Automata Networks without any Letichevsky Criteria [1]

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Abstract: In this paper we investigate some properties of the class of finite automata without any Letichevsky criteria. By our observations, we give a characterization of certain automata networks having very simple communication links.

Keywords: Finite automata, Letichevsky criterion.

1. Introduction

An automata network is a collection of automata connected together according to a directed graph $\mathcal{D}$. The vertices of $\mathcal{D}$ are considered as automata and the edges indicate the existence of communication links. Thus $\mathcal{D}$ has no parallel edges. Each automaton can change its state at discrete time steps as a local transition function of the states and a global input, and synchronous action of the local state transitions defines a global transition on the entire network.

Automata networks are also investigated as 'products of automata', i.e. as compositions of automata obtained by cascading without feedback or with feedback of various restricted types, or, most generally, with the feedback dependencies controlled by an arbitrary directed graph.

The Letichevsky criterion has a central role in the investigations of automata networks and products of automata (see [2],[3],[4],[6]). Automata having semi-Letichevsky criterion and automata without any Letichevsky criteria are also important in the classical result of Z. Ésik and Gy. Horváth (see [3],[4]). In this paper we investigate automata and their networks without any Letichevsky criteria.

2. Preliminaries

We start with some standard concepts and notations. The elements of an *alphabet* X are called *letters* (X is supposed to be finite and nonempty). A *word* over an alphabet X is a finite string consisting of letters of X. The string consisting of zero letters is called the *empty word,* written by λ. The *length* of a word w, in symbols $|w|$, means the number of letters in w when each letter is counted as many times it occurs. By definition, $|\lambda| = 0$. At the same time, for any set H, $|H|$ denotes the cardinality of H. If u and v are words over an alphabet X, then their *catenation* uv is also a word over X. In this case we also say that u is a *prefix* of uv and v is a *suffix* of uv. Catenation is an associative operation and the empty word λ is the identity with respect to catenation: $w\lambda = \lambda w = w$ for any word w.

Let X^* be the set of all words over X, moreover, let $X^+ = X^* \setminus \{\lambda\}$. X^* and X^+ are the *free monoid* and the *free semigroup,* respectively, generated by X under catenation.

By an *automaton* we mean a finite automaton without outputs. Given an automaton $\mathcal{A} = (A, X, \delta)$ with *set of states* A, *set of input letters* X, and *transition* $\delta : A \times X \to A$, it is understood that δ is extended to $\delta^* : A \times X^* \to A$ with $\delta^*(a, \lambda) = a,\ \delta^*(a, xq) = \delta^*(\delta(a, x), q)$. In the sequel, we will

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consider the transition of an automaton in this extended form and thus we will denote it by the same Greek letter δ. Let $\mathcal{A} = (A, X, \delta)$ be an automaton. If for every triplet $a \in A, x, y \in X$ the equality $\delta(a, x) = \delta(a, y)$ holds then we speak about an *autonomous automaton.* It is said that a state $a \in A$ *generates* a state $b \in A$ if $\delta(a, p) = b$ holds for some $p \in X^*$. $A' \subseteq A$ is called *a set of generators in* $\mathcal{A}$ if $\{\delta(a, p) \mid a \in A', p \in X^*\} = A$.

$\mathcal{A}$ is *connected* if it can be generated by one of its states. In other words, $\mathcal{A}$ is connected if it has a state a such that for every state b there exists an input word p having $\delta(a, p) = b$. Then we also say that $\mathcal{A}$ is connected for the state a.

$\mathcal{A}$ is said to be *strongly connected* if it can be generated by each of its states. In other words, $\mathcal{A}$ is strongly connected if for every pair $a, b \in A$ of states there is a word $p \in X^*$ with $\delta(a, p) = b$.

For every state $a \in A$ define the *state subautomaton* $\mathcal{B} = (B, X, \delta')$ *generated by* a such that $B = \{b \mid b = \delta(a, p), p \in X^*\}$, moreover, $\delta'(b, x) = \delta(b, x)$ for every pair $b \in B, x \in X$.

$\mathcal{A}$ is *n-degree nilpotent* if it has a state $a \in A$, called *dead state,* such that for every pair $b \in A, p \in X^*, |p| \geq n$, $\delta(b, p) = a$. $\mathcal{A}$ is *nilpotent* if it is n-degree nilpotent for some n. If n is a minimal nonnegative integer having the above property then we also say that $\mathcal{A}$ is *strictly n-degree nilpotent.*

Take an arbitrary automaton $\mathcal{A} = (A, X, \delta)$. A sequence $a_1, \ldots, a_n$ of pairwise distinct states of $\mathcal{A}$ is a *cycle* if there are input signs $x_1, \ldots, x_n$ such that $\delta(a_i, x_i) = a_{i+1 (\text{mod}\, n)}$ for every $i \in \{1, \ldots, n\}$. Then the positive integer n is the *length* of the cycle.

We say that $\mathcal{A} = (A, X, \delta)$ is a *k-automaton* for some $k \geq 0$, if there are $a \in A, x_1, x_2 \in X, p \in X^*, |p| = k$ such that

(i) $\delta(a, px_1) \neq \delta(a, px_2)$,
(ii) $\delta(a, px_1)$ and $\delta(a, px_2)$ generate autonomous state subautomata of $\mathcal{A}$,
(iii) for every $q_1, q_2 \in X^*$, $\delta(a, px_1q_1) \neq \delta(a, px_2q_2)$.

In addition, we say that $\mathcal{A} = (A, X, \delta)$ is a *(k, ℓ)-automaton* for some $k \geq 0, \ell \geq 1$ if there are $a \in A, x_1, x_2 \in X, p \in X^*, |p| = k$, such that

(i) $\delta(a, px_1) \neq \delta(a, px_2)$,
(ii) $\delta(a, px_1)$ and $\delta(a, px_2)$ generate autonomous state subautomata of $\mathcal{A}$,
(iii) there are $y_1, y_2 \in X, q_1, q_2 \in X^*, |q_1| = |q_2| = \ell - 1$ with $\delta(a, px_1q_1) \neq \delta(a, px_2q_2)$ and $\delta(a, px_1q_1y_1) = \delta(a, px_2q_2y_2)$.

