Convergence of Walsh-Fourier series of functions of bounded oscillation

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The Walsh functions

– The original Walsh system J. L. Walsh (1923)

was generated recursively, it is the Hadamard transform of the Haar system.

– The Walsh-Paley system R. E. A. C. Paley (1932)

is the finite products of Rademacher functions.

– The Walsh-Kaczmarz system A. A. Šneider (1948)

is also the finite products of Rademacher functions, but in different order

Theorem. *The Walsh system is an orthonormal and complete system on* $L^2([0,1[)$*, taking on only the values* $+1$ *and* -1*.*

The Walsh-Paley system

The Rademacher functions: H. A. Rademacher (1922)

$$r_n(x) := r(2^n x), \qquad \text{where } r(x) = \begin{cases} 1 & x \in [0, \frac{1}{2}) \\ -1 & x \in [\frac{1}{2}, 1) \end{cases}, \qquad n \in \mathbf{N}.$$

The binary expansion of n: $(n_0, n_1, \dots)$
Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k 2^k, \qquad \text{where } n_k = 0 \text{ or } n_k = 1.$$

The Walsh-Paley system:

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (x \in [0, 1[).$$

The characters of the Diadic group

The Diadic group $\left(G := \mathop{\times}\limits_{k=0}^{\infty} \mathcal{Z}_2\right)$ is the complete product of cyclic groups of order 2, with discrete topology and assign each singleton the measure $\frac{1}{2}$. G has the product topology and measure. (Haar measure)

The system of characters:
For each $n \in \mathbf{N}$ with binary expansion $(n_0, n_1, \dots)$ let

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi^{n_k}(x_k) \quad (x = (x_0, x_1, \dots) \in G), \qquad \text{where } \varphi(x) = (-1)^x \quad (x \in \mathcal{Z}_2)$$

Theorem. *The system of characters is an orthonormal and complete system on $L^2(G)$.*

The representation of the Diadic group on $[0,1[$

The dyadic rationals: $(\mathbf{Q})$

$$\mathbf{Q} := \left\{\frac{p}{2^n} : 0 \le p < 2^n,\ n,p \in \mathbf{N}\right\} \subset [0,1[.$$

The dyadic topology
is the topology induced by the intervals

$$I(p,n) := \left[\frac{p}{2^n}, \frac{p+1}{2^n}\right) \qquad (0 \le p < 2^n,\, n,p \in \mathbf{N})$$

The Fine's map: N. J. Fine (1949)
For any $x \in [0,1[$ there exists a sequence of numbers 0 and 1 such that

$$x := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \qquad ((x_0, x_1, \dots) \in G),$$

but only the dyadic rationals have two expressions of this form. In this case we have the one which terminates in 0's. Define Fine's map

$$\rho(x) = (x_0, x_1, \dots) \in G.$$

The dyadic group is metrizable by the norm

$$|x| := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \in [0,1[\qquad ((x_0, x_1, \dots) \in G).$$

The dyadic sum on the interval $[0,1[$:

$$x \oplus y := |\rho(x)\rho(y)| \qquad (x, y \in [0,1[).$$

The interval $[0,1[$ is not a group under the new operation.

Fine's map gives a natural relation between the new structure of $[0,1[$ and the structure of G (Harmonic analysis).

Theorem. *Let ρ denote the Fine's map. If f is integrable on G then $f \circ \rho$ is Lebesgue integrable and*

$$\int_G f \, d\mu = \int_0^1 (f \circ \rho)(x) \, dx.$$

Conversely, if g is Lebesgue integrable and f is defined by $f(x) := g(|x|)$ $(x \in G)$ then f is integrable on G and

$$\int_0^1 g(x) \, dx = \int_G f \, d\mu.$$

– The Haar measure corresponds to the Lebesgue measure.
– The system of characters of G corresponds to the Walsh-Paley system.

