A generalization of Walsh and Vilenkin systems

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The Walsh functions

– The original Walsh system J. L. Walsh (1923)

was generated recursively, it is the Hadamard transform of the Haar system.

– The Walsh-Paley system R. E. A. C. Paley (1932)

is the finite products of Rademacher functions.

– The Walsh-Kaczmarz system A. A. Šneider (1948)

is also the finite products of Rademacher functions, but in different order

Theorem. *The Walsh system is an orthonormal and complete system on* $L^2([0,1[)$*, taking on only the values* $+1$ *and* -1*.*

The Walsh-Paley system

The Rademacher functions: H. A. Rademacher (1922)

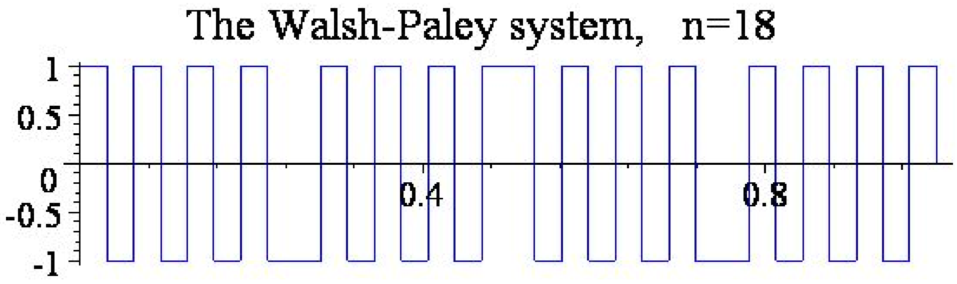
$$r(x) = \operatorname{sgn}(\sin(2^{n+1}\pi x)) \qquad x \in [0,1[.$$

The binary expansion of n: $(n_0, n_1, \dots)$
Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k 2^k, \qquad \text{where } n_k = 0 \text{ or } n_k = 1.$$

The Walsh-Paley system:

$$\omega_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (x \in [0,1[).$$


The Walsh-Paley system, n=18

The characters of the Diadic group

The Diadic group $\left(G := \mathop{\times}_{k=0}^{\infty} \mathcal{Z}_2\right)$ is the complete product of cyclic groups of order 2, with discrete topology and assign each singleton the measure $\frac{1}{2}$. G has the product topology and measure. (Haar measure)

The system of characters:

For each $n \in \mathbf{N}$ with binary expansion $(n_0, n_1, \ldots)$ let

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi^{n_k}(x_k) \quad (x = (x_0, x_1, \ldots) \in G), \qquad \text{where } \varphi(x) = (-1)^x \quad (x \in \mathcal{Z}_2)$$

Theorem. *The system of characters is an orthonormal and complete system on* $L^2(G)$*.*

The representation of the Diadic group on $[0,1[$

The dyadic rationals: ($\mathbf{Q}$)

$$\mathbf{Q} := \left\{ \frac{p}{2^n} : 0 \le p < 2^n, \; n, p \in \mathbf{N} \right\} \subset [0,1[.$$

The Fine's map: N. J. Fine (1949)

For any $x \in [0,1[$ there exists a sequence of numbers 0 and 1 such that

$$x := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \qquad ((x_0, x_1, \dots) \in G),$$

but only the dyadic rationals have two expressions of this form. In this case we have the one which terminates in 0's. Define Fine's map

$$\rho(x) = (x_0, x_1, \dots) \in G.$$

The dyadic group is metrizable by the norm

$$|x| := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \in [0,1[\qquad ((x_0, x_1, \dots) \in G).$$

A new operation on the interval $[0,1[$:

$$x \odot y := |\rho(x)\rho(y)| \qquad (x, y \in [0,1[).$$

The interval $[0,1[$ is not a group under the new operation.

Fine's map gives a natural relation between the new structure of $[0,1[$ and the structure of G (Harmonic analysis).