The following statement is obvious.

Proposition 1 [1] *An automaton is nilpotent if and only if it has both of the following conditions:*

(i) It has only one cycle;
(ii) Its cycle is trivial (having only one element).

□

Each automaton can be considered as an algebra with unary operational symbols (corresponding to each input letter), or, alternatively, as an algebra with two sorts —states and input letters— and one binary operation —the transition function taking a state and a letter to a new state. Therefore, notions such as *subautomaton, homomorphism, isomorphism* is defined in the natural way. Thus $\mathcal{A}' = (A', X', \delta')$ is a *subautomaton* of the automaton $\mathcal{A} = (A, X, \delta)$ if $A' \subseteq A$, $X' \subseteq X$ and δ' is the restriction of δ to $A' \times X'$ (so that $\delta'(a', x') \in A'$ for any $a' \in A'$ and $x' \in X'$). In particular, if $X' = X$ then we speak about an *state-subautomaton.* A pair $\psi = (\psi_1, \psi_2)$ of surjective mappings $\psi_1 : A \to A', \psi_2 : X \to X'$ is a *homomorphism of* $\mathcal{A} = (A, X, \delta)$ onto $\mathcal{A}' = (A', X', \delta')$ if for every $a \in A, x \in X$, one has $\psi_1(\delta(a, x)) = \delta'(\psi_1(a), \psi_2(x))$. In particular, if $X = X'$ and ψ_2 is the identity then sometimes we will also refer to ψ_1 as a *state-homomorphism* (or, if no confusion can result, as a homomorphism).

For every class $\mathcal{K}$ of automata, let us denote, in order, $S(\mathcal{K})$ the class of all subautomata of automata in $\mathcal{K}$ and $H(\mathcal{K})$ the class of all homomorphic images of automata in $\mathcal{K}$. We say that an automaton $\mathcal{A}$ *homomorphically represents* the automaton $\mathcal{B}$ if $\mathcal{B} \in H(S(\{\mathcal{A}\}))$.

Let $\mathcal{A}_i = (A_i, X_i, \delta_i)$ be automata where $i \in \{1, \ldots, n\}$, $n \geq 1$. Take a finite nonvoid set X and a *feedback function* $\varphi_i : A_1 \times \cdots \times A_n \times X \to X_i$ for every $i \in \{1, \ldots, n\}$. The *general product* (or *Gluškov-type product*) of the automata $\mathcal{A}_i$ with respect to the feedback functions φ_i $(i \in \{1, \ldots, n\})$ is defined to be the automaton $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n(X, (\varphi_1, \ldots, \varphi_n))$ with state set $A = A_1 \times \cdots \times A_n$, input set X, transition function δ given by $\delta((a_1, \ldots, a_n), x) = (\delta_1(a_1, \varphi_1(a_1, \ldots, a_n, x)), \ldots, \delta_n(a_n, \varphi_n(a_1, \ldots, a_n, x)))$ for all $(a_1, \ldots, a_n) \in A$ and $x \in X$. In particular, if $\mathcal{A}_1 = \ldots = \mathcal{A}_n$ then we say that $\mathcal{A}$ is a *(general) power.* In the special case $n = 1$, then $\mathcal{A} = \mathcal{A}_1(X, \varphi_1)$, and we speak of a *single factor product.* [2]

We shall use the feedback functions $\varphi_i, i = 1, \ldots, n$ in an extended sense as mappings $\varphi_i^* : A_1 \times \cdots \times A_n \times X^* \to X_i^*$, where $\varphi_i^*(a_1, \ldots, a_n, \lambda) = \lambda$ and $\varphi_i^*(a_1, \ldots, a_n, px) = \varphi_i^*(a_1, \ldots, a_n, p)\varphi_i(\delta_1(a_1, \varphi_1^*(a_1, \ldots, a_n, p)), \ldots, \delta_n(a_n, \varphi_n^*(a_1, \ldots, a_n, p)), x)$, $a_i \in A_i$, $i = 1, \ldots, n, p \in X^*, x \in X$. In the sequel, $\varphi_i^*, i \in \{1, \ldots, n\}$ will also be denoted by φ_i.

We define the *underlying graph* $\mathcal{D} = (V, E)$ $(V = \{1, \ldots, n\}, E \subseteq V \times V)$ of $\mathcal{A}$ such that $(i, j) \in E$ if and only if the feedback function φ_j really depends on its i^{th} state variable. Thus, an underlying graph is a directed graph (or, in short, a digraph) which may contain loop edges.

Let $\mathcal{D} = (V, E)$ be a digraph with $V = \{1, \ldots, n\}$ and, for every $v \in V$, let $\mathcal{A}_v = (A_v, X_v, \delta_v)$ be an automaton. A general product $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n(X, (\varphi_1, \ldots, \varphi_n))$ is a $\mathcal{D}$-*network* if each feedback function φ_v $(v \in V)$ is really independent of its u^{th} $(u \in V)$ state variable whenever $(u, v) \notin E$. If Δ is a nonempty class of digraphs and $\mathcal{D} \in \Delta$ then it is also said that $\mathcal{A}$ is a Δ-*network.* (In the sequel, by a class Δ of digraphs we always mean that Δ is a nonempty class.) Thus, if Δ is the class of all digraphs having neither cycles nor loop edges then the Δ-network is just the *loop-free network* which is further equivalent to the cascade product or, by another name, the α_0-network. In particular, if Δ is the class of all digraphs having neither cycles nor loop edges such that every vertex has at most one incoming edge then we speak about $\alpha_0 - \nu_1$-*network.* (Of course, if all factors are the same then we speak about appropriate types of powers.) If Δ is the class of all digraphs having no edges then the Δ-network is called a *parallel network* or a *quasi-direct network* or, in short, a *q-network.*

If Δ consists of digraphs having only loop edges then we speak about q^ℓ-network (or quasi-direct network of single factor products). In other words, for every $\mathcal{D} = (V, E) \in \Delta$, $E = \{(v, v) \mid v \in V\}$.