The Vilenkin systems

A Vilenkin group $\left(G := \mathop{\times}\limits_{k=0}^{\infty} \mathcal{Z}_{m_k}\right)$ N. Ja. Vilenkin (1947)

is the complete product of cyclic groups of order m_k $(m_k \geq 2,\ k \in \mathbf{N})$, with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. G has the product topology and measure. (Haar measure)

The characters of cyclic groups:

$$\varphi_k^s(x) = \exp(2\pi\imath s x/m_k) \qquad (s \in \{0, \ldots m_k - 1\},\ x \in \mathcal{Z}_{m_k},\ \imath^2 = -1)$$

The m-adic expansion of n: $(n_0, n_1, \ldots)$
Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, $(k \in \mathbf{N})$. Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \leq n_k < m_k).$$

A Vilenkin system is the product system of φ:

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x = (x_0, x_1, \ldots) \in G).$$

Theorem. *The functions of the Vilenkin system are the characters of the Vilenkin group, thus it is an orthonormal and complete system on* $L^2(G)$.

Representative product systems

Denote by $\left(G := \underset{k=0}{\overset{\infty}{\times}} G_k\right)$ the complete product of arbitrary finite groups of order m_k ($m_k \geq 2$, $k \in \mathbf{N}$), with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. G has the product topology and measure. (Haar measure)

The system φ_k:

We order the all normalized coordinate functions of the dual object of the finite group G_k ($\varphi_k^0(x) = 1$) to obtain exactly m_k number of functions.

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_k,\ s = 0, \ldots, m_k - 1),\ \text{where } \sigma \in \Sigma_k,\ i, j \in \{1, ..., d_\sigma\}.$$

The m-adic expansion of n: $(n_0, n_1, \ldots)$

Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, ($k \in \mathbf{N}$). Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \leq n_k < m_k).$$

A representative product systems is the product system of φ:

G. Gát and R. Toledo (1996)

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G).$$

Theorem. *A representative product system is an orthonormal and complete system on* $L^2(G)$.

Characteristics of the system ψ for noncommutative cases:
– It is not uniformly bounded.
– It takes the value 0.

Basic concepts of Fourier analysis

Fourier coefficients:

$$\widehat{f}_k := \int_G f \overline{w}_k \, d\mu \qquad (k \in \mathbf{N})$$

The n-th partial sums of Fourier series:

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}_k w_k \qquad (n \in \mathbf{P})$$

The Dirichlet kernels:

$$D_n(x,y) := \sum_{k=0}^{n-1} w_k(x) \overline{w}_k(y) \qquad (n \in \mathbf{P})$$

The dyadic modulus of continuity:

$$\omega\left(f,\delta\right) = \sup_{0<h<\delta} \sup_{x} \left|f\left(x \oplus h\right) - f\left(x\right)\right|,$$

Uniform convergence

Theorem (Dini-Lipschitz). *N. I. Fine (1949)*

Let f be a W-continuous on $[0,1]$ and suppose that

$$\omega\left(f,\delta\right)=o\left(\log\frac{1}{\delta}\right)^{-1}\quad \textit{as}\quad \delta\to 0+,$$

then S_nf converges uniformly on $[0,1]$.

Theorem. *G. W. Onnewer (1970)*

Let f be a W-continuous on $[0,1]$ and suppose that

$$\sum_{k=1}^{2^n-1}\frac{\left|f\left(x\oplus\frac{2k}{2^{n+1}}\right)-f\left(x\oplus\frac{2k+1}{2^{n+1}}\right)\right|}{k}\to 0\quad \textit{as}\quad n\to\infty$$

uniformly with respect to x, then S_nf converges uniformly on $[0,1]$.

Functions of p-bounded fluctuation $(1\le p<\infty)$

$$\mathfrak{F}\ell_p:=\sup_{n\in\mathbf{P}}\left(\sum_{k=0}^{2^n-1}|\omega(f,|I(k,n)|)|^p\right)^{1/p}<\infty$$

Corollary. *Let f be a W-continuous function on $[0,1]$ of 1-bounded fluctuation. Then S_nf converges uniformly on $[0,1]$.*

Functions of bounded oscillation

The $BO\left(p\left(n\right)\uparrow p\right)$ class:
Let $1 \leq p\left(n\right)\uparrow p$ as $n \to \infty$ where $1 \leq p \leq \infty$. We say that a function f belongs to the $BO\left(p\left(n\right)\uparrow p\right)$ class if

$$O\left(f; p\left(n\right)\uparrow p\right) = \sup_{n} \left\{ \sum_{k=0}^{2^n-1} \sup_{u,t\in I(k,n)} \left|f\left(u\right) - f\left(t\right)\right|^{p(n)} \right\}^{1/p(n)} < \infty,$$

– The $BO(p(n)\uparrow p)$ class is more wider than the class $BV(p(n)\uparrow p)$ which was introduced by Japan Mathematicians Kita and Yoneda in 1990.