Theorem. *Let* ρ *denote the Fine's map. If* f *is integrable on* G *then* $f \circ \rho$ *is Lebesgue integrable and*

$$\int_G f\, d\mu = \int_0^1 (f \circ \rho)(x)\, dx.$$

Conversely, if g *is Lebesgue integrable and* f *is defined by* $f(x) := g(|x|)$ $(x \in G)$ *then* f *is integrable on* G *and*

$$\int_0^1 g(x)\, dx = \int_G f\, d\mu.$$

– The Haar measure corresponds to the Lebesgue measure.
– The system of characters of G corresponds to the Walsh-Paley system.

The Vilenkin groups

A Vilenkin group $\left(G := \mathop{\times}\limits_{k=0}^{\infty} \mathcal{Z}_{m_k}\right)$ N. Ja. Vilenkin (1947)

is the complete product of cyclic groups of order m_k $(m_k \geq 2,\ k \in \mathbf{N})$, with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. G has the product topology and measure. (Haar measure)

Bounded Vilenkin group:

if the sequence $m = (m_0, m_1, \ldots)$ is a bounded sequence.

The generalized Rademacher functions:

$$\varphi_k^s(x) = \exp(2\pi\imath s x/m_k) \qquad (s \in \{0, \ldots m_k - 1\},\ x \in \mathcal{Z}_{m_k},\ \imath^2 = -1)$$

The generalized Rademacher functions are the characters of cyclic groups.

The Vilenkin systems

The m-adic expansion of n: $(n_0, n_1, \dots)$
Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, $(k \in \mathbf{N})$. Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \le n_k < m_k).$$

A Vilenkin system is the product system of φ:

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x = (x_0, x_1, \dots) \in G).$$

Theorem. *The functions of the Vilenkin system are the characters of the Vilenkin group, thus it is an orthonormal and complete system on* $L^2(G)$*.*

The complete product of finite groups

Denote by $\left(G := \mathop{\times}\limits_{k=0}^{\infty} G_k\right)$

the complete product of arbitrary finite groups of order m_k ($m_k \geq 2$, $k \in \mathbf{N}$), with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. G has the product topology and measure. (Haar measure)

The group G is bounded

if the sequence $m = (m_0, m_1, \dots)$ is a bounded sequence.

$\varphi_k^s =?$, $\psi_n =?$ → Harmonic Analysis

Orthonormal systems on finite groups

The dual object (Σ_k) of the finite group G_k $(k \in \mathbf{N})$

is the set of all continuous irreducible unitary representations of the group G_k which are not equivalents.

The Coordinate functions:

For any $\sigma \in \Sigma_k$, let $\{\xi_1, \ldots, \xi_{d_\sigma}\}$ be a fixed basis of the representation space of a representation $U^{(\sigma)}$ in the class σ having the dimension d_σ.The Coordinate functions:

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j \rangle, \quad i, j \in \{1, ..., d_\sigma\}, \sigma \in \Sigma_k$$

The system φ_k:

We order the all normalized coordinate functions of the finite group G_k $(\varphi_k^0(x) = 1)$ to obtain exactly m_k number of functions.

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_k,\ s = 0, \ldots, m_k - 1),\ \text{where } \sigma \in \Sigma_k,\ i, j \in \{1, ..., d_\sigma\}.$$

Theorem. *The system φ_k is an orthonormal and complete system on $L^2(G_k)$.*

Example 1

The permutation group of 3 elements: $\mathcal{S}_3$

	e	(12)	(13)	(23)	(123)	(132)	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

$$\max_{s=0\ldots5} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3}$$

Example 2

The quaternion group of order 8: $\mathcal{Q}_2$

$$\mathcal{Q}_2 := \{[a,b] : a^4 = e,\ b^2 = a^2,\ bab^{-1} = a^3\}.$$

	e	a	a^2	a^3	b	ab	a^2b	a^3b	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1	1	1
φ^2	1	-1	1	-1	1	-1	1	-1	1	1
φ^3	1	-1	1	-1	-1	1	-1	1	1	1
φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}\imath$	$\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$

$$\max_{s=0\ldots7} \|\varphi^s\|_1 \|\varphi^s\|_\infty = 1$$

Representative product systems

The m-adic expansion of n: $(n_0, n_1, \dots)$
Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, $(k \in \mathbf{N})$. Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \le n_k < m_k).$$

A representative product systems is the product system of φ:

G. Gát and R. Toledo (1996)

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G).$$

Theorem. *A representative product system is an orthonormal and complete system on $L^2(G)$.*

Characteristics of the system ψ for noncommutative cases:
– It is not uniformly bounded.
– It takes the value 0.