Take automata $\mathcal{A}_i = (A_i, X_i, \delta_i), i = 1, \ldots, n$. The *direct product* $\mathcal{B} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ *of automata* $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is defined to be the automaton $\mathcal{B} = (B, Y, \delta_\mathcal{B})$, where $B = A_1 \times \cdots \times A_n, Y = X_1 \times \cdots \times X_n$ and $\delta_\mathcal{B}((a_1, \ldots, a_n), (x_1, \ldots, x_n)) = (\delta_1(a_1, x_1), \ldots, \delta_n(a_n, x_n)), (a_1, \ldots, a_n) \in B, (x_1, \ldots, x_n) \in Y$. If $X_1 = \ldots = X_n$ then restricting to the input set in which all letters in the input n-tuple are equal we have the subautomaton $\mathcal{B}' = \mathcal{A}_1 \Delta \cdots \Delta \mathcal{A}_n$, the *diagonal product of* $\mathcal{A}_1, \ldots, \mathcal{A}_n$, whose input alphabet we may naturally identify with X. If $\mathcal{A}_1 = \ldots = \mathcal{A}_n$ then $\mathcal{B}$ is called the n^{th} *direct power* $\mathcal{A}^n = (A^n, X^n, \delta^{(n)})$ of $\mathcal{A}$ and $\mathcal{B}'$ is called the n^{th} *diagonal power* $\mathcal{A}^{\Delta n}$ *of* $\mathcal{A}$. (It is easy to see that the direct product and the diagonal product can also be considered as special general product.) Of course, every diagonal product of single factor products is a q^ℓ-network and vice versa.

We shall use the following simple fact.

Proposition 2 *Every general product of nilpotent automata is a nilpotent automaton. Moreover, if a general product of nilpotent automata is strictly k-degree nilpotent for some nonnegative integer then one of its factors is strictly ℓ-degree nilpotent for some $\ell \geq k$.*

Proof: Let $\mathcal{M} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n(X, \varphi_1, \ldots, \varphi_n)$ be a general product of nilpotent automata $\mathcal{A}_t = (A_t, X_t, \delta_t)$, $t = 1, \ldots, n$. For every $t \in \{1, \ldots, n\}$, let $d_t \in A_t$ be the dead state of $\mathcal{A}_t$. Let s be a large enough positive integer such that each of $\mathcal{A}_t$ is s-degree nilpotent. Obviously, then

[2] Note that a single factor product is different from its factor in general.

for every word $p \in X^*, |p| \geq s$ and state $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$, $\delta_1(a_1, \varphi_1(a_1, \ldots, a_n, p)), \ldots, \delta_n(a_n, \varphi_n(a_1, \ldots, a_n, p)) = (d_1, \ldots, d_n)$. Thus $\mathcal{M}$ is nilpotent.

Let k be the nonnegative integer for which $\mathcal{M}$ is strictly k-degree nilpotent. Obviously, then there are $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n, x_1, \ldots, x_k \in X$ such that $(a_1, \ldots, a_n), (\delta_1(a_1, \varphi_1(a_1, \ldots, a_n, x_1)), \ldots, \delta_n(a_n, \varphi_n(a_1, \ldots, a_n, x_1))), \ldots, (\delta_1(a_1, \varphi_1(a_1, \ldots, a_n, x_1 \cdots x_k)), \ldots, \delta_n(a_n, \varphi_n(a_1, \ldots, a_n, x_1 \cdots x_k)))$ are pairwise distinct states. Because all factors are nilpotent, this implies that there exists an $i \in \{1, \ldots, n\}$ such that $\delta_i(a_i, \varphi_i(a_1, \ldots, a_n, x_1 \cdots x_k)), \ldots, \delta_i(a_i, \varphi_i(a_1, \ldots, a_n, x_1 \cdots x_k))$ are pairwise distinct states. Therefore, there are $a_i \in A_i, p \in X_i^*, |p| \geq k-1$ such that $\delta_i(a_i, p)$ is not the dead state of $\mathcal{A}_i$. Thus $\mathcal{A}_i$ is strictly ℓ-degree nilpotent for some $\ell \geq k$. □

Proposition 3 [1] *Let $\mathcal{A} = (A, X, \delta)$ and $\mathcal{B} = (B, X, \delta')$ be arbitrary automata having the same input set. Suppose that $\mathcal{A}$ has two (distinct) states $a, b \in A$ such that $\delta(a, p) \neq \delta(b, p), p \in X^*$. Then $\mathcal{B}$ can be represented homomorphically by a diagonal product of its connected state subautomata and the automaton $\mathcal{A}$.*

□

Proposition 4 *Given a pair of nonnegative integers k, ℓ with $k \leq \ell$, let $\mathcal{A} = (A, Y, \delta)$ be a strictly ℓ-degree nilpotent automaton and $\mathcal{B} = (B, X, \delta')$ be a strictly k-degree nilpotent automaton. Then $\mathcal{B}$ can be represented homomorphically by a diagonal product of its connected state subautomata and a single factor product of $\mathcal{A}$.*

Proof: If $k = 0$ then our statement is trivial. Therefore, we may assume $k > 0$.

Because $\mathcal{A}$ is strictly ℓ-degree nilpotent (and $\ell \geq k > 0$), there are state $a_0, a \in A$ and input word $y_1 \cdots y_k \in Y^*, y_1, \ldots, y_k \in Y$ such that $\delta(a, y_1 \cdots y_k) = a_0$, and simultaneously, for every $i < k$, $\delta(a, y_1 \cdots y_i) \neq a_0$. First we observe that $\delta(a, y_1), \delta(a, y_1 y_2), \delta(a, y_1 \cdots y_k)$ are distinct states. Indeed, by the above properties, $\delta(a, y_1 \cdots y_k) \notin \{\delta(a, y_1 \cdots y_i) \mid i = 1, \ldots, k-1\}$. On the other hand, if there are $i, j\{1, \ldots, k-1\}, i < j$ with $\delta(a, y_1 \cdots y_i) = \delta(a, y_1 \cdots y_j)$ then for every $m > 0$, $\delta(a, y_1 \cdots y_i (y_{i+1} \cdots y_j)^{m-1} y_{i+1} \cdots y_k) = a_0$ such that $\delta(a, y_1 \cdots y_i (y_{i+1} \cdots y_j)^{m-1} y_{i+1} \cdots y_{k-1}) \neq a_0$. This implies that $\mathcal{A}$ is not k-degree nilpotent (and thus it is not strictly k-degree nilpotent), a contradiction.

