\omega_{[\alpha_1(A,n^{s+1}+k)]^{t^1+1}}(x^1)\omega_{[\alpha_1(A,n^{s+1}+l)]^{t^1+1}}(x^1)$$
$$\times[\alpha_1(A,n^{s+1}+k)]_{t^1}^{\sim}[\alpha_1(A,n^{s+1}+l)]_{t^1}^{\sim}D_{\alpha_2(A,n^{s+1}+k)}(x^2)D_{\alpha_2(A,n^{s+1}+l)}(x^2)dx^1\Bigg]^{1/2}dx^2=:B^1.$$

Since n^{s+1} depends only on $n_{s+1},\dots,n_{A-1}$ (recall that $n_A=1$), then the supremum $\sup_{|n|=A}$ above also depends only on $n_{s+1},\dots,n_{A-1}$. Thus,

$$B^1 \leq \int_{J_{t^2}} 2^{-t^1/2} \left[\sum_{n_{A-1}=0}^{1} \cdots \sum_{n_{s+1}=0}^{1} \int_{J_{t^1}} \sum_{k,l=0}^{2^s-1} \omega_{[\alpha_1(A,n^{s+1}+k)]^{t^1+1}}(x^1)\omega_{[\alpha_1(A,n^{s+1}+l)]^{t^1+1}}(x^1) \right.$$

$$\left. \times [\alpha_1(A,n^{s+1}+k)]_{t^1}^{\sim}[\alpha_1(A,n^{s+1}+l)]_{t^1}^{\sim} D_{\alpha_2(A,n^{s+1}+k)}(x^2)D_{\alpha_2(A,n^{s+1}+l)}(x^2)dx^1 \right]^{1/2} dx^2$$

$$= \int_{J_{t^2}} 2^{-t^1/2} \left[\sum_{n_{A-1}=0}^{1} \cdots \sum_{n_{s+1}=0}^{1} \sum_{k,l=0}^{2^s-1} [\alpha_1(A,n^{s+1}+k)]_{t^1}^{\sim}[\alpha_1(A,n^{s+1}+l)]_{t^1}^{\sim} \right.$$

$$\times D_{\alpha_2(A,n^{s+1}+k)}(x^2)D_{\alpha_2(A,n^{s+1}+l)}(x^2)$$

$$\left. \times \int_{J_{t^1}} \omega_{[\alpha_1(A,n^{s+1}+k)]^{t^1+1}}(x^1)\omega_{[\alpha_1(A,n^{s+1}+l)]^{t^1+1}}(x^1)dx^1 \right]^{1/2} dx^2 =: B^2$$

Discuss the integral

$$\int_{J_{t^1}} \omega_{[\alpha_1(A,n^{s+1}+k)]^{t^1+1}}(x^1)\omega_{[\alpha_1(A,n^{s+1}+l)]^{t^1+1}}(x^1)dx^1.$$

If it differs from zero, then the t^1+1-th, t^1+2-th, ... coordinates of $\alpha_1(A,n^{s+1}+k)$ and $\alpha_1(A,n^{s+1}+l)$ should be equal. Since (2) we have that for every k there exists only a bounded number of l's for which $\alpha_1(A,n^{s+1}+k)=\alpha_1(A,n^{s+1}+l)$. These facts give that for every k there exists - at most - $C2^{t^1}$ number of l's for which this integral is not zero.

Consequently,

$$B^2 \leq C\int_{J_{t^2}} 2^{-t^1/2} \left[\sum_{n_{A-1}=0}^{1} \cdots \sum_{n_{s+1}=0}^{1} 2^{s+t^1}2^{2t^1}2^{2t^2}2^{-t^1} \right]^{1/2} \leq C\sqrt{2^{A+t^1}}.$$

This means

$$\int_{J_{t^1}\times J_{t^2}} \sup_{|n|=A} \left| \sum_{k=0}^{2^s-1} D_{\alpha_1(A,n^{s+1}+k)}(x^1)D_{\alpha_2(A,n^{s+1}+k)}(x^2) \right| dx \leq C\sqrt{2^{A+t^1}}.$$

This inequality immediately gives ($a \vee b = \max(a,b)$)

$$\sum_{t^1=0}^{a}\sum_{t^2=t^1}^{\infty}\int_{J_{t^1}\times J_{t^2}} \sup_{A\geq a\vee(t^2-C)} \sup_{|n|=A} \frac{1}{2^A}\sum_{s=t^1}^{A}\left|\sum_{k=0}^{2^s-1} D_{\alpha_1(A,n^{s+1}+k)}(x^1)D_{\alpha_2(A,n^{s+1}+k)}(x^2)\right| dx$$
$$\leq C\sum_{t^1=0}^{a}\sum_{t^2=t^1}^{\infty}\sum_{A=a\vee(t^2-C)}^{\infty}\sum_{s=t^1}^{A}\sqrt{2^{t^1-A}}$$
$$\leq C\sum_{t^1=0}^{a}\sum_{t^2=t^1}^{\infty}\sum_{A=a\vee(t^2-C)}^{\infty}(A-t^1+1)\sqrt{2^{t^1-A}}$$
$$\leq C\sum_{t^1=0}^{a}\sum_{t^2=t^1}^{\infty}((a\vee t^2)-t^1)\sqrt{2^{t^1-(a\vee t^2)}}$$
$$\leq C\sum_{t^1=0}^{a}\sum_{t^2=t^1}^{a}(a-t^1)\sqrt{2^{t^1-a}} + C\sum_{t^1=0}^{a}\sum_{t^2=a+1}^{\infty}(t^2-t^1)\sqrt{2^{t^1-t^2}} \leq C.$$

This inequality shows that if we want to complete the proof of this lemma, then we have to discuss also the case $\sup_{t^2-C>A\geq a}$. This follows that t^2 should be at least $a+C$. That is, we have to prove that the following integral is bounded.