The representation of G on $[0,1[$

It is similar to the dyadic group, but we need the ordination of the elements of all groups G_k, $(k \in \mathbf{N})$ in some way such that the first is always their identity.

$$G_k \ni x \overset{\text{bijection}}{\longrightarrow} \overline{x} \in \{0, 1, \dots, m_k - 1\}, \qquad \overline{e} = 0.$$

The m-adic rationals: $(\mathbf{Q})$

$$\mathbf{Q} := \left\{ \frac{p}{M_n} : 0 \le p < M_n,\ \ n, p \in \mathbf{N} \right\} \subset [0,1[.$$

The Fine's map:

For any $x \in [0,1[$ there exists a sequence such that

$$x := \sum_{k=0}^{\infty} \frac{\overline{x_k}}{M_{k+1}} \qquad (0 \le \overline{x_k} \le m_k - 1),$$

but only the m-adic rationals have two expressions of this form. In this case we have the one which terminates in 0's. Define Fine's map

$$\rho(x) = (x_0, x_1, \dots) \in G.$$

The dyadic group is metrizable by the norm

$$|x| := \sum_{k=0}^{\infty} \frac{\overline{x_k}}{M_{k+1}} \in [0,1[\qquad ((x_0, x_1, \dots) \in G).$$

A new operation on the interval $[0,1[$:

$$x \odot y := |\rho(x)\rho(y)| \qquad (x, y \in [0,1[).$$

The interval $[0,1[$ is not a group under the new operation.

Fine's map gives a natural relation between the new structure of $[0,1[$ and the structure of G (Harmonic analysis).

Theorem. *Let ρ denote the Fine's map. If f is integrable on G then $f \circ \rho$ is Lebesgue integrable and*

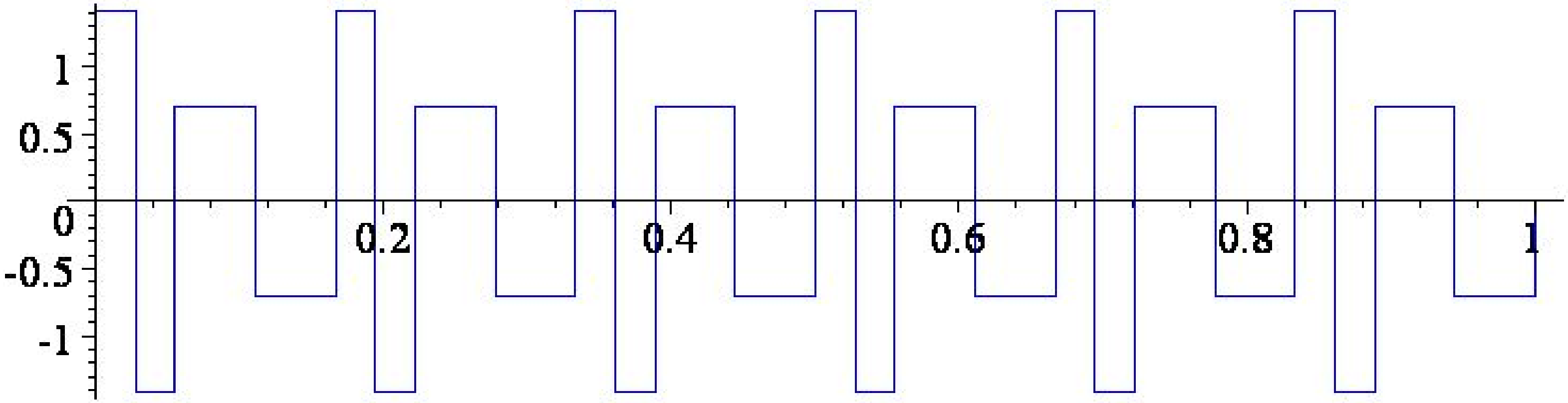
$$\int_G f\,d\mu = \int_0^1 (f \circ \rho)(x)\,dx.$$