Recall that $\delta(a, y_1), \delta(a, y_1 y_2), \delta(a, y_1 \cdots y_k)$ are distinct states and $\delta(a, y_1 \cdots y_k) = a_0$. Therefore, we can define the single factor product $\mathcal{C} = \mathcal{A}(X, \varphi)$ such that for every $x \in X$, $\varphi(a, x) = y_1$ and $\varphi(\delta(a, y_1 \cdots y_i), x) = y_{i+1}, i = 1, \ldots, k-1$. Then for every $p \in X^*$,

$$\delta(a, \varphi(a, p)) = \begin{cases} \delta(a, y_1 \cdots y_i) & \text{if } |p| < |k, \\ a_0 & \text{otherwise.} \end{cases}$$

Let $B' = \{b_1, \ldots, b_n\}$ be a set of generators in $\mathcal{B}$. In other words, suppose that $\{\delta'(b, p) \mid b \in B', p \in X^*\} = B$. Denote by $\mathcal{B}_b$ the state-subautomaton of $\mathcal{B}$ generated by the state $b \in B$ and consider a diagonal product $\mathcal{M} = \mathcal{B}_{b_1} \Delta \cdots \Delta \mathcal{B}_{b_n} \Delta \mathcal{C}^{2n}$ where $\mathcal{C}^{2n}$ denotes the $2n^{th}$ diagonal power of $\mathcal{C}$. Put $H = \{(b_1, \cdots, b_n, c_{1,1}, c_{1,2}, \ldots, c_{n,1}, c_{n,2}) \mid c_{1,1} = c_{1,2} = \ldots = c_{j-1,1} = c_{j-1,2} = c_{j+1,1} = c_{j+1,2} = \ldots = c_{n,1} = c_{n,2} = a_0, c_{j,1} = a_0, c_{j,2} = a, j = 1, \ldots, n\}$.

Clearly, then for every $(b_1, \cdots, b_n, c_{1,1}, c_{1,2}, \ldots, c_{n,1}, c_{n,2}) \in H$, and $p \in X^*$ with $|p| < k$, there exists exactly one $i \in \{1, \ldots, n\}$ such that $\delta(c_{i,1}, \varphi(c_{i,1}, p)) \neq \delta(c_{i,2}, \varphi(c_{i,2}, p))$.

On the other hand, there exists a state $b_0 \in B$ such that for every $b \in B$ and $p \in X^*$ with $|p| \geq k$, $\delta'(b, p) = b_0$. Then for every pair $(b_1, \cdots, b_n, c_{1,1}, c_{1,2}, \ldots, c_{n,1}, c_{n,2}) \in H$ and $p \in X^*$ with $|p| \geq k$, $(\delta'(b_1, p), \ldots, \delta'(b_n, p), \delta(c_{1,1}, \varphi(c_{1,1}, p)), \delta(c_{1,2}, \varphi(c_{1,2}, p)), \ldots,$
$\delta(c_{n,1}, \varphi(c_{n,1}, p)), \delta(c_{n,2}, \varphi(c_{n,2}, p))) = (m_1, \ldots, m_{3n})$, where $m_1 = \cdots = m_n = b_0$ and $m_{n+1} = \cdots = m_{3n} = a_0$. Therefore, it is clear that the following mapping

ψ : $\{(\delta'(b_1,p),\ldots,\delta'(b_n,p),\delta(c_{1,1},\varphi(c_{1,1},p)),\delta(c_{1,2},\varphi(c_{1,2},p)),\ldots,\delta(c_{n,1},\varphi(c_{n,1},p)),$ $\delta(c_{n,2},\varphi(c_{n,2},p))) \mid (b_1,\cdots,b_n,c_{1,1},c_{1,2},\ldots,c_{n,1},c_{n,2}) \in H, p \in X^*\} \to B$

is well-defined and it is a state-homomorphism of a state-subautomaton of $\mathcal{M}$ onto $\mathcal{B}$:

$$\psi((\delta'(b_1,p),\ldots,\delta'(b_n,p),\delta(c_{1,1},\varphi(c_{1,1},p)),\delta(c_{1,2},\varphi(c_{1,2},p)),\ldots,\delta(c_{n,1},\varphi(c_{n,1},p)),$$
$$\delta(c_{n,2},\varphi(c_{n,2},p))) =$$

$$\begin{cases} \delta'(b_i,p) & \text{if there exists } i \in \{1,\ldots,n\} \text{ with } \delta(c_{i,1},\varphi(c_{i,1},p)) \neq \delta(c_{i,2},\varphi(c_{i,2},p)) \\ & \text{and } i \text{ is minimal having this property,} \\ b_0 & \text{if } \delta(c_{1,1},\varphi(c_{1,1},p)) = \delta(c_{1,2},\varphi(c_{1,2},p)),\ldots,\delta(c_{n,1},\varphi(c_{n,1},p)) = \\ & \delta(c_{n,2},\varphi(c_{n,2},p)). \end{cases}$$

□

3. Automata and Letichevsky's criterion

We say that an automaton $\mathcal{A} = (A, X, \delta)$ satisfies *Letichevsky's criterion* if there are a state $a \in A$, input letters $x, y \in X$, input words $p, q \in X^*$ such that $\delta(a, x) \neq \delta(a, y)$ and $\delta(a, xp) = \delta(a, yq) = a$. It is said that $\mathcal{A}$ *satisfies the semi-Letichevsky criterion* if it does not satisfy Letichevsky's criterion but there are a state $a \in A$, input letters $x, y \in X$, an input word $p \in X^*$ such that $\delta(a, x) \neq \delta(a, y), \delta(a, xp) = a$ and for every $q \in X^*$, $\delta(a, yq) \neq a$. If $\mathcal{A}$ do not satisfy either Letichevsky's criterion or the semi-Letichevsky criterion then we say that $\mathcal{A}$ *does not satisfy any Letichevsky criteria* or *is without any Letichevsky criteria.*