$$\sum_{t^1=0}^{a}\sum_{t^2=a+C}^{\infty}\int_{J_{t^1}\times J_{t^2}} \sup_{t^2-C>A\geq a} \sup_{|n|=A} \frac{1}{2^A}\sum_{s=t^1}^{A}\left|\sum_{k=0}^{2^s-1} D_{\alpha_1(A,n^{s+1}+k)}(x^1)D_{\alpha_2(A,n^{s+1}+k)}(x^2)\right| dx =: B^3.$$

The method we are going to use in order to discuss B^3 is the same as we used for the investigation of B^1. The only difference is that in the situation of B^1 we used the estimation $|D_{\alpha_2(A,n^{s+1}+k)}(x^2)| \leq 2^{t^2}$ and in the case of B^3 we use - by the help of (3) and the formula of the Dirichlet kernel D_n (1) - the estimation $|D_{\alpha_2(A,n^{s+1}+k)}(x^2)| \leq C2^A$. The other steps of this process are the same. That is,

$$B^3 \leq C\sum_{t^1=0}^{a}\sum_{t^2=a+C}^{\infty}\int_{J_{t^2}}\sum_{A=a}^{t^2-C}\frac{1}{2^A}\sum_{s=t^1}^{A}2^{-t^1/2}\left[\sum_{n_{A-1}=0}^{1}\cdots\sum_{n_{s+1}=0}^{1}2^{2t^1}2^{2A}2^{s+t^1}2^{-t^1}\right]^{1/2}dx^2$$
$$= C\sum_{t^1=0}^{a}\sum_{t^2=a+C}^{\infty}\sum_{A=a}^{t^2-C}\sum_{s=t^1}^{A}2^{-t^2-A-t^1/2}\sqrt{2^{A-s+2t^1+2A+s}}$$
$$= C\sum_{t^1=0}^{a}\sum_{t^2=a+C}^{\infty}\sum_{A=a}^{t^2-C}\sum_{s=t^1}^{A}2^{A/2+t^1/2-t^2}$$
$$\leq C\sum_{t^1=0}^{a}\sum_{t^2=a+C}^{\infty}\sum_{A=a}^{t^2-C}(A-t^1+1)2^{A/2+t^1/2-t^2}$$
$$\leq C\sum_{t^1=0}^{a}\sum_{t^2=a+C}^{\infty}(t^2-t^1+1)2^{t^1/2-t^2/2} \leq C.$$

This completes the proof of Lemma 4. □

In the sequel with the application of Lemma 4 we prove the main tool with respect to the maximal Marcinkiewicz-like kernel in order to prove that the maximal operator t_*^α is quasi-local (for the definition of quasi-locality see e.g. [10, page 262]) and consequently it is of weak type (L^1, L^1).

Lemma 5. *Let $a \in \mathbb{N}$. Then*

$$\int_{I^2\setminus(I_a\times I_a)} \sup_{n\geq a-C} |M_n^\alpha(x)|dx \leq C.$$

Proof. For $t^1 \leq a-1, t^2 \geq t^1$ and $x \in J_{t^1} \times J_{t^2}$ by (1) and (3) it is clear that

$$|D_{\alpha_1(A,n^{s+1}+k)}(x^1)D_{\alpha_2(A,n^{s+1}+k)}(x^2)| \leq C2^{t^1+(t^2\wedge A)}.$$

This gives

$$\sum_{t^1=0}^{a}\sum_{t^2=t^1}^{\infty}\int_{J_{t^1}\times J_{t^2}} \sup_{A\geq a-C}\sup_{|n|=A}\frac{1}{2^A}\sum_{s=0}^{t^1}\left|\sum_{k=0}^{2^s-1} D_{\alpha_1(A,n^{s+1}+k)}(x^1)D_{\alpha_2(A,n^{s+1}+k)}(x^2)\right|dx$$

$$\leq \sum_{t^1=0}^{a}\sum_{t^2=t^1}^{\infty}\int_{J_{t^1}\times J_{t^2}} \sup_{A\geq a-C}\frac{1}{2^A}\sum_{s=0}^{t^1} 2^{s+t^1+(t^2\wedge A)}dx$$

$$\leq \sum_{t^1=0}^{a}\sum_{t^2=t^1}^{a-C}\frac{1}{2^{t^1+t^2}}\sup_{A\geq a-C} 2^{2t^1+t^2-A} + \sum_{t^1=0}^{a}\sum_{t^2=a-C}^{\infty}\frac{1}{2^{t^1+t^2}}2^{2t^1}$$

$$\leq C\sum_{t^1=0}^{a}\sum_{t^2=t^1}^{a-C} 2^{t^1-a} + C\sum_{t^1=0}^{a}\sum_{t^2=a-C}^{\infty} 2^{t^1-t^2} \leq C.$$

This by equality

$$M_n^\alpha(x) = \frac{1}{n}\sum_{s=0}^{A} n_s \sum_{k=0}^{2^s-1} D_{\alpha_1(A,n^{s+1}+k)}(x^1)D_{\alpha_2(A,n^{s+1}+k)}(x^2)$$

and Lemma 4 immediately gives

$$\sum_{t^1=0}^{a}\sum_{t^2=t^1}^{\infty}\int_{J_{t^1}\times J_{t^2}} \sup_{n\geq a-C}|M_n^\alpha(x)|dx \leq C.$$

Similarly, we can also have

$$\sum_{t^2=0}^{a}\sum_{t^1=t^2}^{\infty}\int_{J_{t^1}\times J_{t^2}} \sup_{n\geq a-C}|M_n^\alpha(x)|dx \leq C.$$