– When $p\left(n\right) = p$ for all n, $BO\left(p\left(n\right)\uparrow p\right) = \mathfrak{F}\ell_p$.

– If $p < \infty$ then $BO\left(p\left(n\right)\uparrow p\right) \subseteq Fl_p$.

– $\bigcup\limits_{p<\infty} Fl_p \subset BO\left(p\left(n\right)\uparrow \infty\right).$

Uniform convergence and $\mathbf{BO(p(n)\uparrow\infty)}$

Theorem. *U. Goginava (2001)*

If f be a function in the class $BO\left(p\left(n\right)\uparrow\infty\right)$ and

$$\omega\left(\delta,f\right)=o\left(\frac{1}{p\left(\left[\log\frac{1}{\delta}\right]\right)\log p\left(\left[\log\frac{1}{\delta}\right]\right)}\right)\quad \text{as } \delta\to 0+$$

then the Walsh-Fourier series of function f converges uniformly in $[0,1]$.

Theorem. *U. Goginava (2001)*

There exists a function in the class $BO\left(p\left(n\right)\uparrow\infty\right)$ for which

$$\omega\left(\delta,f\right)=O\left(\frac{1}{p\left(\left[\log\frac{1}{\delta}\right]\right)\log p\left(\left[\log\frac{1}{\delta}\right]\right)}\right)\quad \text{as } \delta\to 0+$$

and the Walsh-Fourier series diverges at some point.

Pointwise convergence

U. Goginava and R. Toledo

It is well-known that, in general, the condition

$$\sup_{0<h<\delta} \left| f\left(x_0 \oplus h\right) - f\left(x_0\right) \right| = o\left(\frac{1}{\log \frac{1}{\delta}}\right) \quad \text{as } \delta \to 0+,$$

does not guarantee the convergence of $S_n f\left(x_0\right)$.

Lemma. *Let* $p\left(n\right) \log p\left(n\right) = o\left(n\right)$. *If* $f \in BO\left(p\left(n\right) \uparrow \infty\right)$ *and*

$$\sup_{0<h<\delta} \left| f\left(x_0 \oplus h\right) - f\left(x_0\right) \right| = o\left(\frac{1}{\log \frac{1}{\delta}}\right) \quad \textit{as } \delta \to 0+$$

then

$$\lim_{n\to\infty} S_n f\left(x_0\right) = f\left(x_0\right).$$

Theorem. *Let* $p(n)\log p(n) = o(n)$ *and* $p(2n) \leq cp(n)$ *for all* $n \geq 1$, *where* $c > 0$ *is a constant. If* $f \in BO(p(n) \uparrow \infty)$ *and*

$$\sup_{0<h<\delta} |f(x_0 \oplus h) - f(x_0)| = o\left(\frac{1}{p\left(\left[\log \frac{1}{\delta}\right]\right) \log p\left(\left[\log \frac{1}{\delta}\right]\right)}\right) \quad \text{as } \delta \to 0+,$$

then

$$\lim_{n\to\infty} S_n f(x_0) = f(x_0).$$

Estimate of Fourier coefficients

U. Goginava and R. Toledo

Theorem. *Let* $f \in BO\,(p\,(n) \uparrow p)$ *where* $1 \leq p \leq \infty$ *and* $2^n \leq k < 2^{n+1}$. *Thus*

$$|\widehat{f}(k)| \leq \frac{O\,(f; p\,(n) \uparrow p)}{2^{1+n/p(n)}} \qquad (k \in \mathbf{P}).$$

Corollary. *Let* $f \in BO\,(p\,(n) \uparrow p)$ *where* $1 \leq p < \infty$ *and* $2^n \leq k < 2^{n+1}$. *Thus*

$$|\widehat{f}(k)| \leq \frac{O\,(f; p\,(n) \uparrow p)}{\sqrt[p]{k}} \qquad (k \in \mathbf{P}).$$

Corollary. *Let* $f \in BO\,(p\,(n) \uparrow \infty)$ *where* $1 \leq p < \infty$ *and* $2^n \leq k < 2^{n+1}$. *If* $p(n) \log p(n) = o(n)$ *then*

$$|\widehat{f}(k)| \leq o(p(n)^{-1}) \qquad (k \in \mathbf{P}).$$