Conversely, if g is Lebesgue integrable and f is defined by $f(x) := g(|x|)$ $(x \in G)$ then f is integrable on G and

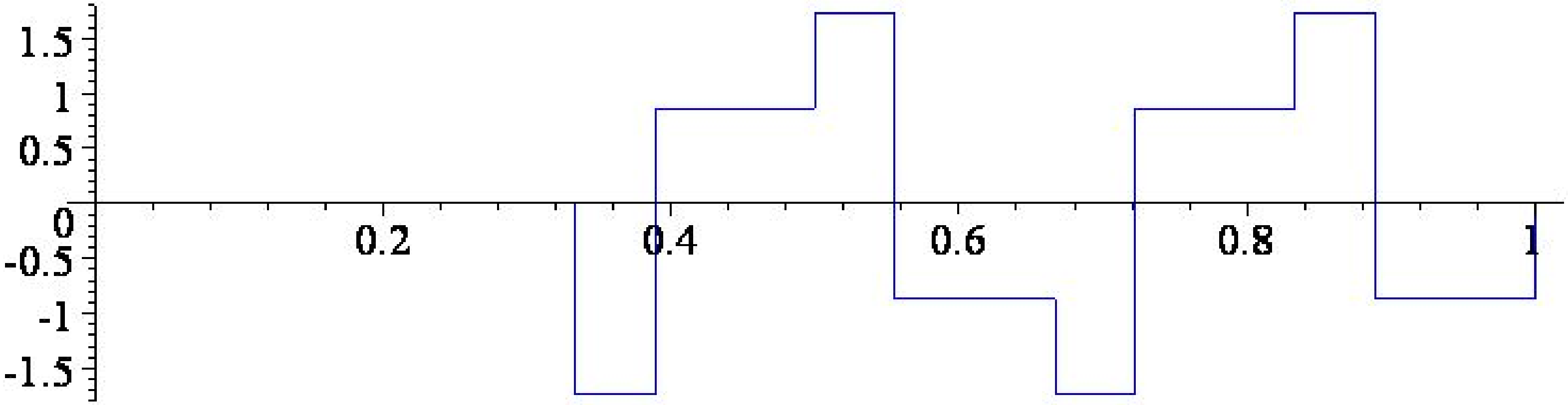
$$\int_0^1 g(x)\,dx = \int_G f\,d\mu.$$

– The Haar measure corresponds to the Lebesgue measure.
– The new systems $\psi_n \circ \rho$ are orthonomal and complete systems on $[0,1[$, but they are not necessary uniformly bounded.

Examples for the complete product of $\mathcal{S}_3$



n=12



n=23

Basic concepts of Fourier analysis

Fourier coefficients:

$$\widehat{f}_k := \int_G f \overline{\psi}_k d\mu \qquad (k \in \mathbf{N})$$

The n-th partial sums of Fourier series:

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \qquad (n \in \mathbf{P})$$

The Dirichlet kernels:

$$D_n(x,y) := \sum_{k=0}^{n-1} \psi_k(x) \overline{\psi}_k(y) \qquad (n \in \mathbf{P})$$

The Fejér means of Fourier series:

