Automorphisms of a non-metacyclic minimal nonabelian p-group, p odd

János Kurdics kurdics@nyf.hu kurdics.uni.hu

Nyíregyháza College, Hungary

09-14 July 2012, Bedlewo (Poland)

Problem: (Y. Berkovich and Z. Janko 2009) Describe the automorphism groups of minimal nonabelian finite p-groups.

Theorem

(L. Rédei 1947) A finite p-group G is a minimal nonabelian group if and only if either it is isomorphic to the quaternion group of order 8, or (A) has the presentation $\langle a, b \mid a^{p^m} = b^{p^n} = 1,\ a^b = a^{1+p^{m-1}} \rangle$, *where* $m \geq 2$, $n \geq 1$, $|G| = p^{m+n}$, *or (B) has the presentation* $\langle a, b \mid a^{p^m} = b^{p^n} = c^p = 1,\ [a, b] = c,\ [a, c] = 1,\ [b, c] = 1 \rangle$, *where* $|G| = p^{m+n+1}$.

Theorem

Let the group H be finite, presented by

$$\langle g_1, \dots, g_r \,|\, w_1(g_1, \dots, g_r), \dots, w_s(g_1, \dots, g_r)\rangle.$$

Then there is a bijective correspondence between automorphisms of the group H and ordered r-tuples $({g'}_1, \dots, {g'}_r)$ of elements generating the group H for which all the relations $w_j({g'}_1, \dots, {g'}_r) = 1$ hold.

Theorem

Let $G = \langle a, b \mid a^{p^m} = 1,\ b^{p^n} = 1,\ a^b = a^{1+p^{m-1}} \rangle$ *with* $m > n$, $n \geq 1$, $|G| = p^{m+n}$. *Let the automorphism* α *be induced by the substitution* $a \mapsto a^t$ *with* t *primitive root modulo* p^m $(1 < t < p^m)$ *and* $b \mapsto b$; *the automorphism* β *by the substitution* $a \mapsto a$, $b \mapsto b^{1+p}$; *the automorphism* γ *by the substitution* $a \mapsto ab$, $b \mapsto b$; *the automorphism* δ *by the substitution* $a \mapsto a$, $b \mapsto a^{p^{m-n}} b$.

Then the automorphism group $\mathrm{Aut}(G)$ is presented with generators α, β, γ and δ, and with generating relations

$$|\alpha| = \varphi(p^m), \ |\beta| = p^{n-1}, \ |\gamma| = p^n, \ |\delta| = p^n, \quad (1)$$

$$\alpha^{-1}\beta\alpha = \beta, \quad (2) \qquad \alpha^{-1}\gamma\alpha = \alpha^s\gamma^t, \quad (3) \qquad \alpha^{-1}\delta\alpha = \delta^{\frac{1}{t}}, \quad (4)$$

$$\beta^{-1}\gamma\beta = \gamma^{\frac{1}{1+p}} \quad (5) \qquad \beta^{-1}\delta\beta = \delta^{1+p}, \quad (6)$$

$$\delta^u\gamma = \alpha^i\beta^j\gamma^{1+up^{m-n}}\delta^{\frac{u}{1+up^{m-n}}}, \quad (7)$$

where $t^s \equiv 1 - \frac{p^{m-1}(t-1)}{2} (\mod p^m)$ (s is an integer between 1 and $\varphi(p^m)$); $\frac{1}{t}$ is the multiplicative inverse of t modulo p^m, an integer between 1 and p^m; $1 \leq u \leq p^n - 1$, $\frac{1}{1+up^{m-n}}$ is the multiplicative inverse of $1 + up^{m-n}$ modulo p^n, an integer between 1 and p^n, $t^i \equiv 1 + up^{m-n} (\mod p^m)$, $(1+p)^j \equiv \frac{1}{1+up^{m-n}} (\mod p^n)$ with i an integer between 1 and $\varphi(p^m)$, j an integer between 1 and p^{n-1}. Furthermore, the factor-group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ of outer automorphisms is isomorphic to a solvable subgroup of order $(p-1)p^{m+3n-4}$ of the group $\mathrm{Aut}(C_{p^{m-1}} \times C_{p^n})$, and the group $\mathrm{Aut}(G)$ has a normal p-Sylow subgroup.

$$aa := f/[f.1^9, f.2^3, f.1 * f.2 * f.1^{-}4 * f.2^{-1}];$$

$27g := f/[f.1^6, f.2^3, f.3^3, f.1^{-1}*f.2*f.1*f.2^{-2}, f.3*f.1*f.3^{-5}*f.1^{-1},$
$f.3 * f.2 * f.3^{-1} * f.2^{-4} * f.1^{-2}, f.3^2 * f.2 * f.3^{-2} * f.2^{-1} * f.1^{-4}];$

54

aut:=AutomorphismGroup(aap)
Group([GroupHomomorphismByImages(
Group([(1,2,4,8,5,7,9,6,3), (2,5,6)(3,7,4)]),
Group([(1,2,4,8,5,7,9,6,3), (2,5,6)(3,7,4)]),
Pcgs([(2,5,6)(3,7,4), ...)
54
CompositionMapping(GroupGeneralMappingByImages(
Group([(2, 3, 4)(5, 8,17)(6, 9,18)(
7,10,19)(11,12,13)(14,15,16), (2, 8,12, 7,16,18)(3, 9,13,
5,14,19)(4,10,11, 6,15,17)(20,21),
(1,20,21)(2,16,12)(4,11,15)(6,10,17)(7,18, 8), (2,12,16)(
3,13,14)(4,11,15)(5,19, 9)(6,17,10)(7,18, 8)]), ...
<action isomorphism>)

