Negative results concerning Fourier series on the complete product of $\mathcal{S}_3$

R. Toledo

September 19-25, 2005

Budapest

The Walsh system J. L. Walsh (1923)

The Walsh-Paley system R. E. A. C. Paley (1932)

as the finite products of Rademacher functions

Diadic group $\left(G := \bigtimes_{k=0}^{\infty} \mathbb{Z}_2\right)$ N. J. Fine (1949)

the Walsh-Paley system is a system of characters of this group.

The Vilenkin group N. Ja. Vilenkin (1947)

Let $\mathbb{Z}_{m_k}$ be the cyclic group of order m_k with discrete topology and assign each singleton the measure $\frac{1}{m_k}$.

$$G := \bigtimes_{k=0}^{\infty} \mathbb{Z}_{m_k}$$

with the product topology and measure

The characters of $\mathcal{Z}_{m_k}$ are the generalized Rademacher functions:

$$\varphi_k^s(x) = \exp(2\pi\imath sx/m_k) \qquad (s \in \{0,\dots m_k - 1\},\ x \in \mathcal{Z}_{m_k},\ \imath^2 = -1)$$

Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, $(k \in \mathbf{N})$.

The expansion of n with respect to m is $(n_0, n_1, \dots)$, where

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \le n_k < m_k).$$

The product system of φ is called the Vilenkin system:

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G).$$

ψ is an orthonormal and complete system in $L^1(G)$.

The complete product of finite groups

G. Gát and R. Toledo, Anal. Math., 1996.

G_k : a finite group with order m_k, ($k \in \mathbf{N}$) with discrete topology and normalized Haar measure

$m := (m_k,\ k \in \mathbf{N})$: a sequence of positive integers ($m_k \geq 2$)

$$G := \mathop{\times}_{k=0}^{\infty} G_k$$

with the product topology and measure

$\varphi_k^s = ?$, $\psi_n = ?$ $\longrightarrow$ Harmonic Analysis

The dual object (Σ_k) of the finite group G_k: the set of all continuous irreducible unitary representations of the group G_k which are not equivalents

For any $\sigma \in \Sigma_k$, let $\{\xi_1, \ldots, \xi_{d_\sigma}\}$ be a fixed basis of the representation space of a representation $U^{(\sigma)}$ in the class σ having the dimension d_σ.

Coordinate functions:

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j \rangle, \quad i, j \in \{1, ..., d_\sigma\}, \sigma \in \Sigma_k$$

We order the all normalized coordinate functions of the finite group G_k to obtain φ: $(\varphi_k^0(x) = 1)$

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_k), \text{ where } \sigma \in \Sigma_k, \ i, j \in \{1, ..., d_\sigma\}.$$

We obtain exactly m_k number of functions.

Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, $(k \in \mathbf{N})$

The expansion of n with respect to m is $(n_0, n_1, \dots)$, where

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \le n_k < m_k).$$

The representative product systems:
Let ψ be the product system of φ_k^s, namely

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G).$$

Weyl-Peter's theorem:
ψ is an orthonormal and complete system in $L^2(G)$.

Characteristics of the system ψ for noncommutative cases:

– It is not uniformly bounded.

– It takes the value 0.

$$\|\psi_n\|_1 \le \|\psi_n\|_2 = 1 \le \|\psi_n\|_\infty$$

The sequence Ψ:

$$\Psi_k(p) := \max_{n<M_k} \|\psi_n\|_p \|\psi_n\|_q \qquad \left(p \ge 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ k \in \mathbf{N}\right)$$

$$\Psi_k := \Psi_k(1) := \max_{n<M_k} \|\psi_n\|_1 \|\psi_n\|_\infty \qquad (k \in \mathbf{N})$$