Proposition 5 *Given an automaton $\mathcal{A} = (A, X, \delta)$, a state $a_0 \in A$, three input words $u, v, w \in X^*$ with $v \neq \lambda$, and two input letters $x_1, x_2 \in X$ under which $\delta(a_0, uv) = \delta(a_0, u)$ and $\delta(a_0, uvwx_1) \neq \delta(a_0, uvwx_2)$. Then $\mathcal{A}$ satisfies either Letichevsky's criterion or the semi-Letichevsky criterion.*

Proof: If $u \neq \lambda$ then we can consider $\delta(a_0, u)$ and λ instead of a_0 and u. Therefore we may suppose $u = \lambda$ and thus we can take a_0 instead of $\delta(a_0, u)$.

By $v \neq \lambda$, there are $y \in X, z \in X^*$ with $v = yz$. On the other hand, $\delta(a_0, uvwx_1) \neq \delta(a_0, uvwx_2)$ implies $x_1 \neq x_2$. Therefore, there exists $i \in \{1, 2\}$ having $x_i \neq y$. Thus we have either Letichevsky's criterion (for $a_0 \in A, y, x_2 \in X$, $z \in X^*$ and some $q \in X^*$, or $a_0 \in A, y, x_1 \in X$, $z \in X^*$ and some $q \in X^*$) or the semi-Letichevsky criterion (for a_0, y, x_2, z or a_0, y, x_1, z). □

The next statement can be found in [1] without proof.

Proposition 6 *$\mathcal{A} = (A, X, \delta)$ is an automaton without any Letichevsky criteria if and only if for every state $a_0 \in A$, input letters $x, y \in X$ and an input word $p \in X^*$ having $\delta(a_0, xp) = a_0$, it holds that $\delta(a_0, x) = \delta(a_0, y)$.* □

Proof: First we suppose that $\mathcal{A} = (A, X, \delta)$ is an automaton without any Letichevsky criteria and consider $a_0 \in A, p \in X^*, x \in X$ with $\delta(a_0, xp) = a_0$. Then $\delta(a_0, x) \neq \delta(a_0, y)$ implies that $\mathcal{A}$ satisfies either Letichevskys criterion or the semi-Letichevsky criterion, a contradiction. Therefore, for every $y \in X$, $\delta(a_0, y) = \delta(a_0, x)$ as we stated.

Conversely, assume that for every state $a_0 \in A$, input letters $x, y \in X$ and an input word $p \in X^*$ having $\delta(a_0, xp) = a_0$, it holds that $\delta(a_0, x) = \delta(a_0, y)$. Then there are no $a_0 \in A, x, y \in X, p \in X^*$ with $\delta(a_0, x) \neq \delta(a_0, y)$ and $\delta(a_0, xp) = a_0$. Thus $\mathcal{A}$ is an automaton without any Letichevsky criteria. □

Now we give an alternative proof of the next statement proved in [1].

Proposition 7 *Let $\mathcal{A} = (A, X, \delta)$ be an automaton without any Letichevsky criteria. There exists a nonnegative integer $n_{\mathcal{A}}$ with $n_{\mathcal{A}} \leq \max(|A| - 2, 0)$ such that for every $p \in X^*$ with $|p| \geq n_{\mathcal{A}}$, each $\delta(a, p)$ $(a \in A)$ generates an autonomous state-subautomaton of $\mathcal{A}$.*

Proof: Take out of consideration the trivial cases. Thus we may assume $|A| > 2$. Let $a \in A$ be an arbitrary state. Of course, if for every $p \in X^*, x_1, x_2 \in X$ we have $\delta(a, x_1) = \delta(a, x_2)$ then this property also holds when $|p| \geq n$. Otherwise there are $x_1, \ldots, x_{m+2} \in X$ having $\delta(a, x_1 \cdots x_m x_{m+1}) \neq \delta(a, x_1 \cdots x_m x_{m+2})$. If $a, \delta(a, x_1), \delta(a, x_1x_2), \ldots, \delta(a, x_1 \cdots x_m), \delta(a, x_1 \cdots x_m x_{m+1})$, $\delta(a, x_1 \cdots x_m x_{m+2})$ are not distinct states then there are a state $a \in A$, three input words $u, v, w \in X^*$ with $v \neq \lambda$, and two input letters $x_{m+1}, x_{m+2} \in X$ under which $\delta(a, uv) = \delta(a, u)$ and $\delta(a, uvwx_{m+1}) \neq \delta(a, uvwx_{m+2})$. Applying Proposition 5, this means that $\mathcal{A}$ satisfies either Letichevsky's criterion or the semi-Letichevsky criterion, a contradiction. Hence, $m \leq |A| - 3$. Thus $n_{\mathcal{A}} \leq |A| - 2$. □

4. q^ℓ-Networks of Automata without any Letichevky Criteria

Proposition 8 *Let $\mathcal{M} = \mathcal{A}_1 \times \mathcal{A}_n(X, \varphi_1, \ldots, \varphi_n)$ be a general product of automata without any Letichevsky criteria. There exists a factor $\mathcal{A}_t, t \in \{1, \ldots, n\}$ of $\mathcal{M}$, a general single factor product $\mathcal{C} = \mathcal{A}_t(X, \varphi)$ of $\mathcal{A}_t$ such that $\mathcal{M}$ can be represented homomorphically by a diagonal product of its connected state subautomata and $\mathcal{C}$.*

Proof: First we suppose that there exists a factor $\mathcal{A}_t = (A_t, X_t, \delta_t), t \in \{1, \ldots, n\}$ of $\mathcal{M}$, a pair of states $a, b \in A_t$ having $\delta_t(a, p) \neq \delta_t(b, p), p \in X^*$. Let $\mathcal{C} = \mathcal{A}_t(X, \varphi)$ be defined by an arbitrarily fixed mapping $\varphi : A_t \times X_t \to X$. Obviously, then $\mathcal{C}$ preserves the property that for every $p \in X^*, \delta(a, \varphi(a, p)) \neq \delta(b, \varphi(b, p))$. Thus we can apply Proposition 3.