If we prove the almost everywhere relation

$$I^2 \setminus (I_a\times I_a) \subset \left(\bigcup_{t^1=0}^{a}\bigcup_{t^2=t^1}^{\infty} J_{t^1}\times J_{t^2}\right)\bigcup\left(\bigcup_{t^2=0}^{a}\bigcup_{t^1=t^2}^{\infty} J_{t^1}\times J_{t^2}\right) =: J^1\bigcup J^2,$$

then the proof of Lemma 5 would be complete. This will be quite easy. Let $x = (x^1, x^2) \in I^2 \setminus (I_a \times I_a)$. Then, either x^1 or x^2 (or both) is not an element of I_a. Say, $x^1 \notin I_a$. Then $x \in J_{t^1}$ for some $t^1 < a$. If $x^2 \in I_a$ and $x^2 \neq 0$, then $x \in J^1$. If $x^1 \in J_{t^1}$ and $x^2 \notin I_a$, then $x^1 \in J_{t^1}$ and $x^2 \in J_{t^2}$ for some $t^1, t^2 < a$. For $t^2 \geq t^1$ we have $x \in J^1$ and for $t^1 \geq t^2$ we have

$x \in J^2$. This procedure can be done if $x^1, x^2 \neq 0$. The set of the points $x = (x^1, x^2)$, where either $x^1 = 0$ or $x^2 = 0$ is a zero measure set, so the a.e. relation $I^2 \setminus (I_a \times I_a) \subset J^1 \times J^2$ is proved. That is, the proof of Lemma 5 is really complete. □

Corollary 6. *Let $n \in \mathbb{P}$. Then*

$$\|M_n^\alpha\|_1 \leq C.$$

Proof. By Lemma 5 we have

$$\int_{I^2 \setminus (I_{|n|} \times I_{|n|})} M_n^\alpha \leq C.$$

Besides, (3) and (1) gives

$$|M_n^\alpha(x)| \leq \frac{1}{n} \sum_{k=0}^{n-1} |D_{\alpha_1(|n|,k)}(x^1)||D_{\alpha_2(|n|,k)}(x^2)| \leq C \frac{1}{n} \sum_{k=0}^{n-1} 2^{|n|} \cdot 2^{|n|} \leq C 2^{2|n|}.$$

Hence,

$$\int_{I_{|n|} \times I_{|n|}} M_n^\alpha \leq C$$

and this completes the proof of Corollary 6. □

Now, we can prove that the maximal operator t_*^α is quasi-local (for the definition of quasi-locality see e.g. [10, page 262]) and then a bit later the fact that it is of weak type (L^1, L^1). In other words:

Lemma 7. *Let $f \in L^1(I^2)$, $\operatorname{supp} f \subset I_a(u^1) \times I_a(u^2)$, $\int f = 0$ for some $u \in I^2$ and $a \in \mathbb{N}$. Then*

$$\int_{I^2 \setminus (I_a(u^1) \times I_a(u^2))} t_*^\alpha f(x) dx \leq C \|f\|_1.$$

Proof. From the shift invariancy of the Lebesgue measure we can suppose that $u^1 = u^2 = 0$. If $|n| \leq a - C$ for some fixed constant $C > 0$ depending only on α_1, α_2, then we have by (3) that $\alpha_1(|n|, k), \alpha_1(|n|, k) < 2^a$ for every $k < n$. Consequently, the kernel $M_n^\alpha(x^1, x^2)$ (which is a linear combination of two-dimensional Walsh-Paley functions $\omega_{j,k}$ with $j, k < 2^a$) is $\mathcal{A}_{a,a}$ measurable. This implies

$$t_n^\alpha f(y) = \int_{I_a \times I_a} f(x) M_n^\alpha(y + x) dx = M_n^\alpha(y) \int_{I_a \times I_a} f(x) dx = 0.$$

That is, $|n| \geq a - C$ can be supposed. By the theorem of Fubini and Lemma 5 we get

$$\begin{aligned}
\int_{I^2 \setminus I_a^2} t_*^\alpha f &= \int_{I^2 \setminus I_a^2} \sup_{|n| \geq a - C} |t_n^\alpha f| \\
&= \int_{I^2 \setminus I_a^2} \sup_{|n| \geq a - C} \left| \int_{I_a^2} f(x) M_n^\alpha(y + x) dx \right| dy \\
&\leq \int_{I_a^2} |f(x)| \int_{I^2 \setminus I_a^2} \sup_{|n| \geq a - C} |M_n^\alpha(z) dz| dx \leq C \int_{I_a^2} |f(x)| dx = C \|f\|_1.
\end{aligned}$$

This completes the proof of Lemma 7. □

Theorem 8. *The operator t_*^α is of weak type (L^1, L^1) and it is also of type (L^p, L^p) for all $1 < p \leq \infty$.*

Proof. Now, we know that operator t_*^α is of type (L^∞, L^∞) which is given by Corollary 6 and it is quasi-local (Lemma 7). Consequently, to prove that operator t_*^α is of weak type (L^1, L^1) is nothing else but to follow the standard argument (see e.g. [10]). Finally, the interpolation lemma of Marcinkiewicz (see e.g. [10]) gives that it is also of type (L^p, L^p) for all $1 < p \le \infty$. □

Proof of Theorem 1. Next, we turn our attention to the proof of the theorem of convergence, that is, Theorem 1. This is also a trivial consequence of the fact that the maximal operator t_*^α is of weak type (L^1, L^1) and the fact that Theorem 1 holds for each two-dimensional Walsh-Paley polynomial (which is also very easy to see). □

Next we turn our attention to the divergence project. In the sequel we give some necessary preliminary assumptions.