Theorem

Let $G = \langle a, b \mid a^{p^m} = 1,\ b^{p^n} = 1,\ a^b = a^{1+p^{m-1}} \rangle$ *with* $n \geq m > 1$, $|G| = p^{m+n}$. *Let the automorphism* α *be induced by the substitution* $a \mapsto a^t$ *with* t *primitive root modulo* p^m $(1 < t < p^m)$ *and* $b \mapsto b$*; the automorphism* β *by the substitution* $a \mapsto a$, $b \mapsto b^{1+p}$*; the automorphism* γ *by the substitution* $a \mapsto ab^{p^{n-m+1}}$, $b \mapsto b$*; the automorphism* δ *by the substitution* $a \mapsto a$, $b \mapsto ab$.

Then the automorphism group $\mathrm{Aut}(G)$ is presented with generators α, β, γ and δ, and with generating relations

$$|\alpha| = \varphi(p^m),\ |\beta| = p^{n-1},\ |\gamma| = p^{m-1},\ |\delta| = p^m, \quad (1)$$

$$\alpha^{-1}\beta\alpha = \beta, \quad (2) \qquad \alpha^{-1}\gamma\alpha = \gamma^t, \quad (3) \qquad \alpha^{-1}\delta\alpha = \delta^{\frac{1}{t}}, \quad (4)$$

$$\beta^{-1}\gamma\beta = \gamma^{\frac{1}{1+p}}, \quad (5) \qquad \beta^{-1}\delta\beta = \delta^{1+p}, \quad (6)$$

$$\delta^u\gamma = \alpha^i\beta^j\gamma^{1+up^{n-m+1}}\delta^{\frac{u}{1+up^{n-m+1}}}, \quad (7)$$

where $\frac{1}{t}$ is the multiplicative inverse of t modulo p^m, an integer between 1 and p^m; $\frac{1}{1+p}$ is the multiplicative inverse of $1+p$ modulo p^{m-1}, an integer between 1 and p^{m-1}; $1 \le u \le p^m - 1$, $\frac{1}{1+up^{n-m+1}}$ is the multiplicative inverse of $1+up^{n-m+1}$ modulo p^n, an integer between 1 and p^n, $t^i \equiv 1 + up^{n-m+1} (\mod p^m)$, $(1+p)^j \equiv \frac{1}{1+up^{n-m+1}} (\mod p^n)$, $1 \le i \le p^m$ and $1 \le i \le p^{n-1}$. Furthermore, the factor-group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ of outer automorphisms is isomorphic to a subgroup of order $(p-1)p^{3m+n-5}$ of the solvable group $\mathrm{Aut}(C_{p^{m-1}} \times C_{p^n})$, and the group $\mathrm{Aut}(G)$ has a normal p-Sylow subgroup.

$$aa := f/[f.1^9, f.2^9, f.2^{-1} * f.1 * f.2 * f.1^{-4}];\ 81$$

$$g := f/[f.1^6, f.2^3, f.3^3, f.4^9, f.1^{-1}*f.2^{-1}*f.1*f.2, f.1^{-1}*f.3*f.1*f.3^{-2},$$

$$f.2^{-1} * f.3 * f.2 * f.3^{-1}, f.2^{-1} * f.4 * f.2 * f.4^{-4},$$

$$f.4*f.3*f.4^{-7}*f.3^{-4}*f.2^{-2}*f.1^{-2}, f.4^2*f.3*f.4^{-8}*f.3^{-7}*f.2^{-}1*f.1^{-}4,$$

$$f.4^3*f.3*f.4^{-3}*f.3^{-1}*f.2^{-3}*f.1^{-6}, f.4^4*f.3*f.4^{-1}*f.3^{-4}*f.2^{-2}*f.1^{-2},$$

$$f.4^5*f.3*f.4^{-2}*f.3^{-7}*f.2^{-1}*f.1^{-4}, f.4^6*f.3*f.4^{-6}*f.3^{-1}*f.2^{-3}*f.1^{-6},$$

$$f.4^7*f.3*f.4^{-4}*f.3^{-4}*f.2^{-2}*f.1^{-2}, f.4^8*f.3*f.4^{-5}*f.3^{-7}*f.2^{-1}*f.1^{-4}];$$

486

Group([GroupHomomorphismByImages(Group([(1, 2, 6,18,37,39,24, 9, 3)(4,12,30, 20, 7,10,26,32,13)(5,15,11,28,25,19,22, 8,16) (14,34,27,44,31,33,45, 21,35)(17,41,40,48,23,29,47,38,42)(36,46,53,62,54,55,63,52,50) (43,49,60,66,57,58,65,59,51)(56,70,73,78,64,68,79,72,71) ......)

486

CompositionMapping(GroupGeneralMappingByImages(Group([(2, 5,12, 3, 4,11)(7, 9)(8,16,13,10,15,14)(17,18)(20,23,24,21,22,25) (26,27), (2, 8,20)(3,21,10)(4,15,22)(5,23,16)(11,25,14)(12,13,24), (1,24,10, 7,22,16, 9,20,14)(2,23,18,12,25,17, 4,21, 6)(3,27,13, 5,19,15, 11,26, 8), ...),

<action isomorphism>)