$$\sigma_n f = \frac{1}{n} \sum_{k=1}^{n-1} S_k f \qquad (n \in \mathbf{P})$$

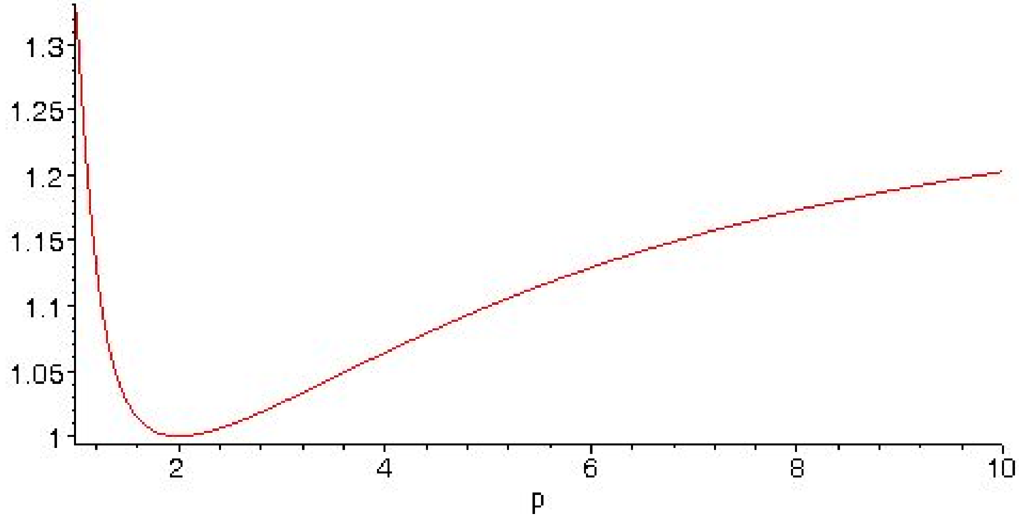
The sequence Ψ

Let $p \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $k \in \mathbf{N}$.

$$\Psi_k(p) := \max_{n<M_k} \|\psi_n\|_p \|\psi_n\|_q = \prod_{i=0}^{k-1} \max_{s<m_i} \|\varphi_i^s\|_p \|\varphi_i^s\|_q$$

$$\Psi_k := \Psi_k(1) := \max_{n<M_k} \|\psi_n\|_1 \|\psi_n\|_\infty$$

$\Psi_1(p)$ for the complete product of $\mathcal{S}_3$



Convergence of Fourier series in L^p-norm

The problem

Which are the values of p ($1 \leq p < \infty$) such that for all function $f \in L^p(G)$ the sequence of partial sums $S_n f$ of the Fourier series of f converges to the function f in L^p-norm?

For $\mathbf{p = 2}$ the statement is true. ($L^2(G)$ is a Hilbert space)

R. Toledo (1999)

Theorem (For $\mathbf{p = 1}$). *For an arbitrary group G there exists a function $f \in L^1(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^1-norm.*

P. Simon, F. Schipp and W. S. Young (1976)

Theorem (For Vilenkin). *Let G be a Vilenkin group and $1 < p < \infty$. Then for all function $f \in L^p(G)$ the sequence of partial sums $S_n f$ of the Fourier series of f converges to the function f in L^p-norm.*

Theorem. *Let* p *be a fix number in the interval* $(1,2)$ *and* $\frac{1}{p}+\frac{1}{q}=1$*. If* G *is a group with unbounded sequence*

$$\Psi_k(p) := \max_{n<M_k} \|\psi_n\|_p \|\psi_n\|_q \qquad (k \in \mathbf{N}),$$

then there exists a function $f \in L^p(G)$ *such that the sequence of partial sums* $S_n f$ *of the Fourier series of* f *does not converge to the function* f *in* L^p*-norm.*

Corollary. *If* G *is a bounded group with unbounded sequence* Ψ*, then for all* $p \neq 2$ *there exists a function* $f \in L^p(G)$ *such that the sequence of partial sums* $S_n f$ *of the Fourier series of* f *does not converge to the function* f *in* L^p*-norm.*

Corollary. *If* G *is the complete product of* $\mathcal{S}_3$*, then for all* $p \neq 2$ *there exists a function* $f \in L^p(G)$ *such that the sequence of partial sums* $S_n f$ *of the Fourier series of* f *does not converge to the function* f *in* L^p*-norm.*

Convergence of Fejér means in L^p-norm

G. Gát and R. Toledo (1996)

Theorem. *If G is a bounded group and $f \in L^p(G)$, $1 < p < \infty$, then $\sigma_n f \to f$ in L^p-norm.*