In order to prove the divergence theorem, that is Theorem 2 we need several lemmas and some preliminary assumptions. In one of the final steps of the construction of the counterexample function we will need an almost everywhere convergence result with respect to some „absolute Marcinkiewicz" means. Recall that $M_n = \frac{1}{n}\sum_{k=0}^{n-1} D_{k,k}$ denotes the nth (original) Marcinkiewicz kernel and $t_n f = f * M_n$ the nth Marcinkiewicz mean of function f. Set $\tilde{t}_n f := f * |M_{2^n}|$. We prove for the maximal operator $\tilde{t}_* f := \sup_L |\tilde{t}_{2^L} f|$ the following lemma.

Lemma 9. *Operator $\tilde{t}_*$ is of weak type (L^1, L^1) and it is of type (L^p, L^p) for $1 < p \le \infty$.*

Proof. The proof that $\tilde{t}_*$ is of type (L^∞, L^∞) is quite easy. It comes from the fact that $\|M_{2^L}\|_1 \le C$ (see e.g. [3]) as

$$\|\tilde{t}_* f\|_\infty \le \|f\|_\infty \|M_{2^L}\|_1 \le C\|f\|_\infty.$$

Next, we prove that $\tilde{t}_*$ is quasi-local. Recall the definition of quasi-locality. Let $f \in L^1(I^2)$, $\operatorname{supp} f \subset I_k(u^1) \times I_k(u^2)$, $\int_{I^2} f = 0$ for some $u = (u^1, u^2) \in I^2$. Then we have prove

$$\int_{I^2 \setminus [I_k(u^1) \times I_k(u^2)]} \tilde{t}_* f \le C\|f\|_1.$$

(For more on quasi-locality see the book [10, page 262].) This means that $\tilde{t}_*$ is quasi-local. By the shift invariancy of the Lebesgue measure it can be supposed that $u^1 = u^2 = 0$. If $L \le k$, then

$$\tilde{t}_{2^L} f(y) = \int_{I^2} f(x)|M_{2^L}(y+x)|dx = \int_{I_k \times I_k} f(x)|M_{2^L}(y+x)|dx = |M_{2^L}(y)| \int_{I_k \times I_k} f(x)dx = 0.$$

In [3, Lemma 3] one can find

$$\int_{I^2 \setminus I_k^2} \sup_{n \ge 2^k} |M_n| \le C$$

and consequently,

$$\int_{I^2 \setminus I_k^2} \sup_{L \ge k} |M_{2^L}| \le C$$

of course. By the above written $L > k$ can be supposed. This gives

$$\begin{aligned}\int_{I^2\setminus I_k^2} \tilde{t}_* f &= \int_{I^2\setminus I_k^2} \sup_{L\geq k} \left| \int_{I_k^2} f(x)|M_{2^L}(y+x)|dx \right| dy \\ &\leq \int_{I^2\setminus I_k^2} \int_{I_k^2} |f(x)| \sup_{L\geq k} |M_{2^L}(y+x)|dxdy \\ &= \int_{I_k^2} |f(x)| \int_{I^2\setminus I_k^2} \sup_{L\geq k} |M_{2^L}(z)|dzdx \leq C\|f\|_1.\end{aligned}$$

That is, the sublinear operator $\tilde{t}_*$ is quasi-local and it is of type (L^∞, L^∞). By standard argument [10] it follows that it is of weak type (L^1, L^1). The Marcinkiewicz interpolation theorem gives that $\tilde{t}_*$ is of type (L^p, L^p) for every $1 < p \leq \infty$. This completes the proof of this lemma. □

For $L \in \mathbb{N}$ let $\phi_L : I^2 \times I^2 \to \{z \in \mathbb{C} : |z| = 1\}$ be Lebesgue measurable and also suppose that $E_{L,L}\phi_L(x, y) = 0$ for all $x, y \in I^2$. (The conditional expectation operator $E_{L,L}$ is taken with respect to variable $x \in I^2$.)

Define the following operator:

$$t_{2^L}^{\phi_L} f(y) := \int_{I^2} f(x)\phi_L(x, y)M_{2^L}(y + x)dx$$

and its maximal function $t_*^\phi f := \sup_L |t_{2^L}^{\phi_L} f|$. In the sequel we prove

Lemma 10. *Operator $t_*^\phi f$ is weak type (L^1, L^1) and it is of type (L^p, L^p) for every $1 < p \leq \infty$. Moreover, $\lim_{L\to+\infty} t_{2^L}^{\phi_L} f = 0$ a.e. for all $f \in L^1(I^2)$.*

Proof. Since $|\phi_L| = 1$, then we have

$$t_*^\phi f(y) \leq \sup_L \int_{I^2} |f(x)||M_{2^L}(y + x)|dx = \tilde{t}_*|f|(y)$$

and by Lemma 9 we have

$$\text{mes}(t_*^\phi f > \lambda) \leq \text{mes}(\tilde{t}_*|f| > \lambda) \leq C\||f|\|_1/\lambda = C\|f\|_1/\lambda$$

and

$$\|t_*^\phi f\|_p \leq \|\tilde{t}_*|f|\|_p \leq C_p\||f|\|_p = C_p\|f\|_p.$$

That is, operator $t_*^\phi f$ is of weak type (L^1, L^1) and it is of type (L^p, L^p) for every $1 < p \leq \infty$. Now, we are to prove the a.e. equality $\lim_{L\to+\infty} t_{2^L}^{\phi_L} f = 0$.