Now we assume that for every factor $\mathcal{A}_t = (A_t, X_t, \delta_t), t \in \{1, \ldots, n\}$ of $\mathcal{M}$, and distinct states $a, b \in A_t$, there exists a $p \in X^*$ having $\delta(a, p) = \delta(b, p)$. By Proposition 6, this implies that both of a and b can not be element of cycles in $\mathcal{A}_t$. But then every factor $\mathcal{A}_t = (A_t, X_t, \delta_t), t \in \{1, \ldots, n\}$ of $\mathcal{M}$ has only one cycle and its cycle is trivial (having only one element). Applying Proposition 1, we obtain that all factors of $\mathcal{M}$ are nilpotent automata, moreover, there are nonnegative integers k, ℓ with $k \leq \ell$ such that $\mathcal{M}$ is k-degree nilpotent and one of its factor is ℓ-degree nilpotent. Therefore, applying Proposition 4, the proof is complete. □

Given a nonnegative integer k, a positive integer ℓ, define the automata $\mathcal{A}_{X,k} = (A_{X,k}, X, \delta_{X,k})$ $\mathcal{A}_{X,k,0} = (A_{X,k,0}, X, \delta_{X,k,0})$ and $\mathcal{A}_{X,k,\ell} = (A_{X,k,\ell}, X, \delta_{X,k,\ell})$ in the following manner:

$$A_{X,k} = \{p \in X^* \mid |p| \leq k\},$$

$$\delta_{X,k}(u, x) = \begin{cases} px & \text{if } u = p \in X^*, |p| < k, \\ p & \text{if } u = p \in X^*, |p| = k, \end{cases}$$

$$A_{X,k,\ell} = \{p \in X^* \mid |p| < k\} \cup \{p \in X^* \mid |p| = k\} \times \{1, 2, \ldots, \ell\} \cup \{*\},$$

$$\delta_{X,k,\ell}(u, x) = \begin{cases} px & \text{if } u = p \in X^*, |p| < k, \\ (p, 1) & \text{if } u = p \in X^*, |p| = k, \\ (p, i+1) & \text{if } u = (p, i) \in X^*, |p| = k, 1 \leq i \leq \ell - 1, \\ * & \text{if } u = (p, \ell) \in X^*, |p| = k. \end{cases}$$

Moreover, let $\mathcal{A}_{X,0,0} = (A_{X,0,0}, X, \delta_{X,0,0})$ be an automaton with a singleton state set $\{\lambda\}$ (where $\delta_{X,0,0}(\lambda, x) = \lambda, x \in X$).

Lemma 9 *Let $\mathcal{A}$ be a k-automaton for some $k \geq 0$. There exist single factor products $\mathcal{C}_t = \mathcal{A}(X_t, \varphi_t), t = 1, \ldots, n$ such that $A_{X,k+1}$ can be represented homomorphically by the diagonal product $\mathcal{C}_1 \Delta \cdots \Delta \mathcal{C}_n$.*

Proof: Consider $\mathcal{A} = (A, Y, \delta)$, $a \in A, y_1, y_2 \in Y, p \in Y^*, |p| = k$ such that

(i) $\delta(a, py_1) \neq \delta(a, py_2)$,

(ii)$\delta(a, py_1)$ and $\delta(a, py_2)$ generate autonomous state subautomata of $\mathcal{A}$,
(iii) for every $q_1, q_2 \in Y^*$, $\delta(a, py_1q_1) \neq \delta(a, py_2q_2)$.

For every $x \in X$ define $\mathcal{C}_{x,1} = \mathcal{A}(X, \varphi_{x,1})$ in the following manner:

$$\varphi_x(d, z) = \begin{cases} z' & \text{if } d = \delta(a, q), \text{ such that } q \text{ and } qz' \text{ are prefices of } p, \\ y_1 & \text{if } d = \delta(a, p), z = x, \\ y_2 & \text{if } d = \delta(a, p), z \neq x. \end{cases}$$

Moreover, let $\mathcal{C}_{x,2} = \mathcal{A}(X, \varphi_{x,2}), x \in X$ such that

$$\varphi_x(d, z) = \begin{cases} z' & \text{if } d = \delta(a, q), \text{ such that } q \text{ and } qz' \text{ are prefices of } p, \\ y_2 & \text{if } d = \delta(a, p). \end{cases}$$

Let $X = \{x_1, \ldots, x_m\}$ and consider $\mathcal{M} = (\mathcal{C}_{x_1,1} \Delta \mathcal{C}_{x_1,2} \Delta \cdots \Delta \mathcal{C}_{x_m,1} \Delta \mathcal{C}_{x_m,2})^{|p|+1}$. Let $\mathcal{C} = (C, X, \delta_{\mathcal{C}})$ denote the state subautomaton of $\mathcal{M}$ generated by $(c_{1,1,1}, c_{1,1,2}, \ldots, c_{1,m,1}, c_{1,m,2}, \ldots, c_{|p|+1,1,1} \ldots, c_{|p|+1,m,2})$, where $(c_{i,1,1}, c_{i,1,2} \ldots, c_{i,m,1}, c_{i,m,2}) = (\delta(a, r), \ldots, \delta(a, r))$ for which r is a prefix of p having $|r| = i - 1$.