This equality holds for every two-dimensional Walsh polynomial P since if for instance $P \in \mathcal{P}_{2^L,2^L}$ ($\mathcal{P}_{N,M} = \{\sum_{j<N,k<M} c_{j,k}\omega_{j,k}\}$), that is, P is $\mathcal{A}_{L,L}$ measurable, then for $N \geq L$ we have

$$\begin{aligned}t_{2^N}^{\phi_N} P(y) &= E_{0,0}[P(.)\phi_N(., y)M_{2^N}(y + .)] = E_{0,0}(E_{N,N}[P(.)\phi_N(., y)M_{2^N}(y + .)]) \\ &= E_{0,0}\left(P(.)M_{2^N}(y + .)\left(E_{N,N}\phi_N(., y)\right)\right) = 0.\end{aligned}$$

Since the maximal operator $\sup_L |t^{\phi_L} f|$ is of weak type (L^1, L^1), then by standard argument [10] we have that $\lim_L t^{\phi_L} f = 0$ a.e. for every $f \in L^1(I^2)$. □

Let $a/2, \tau \in \mathbb{N}$ and define a subset of the set of two-dimensional intervals $\mathcal{I} \times \mathcal{I}$:

$$\mathcal{I}_{a,\tau}(x) := \left\{ I_L(x^1) \times I_{a-L}(x^2) : \frac{a}{2} - \tau \le L \le \frac{a}{2} \right\} \quad (x \in I^2).$$

It is easy to have $\bigcap \mathcal{I}_{a,\tau}(x) = I_{\frac{a}{2}}(x^1) \times I_{\frac{a}{2}+\tau}(x^2)$, $\operatorname{mes}(\bigcap \mathcal{I}_{a,\tau}(x)) = 2^{-a-\tau}$. Moreover, $F \in \mathcal{I}_{a,\tau}(x)$ implies $\operatorname{mes}(F) = 2^{-a}$.

We remark that since

$$\bigcup_{L=a/2-\tau}^{a/2} I_L(x^1) = I_{a/2-\tau}(x^1), \qquad \bigcup_{L=a/2-\tau}^{a/2} I_{a-L}(x^2) = I_{a/2}(x^2),$$

then the smallest dyadic rectangle containing $\bigcup \mathcal{I}_{a,\tau}(x)$ is $I_{a/2-\tau}(x^1) \times I_{a/2}(x^2)$. Then, we prove:

Lemma 11. $\operatorname{mes}(\bigcup \mathcal{I}_{a,\tau}(x)) = \frac{1+\tau/2}{2^a}$.

Proof. Denote (for the sake of this proof, only)

$$\mu_k = \operatorname{mes}\left(\bigcup_{L=a/2-\tau}^{a/2-\tau+k} I_L(x^1) \times I_{a-L}(x^2) \right) \quad (k = 0, 1, \dots, \tau).$$

Then for $k \ge 1$ we have

$$\mu_k = \mu_{k-1} + \operatorname{mes}\left(I_{a/2-\tau+k}(x^1) \times I_{a/2+\tau-k}(x^2)\right) - \operatorname{mes}\left(\bigcup_{L=a/2-\tau}^{a/2-\tau+k-1} I_{a/2-\tau+k}(x^1) \times I_{a-L}(x^2) \right)$$

$$= \mu_{k-1} + 2^{-a} - \operatorname{mes}(I_{a/2-\tau+k}(x^1) \times I_{a/2+\tau-k+1}(x^2))$$

$$= \mu_{k-1} + 2^{-a} - 2^{-a-1} = \mu_{k-1} + 2^{-a-1}.$$

This gives

$$\operatorname{mes}(\bigcup \mathcal{I}_{a,\tau}(x)) = \mu_\tau = \mu_0 + \tau 2^{-a-1} = 2^{-a} + \tau 2^{-a-1} = \frac{1+\tau/2}{2^a}.$$

This completes the proof of Lemma 11. □

Let (a_n) and (τ_n) be strictly monotone increasing sequences of natural numbers (besides, a_n is even) satisfying

$$a_{n-1}/2 < a_n/2 - \tau_n \quad (1 \le n \in \mathbb{N}). \tag{4}$$

Define the sets $J_{a_n,\tau_n}(t), \Omega_{a_n,\tau_n}(t)$ recursively ($t \in I^2, n \in \mathbb{N}$):

$$J_{a_0,\tau_0} := \{t\}, \quad \Omega_{a_0,\tau_0}(t) := \bigcup \mathcal{I}_{a_0,\tau_0}(t) = \bigcup_{L=a_0/2-\tau_0}^{a_0/2} I_L(t^1) \times I_{a_0-L}(t^2).$$

Suppose that $J_{a_j,\tau_j}(t), \Omega_{a_j,\tau_j}(t)$ are defined for $j < n$. Then decompose

$$\left(I_{a_0/2-\tau_0}(t^1) \times I_{a_0/2}(t^2)\right) \setminus \bigcup_{j=0}^{n-1} \Omega_{a_j,\tau_j}(t)$$

as the disjoint union of dyadic rectangles of the form $I_{a_n/2-\tau_n}(x^1) \times I_{a_n/2}(x^2)$. (By (4) it is possible to do.) Take from each dyadic rectangle an element to represent. The set of x's corresponding to these rectangles is $J_{a_n,\tau_n}(t)$. That is,

$$\big(I_{a_0/2-\tau_0}(t^1) \times I_{a_0/2}(t^2)\big) \setminus \bigcup_{j=0}^{n-1} \Omega_{a_j,\tau_j}(t) = \bigcup_{x\in J_{a_n,\tau_n}(t)} I_{a_n/2-\tau_n}(x^1) \times I_{a_n/2}(x^2)$$

Then,

$$\Omega_{a_n,\tau_n}(t) := \bigcup_{x\in J_{a_n,\tau_n}(t)} \bigcup \mathfrak{I}_{a_n,\tau_n}(x).$$

In the sequel we prove the following a.e. relation:

Lemma 12. *If $\sum_{n=0}^{\infty} \frac{\tau_n}{2^{\tau_n}} = +\infty$, then the following a.e. equality holds*

$$I_{a_0/2-\tau_0}(t^1) \times I_{a_0/2}(t^2) = \bigcup_{n=0}^{\infty} \Omega_{a_n,\tau_n}(t).$$