Define the mapping $\psi : C \to A_{X,k}$ such that
$\psi((c_{1,1,1}, c_{1,1,2}, \ldots, c_{1,m,1}, c_{1,m,2}, \ldots, c_{|p|+1,1,1} \ldots, c_{|p|+1,m,2})) = \lambda$, moreover for every
$(c'_{1,1,1}, c'_{1,1,2}, \ldots, c'_{1,m,1}, c'_{1,m,2}, \ldots, c'_{|p|+1,1,1} \ldots, c'_{|p|+1,m,2}) \in C$,
$\psi((c'_{1,1,1}, c'_{1,1,2}, \ldots, c'_{1,m,1}, c'_{1,m,2}, \ldots, c'_{|p|+1,1,1} \ldots, c'_{|p|+1,m,2})) = x_{i_1} \cdots x_{i_j}$,
whenever $c'_{|p|+1,i_1,1} \neq c'_{|p|+1,i_1,2}, c'_{|p|,i_2,1} \neq c'_{|p|,i_2,2}, \ldots, c'_{|p|-s-2,i_j,1} \neq c'_{|p|-s-2,i_j,2}$, and
$c'_{s,t,1} = c'_{s,t,2}$ holds for all the other indices $s \in \{1, \ldots, |p| + 1\}, t \in \{1, \ldots, m\}$.

By an elementary computation we obtain that ψ is a state homomorphism of $\mathcal{C}$ onto $\mathcal{A}_{X,k+1}$. $\square$

By an analogous treatment we can prove the following statement.

Lemma 10 *Consider integers $k \geq 0, \ell \geq 1$ and let for every $i \in \{1, \ldots, k+1\}$, $\mathcal{A}_i$ be either $i-1$-automata or $(i-1, \ell_i)$-automata with $\ell_i \geq k+\ell-i+1$. There are single factor products $\mathcal{C}_t$, $t = 1, \ldots, n$ of automata in $\{\mathcal{A}_i \mid i = 1, \ldots, k\}$ such that $A_{X,k+1,\ell}$ can be represented homomorphically by the diagonal product $\mathcal{C}_1 \Delta \cdots \Delta \mathcal{C}_n$.*

In [1] there is a characterization of q^ℓ-products of automata without any Letichevsky criteria. Now we give an alternative characterization.

Theorem 11 *Let $\mathcal{K}$ be a class of automata without any Letichevsky criteria. The following statements are equivalent:*

(i) Every general product of factors from $\mathcal{K}$ can also be represented homomorphically by a q^ℓ-network of the components from $\mathcal{K}$;
(ii) For every (k, ℓ)-automaton in $\mathcal{K}$ (with $k \geq 0, \ell \geq 1$) and for every $k' = 0, \ldots, k-1$ there exists either a k'-automaton or a (k', ℓ')-automaton with $k' + \ell' \geq k + \ell$ in $\mathcal{K}$.

Proof: First we assume that we have the condition (ii) of our statement and let $\mathcal{M} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n(X, \varphi_1, \ldots, \varphi_n)$ be a product of automata $\mathcal{A}_t = (A_t, X_t, \delta_t), t = 1, \ldots, n$ (such that each of $\mathcal{A}_t, t = 1, \ldots, n$ is without any Letichevsky criteria). It is enough to prove that $\mathcal{M}$ can be represented homomorphically by a diagonal product of automata such that each of its factors is a single factor product of an appropriate automaton in $\mathcal{K}$. By Proposition 8, we can restrict ourselves to prove our statement to the connected state subautomata of $\mathcal{M}$.

Consider an arbitrary connected state subautomaton $\mathcal{C} = (C, X, \delta_{\mathcal{C}})$ of $\mathcal{M}$ and denote $(a_1, \ldots, a_n)$ the state of $\mathcal{M}$ which generates $\mathcal{C}$. Let m be the maximum of $n_{\mathcal{A}_1}, \ldots, n_{\mathcal{A}_n}$, where $n_{\mathcal{A}_t}, t = 1, \ldots, n$ denotes the minimal nonnegative integer such that for every pair $a \in A_t, p \in X_t$, the state subautomaton of $\mathcal{A}_t$ generated by $\delta_t(a, p)$ is an autonomous automaton. (Recall that Proposition 7 shows the existence of $n_{\mathcal{A}_t}, t = 1, \ldots, n$.)

For every $p \in X^*, |p| = n$ and $t \in \{1, \ldots, n\}$ define the single factor product $\mathcal{B}_{t,p} = \mathcal{A}_t(X, \varphi'_{t,p})$ in the following manner.

Let $\varphi'_{t,p}$ be arbitrarily fixed if $p = \lambda$. Otherwise, for every $x \in X$ and prefix $p'y$ of p with $y \in X$, let $\varphi'_{t,p}(\delta(a_t, p'), x) = y$. Obviously, then for every $t \in \{1, \ldots, n\}, p, r \in X^*, |p| = n$,

$$\delta_{t,p}(a, r) = \begin{cases} \delta_t(a_t, q) & \text{if } r \leq m \text{ and } p' \text{ is a prefix of } p \text{ with } |p'| = |r|, \\ \delta_t(a_t, pr') & \text{if } |r| > m \text{ and } r' \in X^* \text{ is an arbitrary word with } |pr'| = r. \end{cases}$$

Let k be the maximal nonnegative integer for which there exists a k-automaton in $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$.

Moreover, let $\{(k_1, \ell_1), \ldots, (k_s, \ell_s)\} = \{(-1, 0)\}$ if there exist no (k, ℓ)-automaton in $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ for some $k \geq 0, \ell \geq 1$.

Otherwise let $k_1, \ldots, k_s$ be all nonnegative integers for which there are (k_i, ℓ_i)-automata ($i = 1, \ldots, s$) in $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ and for every $k_i, i = 1, \ldots, s$ let ℓ_i be the maximal positive integer having this property.

By Lemma 9 the automaton $A_{X,k+1}$ can be represented homomorphically by a diagonal product of single factor products in $\mathcal{K}$. Using the condition (ii) and Lemma 10, for every $(k_i, \ell_i), i = 1, \ldots, s$ with $(k_i, \ell_i) \neq (0, 0)$, the automaton A_{X,k_i+1,ℓ_i} can also be represented homomorphically by a diagonal product of single factor products in $\mathcal{K}$. Moreover, it is trivial that $A_{X,0,0}$ has this property (even if $\{(k_1, \ell_1), \ldots, (k_s, \ell_s)\} = \{(-1, 0)\}$).