Proof. Determine the „filling ratio” of the set $\Omega_{a_n,\tau_n}(t)$. For $n = 0$ this is equal to

$$\frac{\mathrm{mes}(\bigcup \mathfrak{I}_{a_0,\tau_0}(t))}{\mathrm{mes}(I_{a_0/2-\tau_0}(t^1) \times I_{a_0/2}(t^2))} = \frac{\frac{1+\tau_0/2}{2^{a_0}}}{\frac{2^{\tau_0}}{2^{a_0}}} = \frac{2+\tau_0}{2^{\tau_0+1}}.$$

Taking account the procedure of the construction of the sets $\Omega_{a_n,\tau_n}(t)$ we have that when we step from $n-1$ to n, we decompose the „remaining part” of the rectangle $I_{a_0/2-\tau_0}(t^1)\times I_{a_0/2}(t^2)$. In other words, denoting

$$\beta_n = \frac{2+\tau_n}{2^{\tau_n+1}}, \quad \mu_n = \mathrm{mes}(\Omega_{a_n,\tau_n}(t)) \quad (n \in \mathbb{N})$$

we get

$$\begin{aligned}
\mu_0 &= \beta_0 \,\mathrm{mes}(I_{a_0/2-\tau_0}(t^1) \times I_{a_0/2}(t^2)) = \beta_0 \frac{2^{\tau_0}}{2^{a_0}} =: \beta_0 \mu_{-1}\\
\mu_1 &= \beta_1(\mu_{-1} - \mu_0)\\
\mu_2 &= \beta_2(\mu_{-1} - \mu_0 - \mu_1), \ldots\\
\mu_n &= \beta_n(\mu_{-1} - \mu_0 - \cdots - \mu_{n-1}) \quad (1 \le n \in \mathbb{N}).
\end{aligned}$$

By induction (with respect to n) we prove that

$$\mu_{-1} - \sum_{k=0}^{n-1} \mu_k = \prod_{k=0}^{n-1} (1-\beta_k) \cdot \mu_{-1}.$$

If $n = 0$, then both sides are μ_{-1} (empty sum is zero, empty product is one). Suppose that we have this equality for every nonnegative integer up to $n-1$. Then prove it for n in the following way:

$$\mu_n = \beta_n(\mu_{-1} - \sum_{k=0}^{n-1} \mu_k) = \beta_n \prod_{k=0}^{n-1}(1-\beta_k)\mu_{-1}.$$

Thus,

$$\mu_{-1} - \sum_{k=0}^{n} \mu_k = \mu_{-1} - \sum_{k=0}^{n-1} \mu_k - \mu_n = \prod_{k=0}^{n-1}(1-\beta_k)\mu_{-1} - \beta_n \prod_{k=0}^{n-1}(1-\beta_k)\mu_{-1} = \prod_{k=0}^{n}(1-\beta_k)\mu_{-1}.$$

That is, the equality is proved.

Since $\sum_{k=0}^{\infty}\beta_k = \sum_{k=0}^{\infty}\frac{2+\tau_k}{2^{\tau_k+1}} = +\infty$, then we have $\prod_{k=0}^{\infty}(1-\beta_k) = 0$ and consequently $\lim_{n\to\infty}(\mu_{-1} - \sum_{k=0}^{n-1}\mu_k) = 0$. This can be verified easily for instance as ($\beta_n \to 0$):

$$0 \le \prod_{k=0}^{n}(1-\beta_k) = \frac{1}{\prod_{k=0}^{n}\frac{1}{1-\beta_k}} = \frac{1}{\prod_{k=0}^{n}(1+\frac{\beta_k}{1-\beta_k})} \le \frac{1}{\prod_{k=0}^{n}(1+\beta_k)} \le \frac{1}{\sum_{k=0}^{n}\beta_k} \to 0.$$

Recall that $\mu_n = \operatorname{mes}(\Omega_{a_n,\tau_n}(t))$ and also the fact that sets $\Omega_{a_n,\tau_n}(t)$ $(n \in \mathbb{N})$ are disjoint. This gives

$$\operatorname{mes}\left(I_{a_0/2-\tau_0}(t^1) \times I_{a_0/2}(t^2) \setminus \bigcup_{n=0}^{\infty}\Omega_{a_n,\tau_n}(t)\right) = \mu_{-1} - \sum_{k=0}^{\infty}\mu_k = 0.$$

This completes the proof of Lemma 12. □

Let $t = (\sum_{j=0}^{a_0/2-\tau_0-1} t_j^1 e_j^1, \sum_{k=0}^{a_0/2-1} t_k^2 e_k^2) \in I^2$. Lemma 12 gives

$$I^2 = \bigcup_{\substack{t_j^1, t_k^2 \in \{0,1\} \\ j<a_0/2-\tau_0,\, k<a_0/2}} \bigcup_{n=0}^{\infty}\Omega_{a_n,\tau_n}(t) =: \bigcup_{t\in T}\bigcup_{n=0}^{\infty}\Omega_{a_n,\tau_n}(t).$$

After then, we define functions $f_{a,\tau}(x) : I^2 \to [0,+\infty)$ as

$$f_{a,\tau}(x) := \begin{cases} \tau_n 2^{\tau_n}, & \text{if there exists a } \ t \in T, n \in \mathbb{N}, y \in J_{a_n,\tau_n}(t) \text{ such that } \ x \in \bigcap \mathfrak{I}_{a_n,\tau_n}(y), \\ 0, & \text{otherwise.} \end{cases}$$

In other words, denoting by 1_B the characteristic function of set $B \subset I^2$ we have

$$f_{a,\tau}(x) = \sum_{t\in T}\sum_{n=0}^{\infty}\sum_{y\in J_{a_n,\tau_n}(t)} \tau_n 2^{\tau_n} 1_{I_{a_n/2}(y^1)\times I_{a_n/2+\tau_n}(y^2)}(x).$$

Since functions $f_{a,\tau}$ will serve as fundamental functions in the counterexample function, then it is necessary prove that they are Lebesgue integrable.