Obviously, by (ii), it is enough to prove that the connected state subautomaton $\mathcal{B}$ of $\mathcal{M}$ can be represented homomorphically by a diagonal product

$\mathcal{P} = A_{X,k+1} \Delta A_{X,k_1+1,\ell_1} \Delta \ldots \Delta A_{X,k_s+1,\ell_s} \Delta \mathcal{B}_1 \Delta \cdots \Delta \mathcal{B}_j$,

where $\{\mathcal{B}_t \mid t = 1, \ldots, j\} = \{\mathcal{B}_{t,p} \mid p \in X^*, |p| = m), t = 1, \ldots, n\}$.

Let $\mathcal{B} = (B, X, \delta_{\mathcal{B}})$ be the state subautomaton of $\mathcal{P}$ generated by the state $(\lambda, \lambda, \ldots, \lambda, d_1, \ldots, d_j)$, where for every $i = 1, \ldots, j$, $d_i = b_t$ if $\mathcal{B}_i = \mathcal{B}_{t,q}$ for some $q \in X^*, |q| = m$.

Clearly, for every $p \in X^*$, $\delta_{\mathcal{B}}(\lambda, \lambda, \ldots, \lambda, d_1, \ldots, d_j) = (r, r_1, \ldots, r_s, b_1, \ldots, b_j)$, where $r \in X^*, |r| \leq k+1$, $r_i = \lambda$ if $(k_1, \ell_1) = (-1, 0)$ (with $s = 1$), and otherwise $r_i \in \{p_i \in X^* \mid 0 \leq |p_i| \leq k_i\} \cup \{(p_i, h_i) \mid p_i \in X^*, |p_i| = k_i + 1, 1 \leq h_i \leq \ell_i\}, i = 1, \ldots, s$ such that all of $r, r_i, i = 1, \ldots, s$ are prefixes of $q \in X^*$ for some $q \in \{r, p_1, \ldots, p_s\}$. Let $\varrho((r, r_1, \ldots, r_s)$ denote this q.

Note that $q = r$ if $|r| \leq k + 1$ and $q = p_i$ for some $i = 1, \ldots, s$ if $|q| \leq k_i + 1$.

Consider a pair $w, z \in X^*$ with $|w| = |z| = m$ and assume that $\varrho((r, r_1, \ldots, r_s)$ is a prefix both of them for some $(r, r_1, \ldots, r_s, b_1, \ldots, b_j) \in B$. Observe that $\mathcal{B}_f = \mathcal{B}_{g,w}$ and $\mathcal{B}_h = \mathcal{B}_{i,z}$ imply $b_f = b_h$. Therefore the following mapping $\psi : B \to C$ is unambigously determined:

$\psi(r, r_1, \ldots, r_s, b_1, \ldots, b_j) = (b_{g_1}, \ldots, b_{g_n})$, where $\mathcal{B}_{g_i} = \mathcal{B}_{i,w}, i = 1, \ldots, n$, such that $\varrho(r, r_1, \ldots, r_s)$ is a prefix of w.

By an elementary computation one can show that ψ is a homomorphism of $\mathcal{B}'$ onto $\mathcal{B}$.

Now we assume that (ii) does not hold. This implies that there are a (k, ℓ)-automaton in $\mathcal{K}$ (with $k \geq 0, \ell \geq 1$) and a nonnegative integer k' with $k' < k$ such that

there exist neither a k'-automaton nor a (k', ℓ')-automaton with $k' + \ell' \geq k + \ell$ in $\mathcal{K}$.

Then there are two possible cases:

(a) for any nonnegative integer k'', there is no k''-automaton in $\mathcal{K}$;

(b) for every k''-automaton in $\mathcal{K}$, $k'' < k'$.

Therefore, for every single factor product $\mathcal{A} = (A, X, \delta)$ with a factor in $\mathcal{K}$, state $a \in A$, input letters x_1, x_2, word $p \in X^*$ with $|p| = k'$,

- either $\delta(a, px_1) = \delta(a, px_2)$

- or for every $q_1, q_2 \in X^*$ with $|q_1| = |q_2| = \ell, \delta(a, px_1q) = \delta(a, px_2q)$. In other words, $\delta(a, px_1q) = \delta(a, px_2q)$ in both cases. But then we can give an automaton which can not be represented homomorphically by a diagonal product of single factor products in $\mathcal{K}$ in the following manner.

$\mathcal{B} = (\{b_1, \ldots, b_{k'+1}, b_{k'+2,1}, b_{k'+2,2}, \ldots, b_{k+\ell+1,1}, b_{k+\ell+1,2}, *\}, \{x_1, x_2, \delta_{\mathcal{B}}),$

$$\delta_{\mathcal{B}}(b,x) = \begin{cases} b_{j+1} & \text{if } b = b_j, 1 \le j \le k' \\ b_{k'+2,1} & \text{if } b = b_{k'+1}, x = x_1, \\ b_{k'+2,2} & \text{if } b = b_{k'+1}, x = x_2, \\ b_{j+1,1} & \text{if } b = b_{j,1}, k'+2 \le j \le k+\ell, \\ b_{j+2,2} & \text{if } b = b_{j,2}, k'+2 \le j \le k+\ell, \\ * & \text{if } b \in \{b_{k+\ell,1}, b_{k+\ell,2}\}. \end{cases} \qquad \square$$

We remark that Theorem 11 and the following result show that the concept of α_0-ν_1-network of automata without any Letichevsky criteria is more general than the concept of q^ℓ-network of this like automata.

Theorem 12 [5] *Let $\mathcal{K}$ be a class of automata without any Letichevsky criteria. Then every general product of factors from $\mathcal{K}$ can be represented homomorphicaly by an α_0-ν_1-network of the same components.*

Finally, we recall a connection between single factor products and α_0-ν_1-network of automata without any Letichevsky criteria.

Lemma 13 [1] *If $\mathcal{A}$ is an automaton without any Letichevsky criteria then every single factor product of $\mathcal{A}$ can be represented homomorphically by an α_0-ν_1-network of $\mathcal{A}$.*

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