Lemma 13. *For all $(a_n), (\tau_n)$ above we have*

$$\|f_{a,\tau}\|_1 \le 2.$$

Proof.

$$\begin{aligned}
\int_{I^2} |f_{a,\tau}(x)|dx &= \sum_{t\in T}\sum_{n=0}^{\infty}\sum_{y\in J_{a_n,\tau_n}(t)} \tau_n 2^{\tau_n}\operatorname{mes}(1_{I_{a_n/2}(y^1)\times I_{a_n/2+\tau_n}(y^2)} = 1)\\
&= \sum_{t\in T}\sum_{n=0}^{\infty}\sum_{y\in J_{a_n,\tau_n}(t)} \tau_n 2^{\tau_n}\operatorname{mes}(\bigcap \mathcal{J}_{a_n,\tau_n}(y))\\
&= \sum_{t\in T}\sum_{n=0}^{\infty}\sum_{y\in J_{a_n,\tau_n}(t)} \frac{2}{1+\frac{2}{\tau_n}}\operatorname{mes}(\bigcup \mathcal{J}_{a_n,\tau_n}(y))\\
&\le \sum_{t\in T}\sum_{n=0}^{\infty}\sum_{y\in J_{a_n,\tau_n}(t)} 2\operatorname{mes}(\bigcup \mathcal{J}_{a_n,\tau_n}(y))\\
&= 2\sum_{t\in T}\sum_{n=0}^{\infty}\operatorname{mes}(\Omega_{a_n,\tau_n}(t))\\
&\le 2\sum_{t\in T}\operatorname{mes}(I_{a_0/2-\tau_0}(t^1)\times I_{a_0/2}(t^2)) = 2\operatorname{mes}(I^2) = 2.
\end{aligned}$$

□

The following lemma also plays a prominent role in verifying that the counterexample function (will be given later) is really a counterexample function.

Lemma 14. *Let $(a_n), (\tau_n)$ and set T as above. Then, for almost every $x \in I^2$ there exists a unique $t \in T, n \in \mathbb{N}, y \in J_{a_n,\tau_n}(t)$ such that $x \in I_L(y^1) \times I_{a_n-L}(y^2)$ for an $L \in \{a_n/2 - \tau_n, a_n/2 - \tau_n + 1, \dots, a_n/2\}$. Besides,*

$$\frac{1}{2^L}\sum_{k=0}^{2^L-1} S_{k,2^{a_n-L}} f_{a,\tau}(x) \ge \tau_n/4 \ge \tau_0/4. \tag{5}$$

Proof. Recall the a.e. equality

$$x \in \bigcup_{t\in T}\bigcup_{n=0}^{\infty}\Omega_{a_n,\tau_n}(t).$$

The construction of the sets $\Omega_{a_n,\tau_n}(t)$ immediately gives that $x \in I_L(y^1) \times I_{a_n-L}(y^2)$ for an $L \in \{a_n/2 - \tau_n, a_n/2 - \tau_n + 1, \dots, a_n/2\}$ for a unique $t \in T, n \in \mathbb{N}, y \in J_{a_n,\tau_n}(t)$. The only thing to prove is relation (5). This will be an easy consequence of the fact that $f_{a,\tau}$ is nonnegative everywhere and so do the one-dimensional kernels $K_{2^L}, D_{2^{a_n-L}}$ (for this see e.g. [10, page 47]).

(6)

$$\begin{aligned}
&\frac{1}{2^L}\sum_{k=0}^{2^L-1} S_{k,2^{a_n-L}} f_{a,\tau}(x)\\
&= \int_{I\times I} f_{a,\tau}(z) K_{2^L}(x^1+z^1) D_{2^{a_n-L}}(x^2+z^2)dz^1dz^2\\
&= \sum_{t\in T}\sum_{n=0}^{\infty}\sum_{v\in J_{a_n,\tau_n}(t)} \tau_n 2^{\tau_n}\int_{I\times I} 1_{I_{a_n/2}(v^1)\times I_{a_n/2+\tau_n}(v^2)}(z) K_{2^L}(x^1+z^1) D_{2^{a_n-L}}(x^2+z^2)dz^1dz^2\\
&\geq \tau_n 2^{\tau_n}\int_{I\times I} 1_{I_{a_n/2}(y^1)\times I_{a_n/2+\tau_n}(y^2)}(z) K_{2^L}(x^1+z^1) D_{2^{a_n-L}}(x^2+z^2)dz^1dz^2\\
&= \tau_n 2^{\tau_n}\int_{I_{a_n/2}(y^1)\times I_{a_n/2+\tau_n}(y^2)} K_{2^L}(x^1+z^1) D_{2^{a_n-L}}(x^2+z^2)dz^1dz^2.
\end{aligned}$$

Since $x^1+z^1\in I_L$, then we have $K_{2^L}(x^1+z^1)=\frac{2^L-1}{2}$ (see [10, page 47]). Consequently the right side of (6) equals with

$$\tau_n 2^{\tau_n}\int_{I_{a_n/2}(y^1)\times I_{a_n/2+\tau_n}(y^2)} \frac{2^L-1}{2}2^{a_n-L}dz^1dz^2 \geq \tau_n/4.$$

Recall that $a_n/2-\tau_n\leq L\leq a_n/2$ and consequently, $a_n-L\leq a_n/2+\tau_n$ Thus $x^2+z^2\in I_{a_n-L}, D_{2^{a_n-L}}(x^2+z^2)=2^{a_n-L}$. This completes the proof of Lemma 14. □

Finally, we give the construction of the divergence example.

The proof of Theorem 2. Let sequences $(a_n),(\tau_n)$ be defined as above and define the following sequences:

$$\tau^0 := (\tau_0,\tau_1,\dots),\ a^0 := (a_0,a_1,\dots),$$

$$\tau^i := (\tau_k,\tau_{k+1},\dots),\ a^i := (a_k,a_{k+1},\dots),\text{ where }\quad k:=\min\left\{j\in\mathbb{N}:\tau_j>i^3\right\}, i\in\mathbb{P}.$$

It is very obvious by Lemma 13 that $f:=\sum_{i=1}^{\infty}\frac{1}{i^2}f_{a^i,\tau^i}\in L^1(I^2)$. Moreover, $\|f\|_1\leq 2\sum_{i=1}^{\infty}\frac{1}{i^2}<4$.

Fix an $i\in\mathbb{P}$. Use the notation $\tau^i=(\tau_0^i,\tau_1^i,\dots), a^i=(a_0^i,a_1^i,\dots)$. Then let

$$T^i=\left\{t=(\sum_{j=0}^{a_0^i/2-\tau_0^i-1}t_j^1e_j^1, \sum_{k=0}^{a_0^i/2-1}t_k^2e_k^2) : t_j^1,t_k^2\in\{0,1\}, j<a_0^i/2-\tau_0^i, k<a_0^i/2\right\}.$$

Apply Lemma 14 with respect to sequence $a=a^i$. Then, for a.e. $x\in I^2$ (we have $I^2=\bigcup_{t\in T^i}\bigcup_{n=0}^{\infty}\Omega_{a_n^i,\tau_n^i}(t)$ a.e.) there exists a unique $t\in T^i, n\in\mathbb{N}, y\in J_{a_n^i,\tau_n^i}(t)$ such that $x\in I_L(y^1)\times I_{a_n^i-L}(y^2)$ for a $a_n^i/2-\tau_n^i\leq L\leq a_n^i/2$.

By using the fact that each f_{a^i,τ^i} and one-dimensional kernels $K_{2^L}, D_{2^{a_n^i-L}}$ are nonnegative everywhere by (5) in Lemma 14 we have

$$\frac{1}{2^L}\sum_{k=0}^{2^L-1}S_{k,2^{a_n^i-L}}f(x)\geq\frac{1}{i^2}\frac{1}{2^L}\sum_{k=0}^{2^L-1}S_{k,2^{a_n^i-L}}f_{a^i,\tau^i}(x)\geq\frac{1}{i^2}\tau_0^i/4>i/4.$$

Since $a_n^i = a_k, \tau_n^i = \tau_k$ for some $k \in \mathbb{N}$ and

$$a_{k-1}/2 < a_k/2 - \tau_k = a_n^i/2 - \tau_n^i \le L \le a_n^i/2 = a_k/2 < a_{k+1}/2 - \tau_{k+1},$$

then we have that for each $L \in \mathbb{N}$ there is maximum one a_n^i such that $a_n^i/2 - \tau_n^i \le L \le a_n^i/2$. Therefore, this a_n^i can be denoted as $A(L)$. Denote the set of L's arisen from this process above by Λ.

Consequently, we get

$$\limsup_{L\in\Lambda, L\to+\infty} \frac{1}{2^L} \sum_{k=0}^{2^L-1} S_{k,2^{A(L)-L}} f(x) = +\infty \tag{7}$$

almost everywhere. Let $\alpha_1(L,k) = k$, $\alpha_2(L,k) = 2^{A(L)-L} + k$ $(0 \le k < 2^L, L \in \Lambda)$. For the Marcinkiewicz-like kernel $M_{2^L}^\alpha(z)$ we have

$$\begin{aligned} M_{2^L}^\alpha(z) &= \frac{1}{2^L} \sum_{k=0}^{2^L-1} D_k(z^1) D_{2^{A(L)-L}+k}(z^2) \\ &= \frac{1}{2^L} \sum_{k=0}^{2^L-1} D_k(z^1) D_{2^{A(L)-L}}(z^2) + \frac{1}{2^L} \sum_{k=0}^{2^L-1} D_k(z^1) r_{A(L)-L}(z^2) D_k(z^2). \end{aligned}$$

Recall that $A(L) - L = a_n^i - L \ge L$ and therefore $E_{L,L} r_{A(L)-L}(x+z) = 0$. By Lemma 10 we have that

$$\lim_{L\in\Lambda, L\to+\infty} \int_{I^2} f(z) r_{A(L)-L}(x+z) M_{2^L}(x+z) dz = 0$$

a.e. for each $f \in L^1(I^2)$. This equality with (7) gives

$$\limsup_{L\in\Lambda, L\to+\infty} |t_{2^L}^\alpha f| = +\infty$$

a.e. Moreover, let $\alpha_j(L,k) = k$ for $L \notin \Lambda, k < 2^L$ $(j = 1,2)$. For $L \in \Lambda$ we also have $\frac{\alpha_2(L,k)}{2^L} \le \frac{2^{A(L)-L}+k}{2^L} < 2^{a_n^i - 2L} + 1 \le 2^{2\tau_n^i} + 1$. Since $\tau_n^i = \tau_k$ for some k, $a_n^i/2 - \tau_n^i \le L \le a_n^i/2$ and the strictly monotone increasing sequence a can be as big as we want with respect to τ, then they can be given as $2^{2\tau_n^i} + 1 \le \gamma(2^L)$. That is, $\frac{\alpha_j(L,k)}{2^L} \le \gamma(2^L)$ for all $L \in \mathbb{N}$ and $j = 1,2$. This completes the proof of the divergence Theorem. □

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