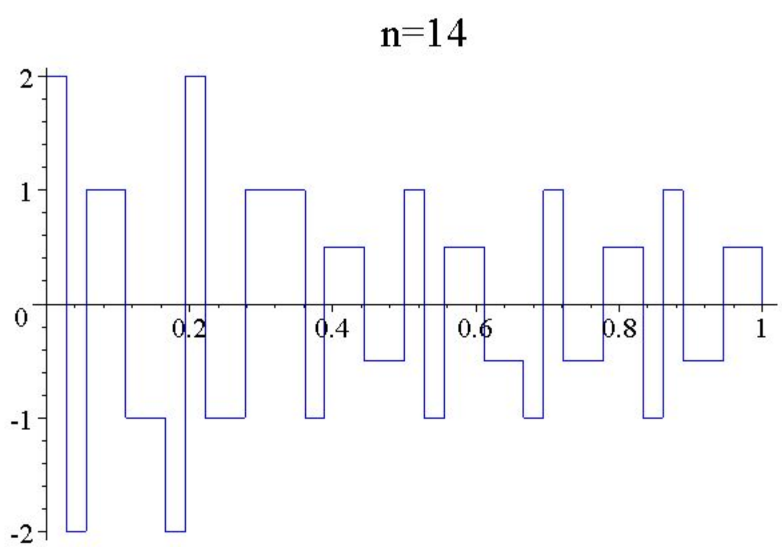
Example 1

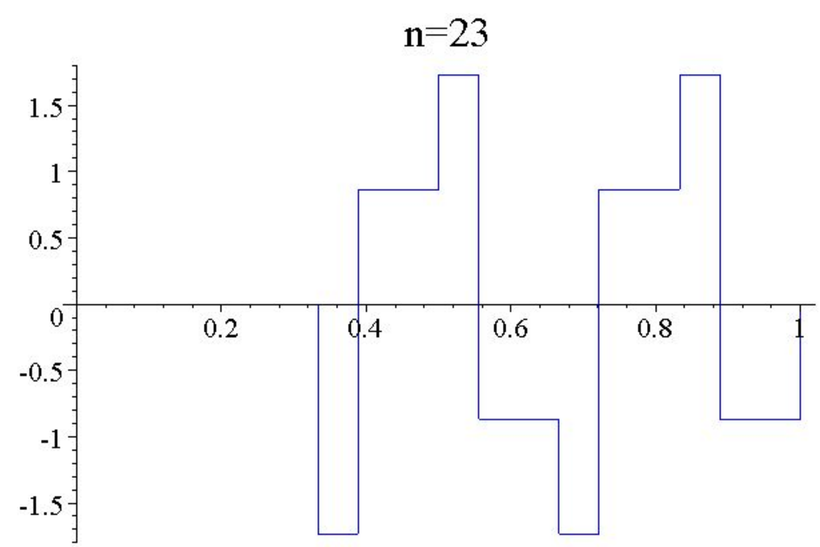
The complete product of $\mathcal{S}_3$

	e	(12)	(13)	(23)	(123)	(132)	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

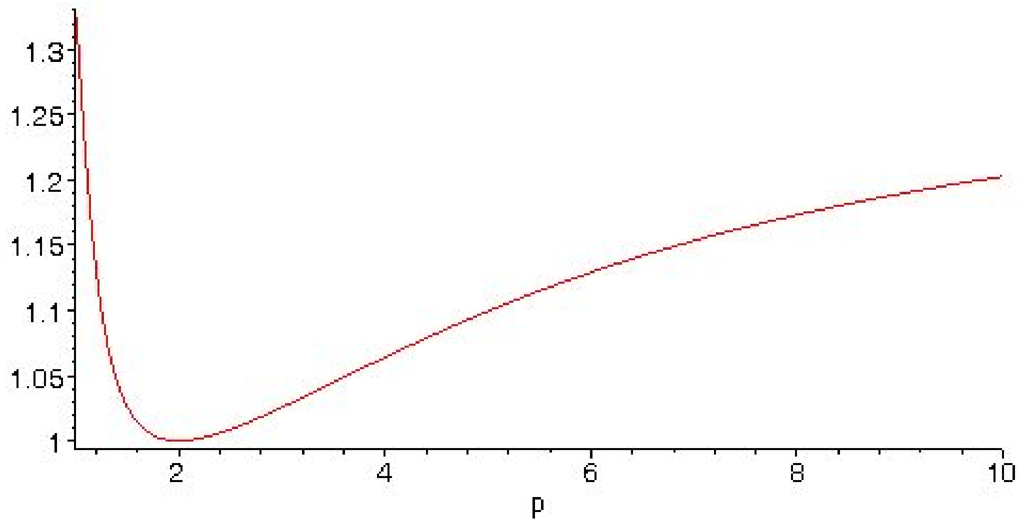
$$\Psi_k = \left(\frac{4}{3}\right)^k \to \infty$$

Representation on the interval $[0, 1]$





$\Psi_1(p)$ for the complete product of $\mathcal{S}_3$



Example 2

The complete product of $\mathcal{Q}_2$

	e	a	a^2	a^3	b	ab	a^2b	a^3b	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1	1	1
φ^2	1	-1	1	-1	1	-1	1	-1	1	1
φ^3	1	-1	1	-1	-1	1	-1	1	1	1
φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}\imath$	$\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$

$$\Psi_k(p) \equiv 1 \qquad 1 \le p \le \infty$$

Fourier coefficients:

$$\widehat{f}_k := \int_G f\overline{\psi}_k d\mu \qquad (k \in \mathbf{N})$$

The n-th partial sums of Fourier series:

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \qquad (n \in \mathbf{P})$$

The Dirichlet kernels:

$$D_n(x,y) := \sum_{k=0}^{n-1} \psi_k(x)\overline{\psi}_k(y) \qquad (n \in \mathbf{P})$$

The Fejér means of Fourier series

$$\sigma_n f = \frac{1}{n} \sum_{k=1}^{n-1} S_k f \qquad (n \in \mathbf{P})$$

The intervals: $I_n(x) := \{y \in G : y_k = x_k,$ for $0 \le k < n\}$

$$I_0(x) := G, \qquad I_n := I_n(e).$$

Lemma. *(Paley lemma) If $n \in \mathbf{N}$ and $x, y \in G$, then*

$$D_{M_n}(x,y) = \begin{cases} M_n & \text{for } x \in I_n(y), \\ 0 & \text{for } x \notin I_n(y) \end{cases}$$

Corollary. *For each $f \in L^p(G)$, $p \ge 1$ and $n \in \mathbf{N}$ the $S_{M_n}f$ sums converge to f in L^p-norm and a.e.*

R. Toledo, Proc. of Alexits Memorial Conference, 1999.

Theorem. *For an arbitrary group G there exists a function $f \in L^1(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^1-norm.*

G. Gát and R. Toledo, Anal. Math., 1996.

Theorem. *If G is a bounded group with unbounded sequence Ψ, then there exist a $1 < p < 2$ and a function $f \in L^p(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^p-norm.*

The problem is open for the general case.

Theorem. *Let p be a fix number in the interval $(1,2)$ and $\frac{1}{p}+\frac{1}{q}=1$. If G is a group with unbounded sequence*

$$\Psi_k(p) := \max_{n<M_k} \|\psi_n\|_p \|\psi_n\|_q \qquad (k \in \mathbf{N}),$$

then there exists a function $f \in L^p(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^p-norm.

Corollary. *If G is a bounded group with unbounded sequence Ψ, then for all $p \neq 2$ there exists a function $f \in L^p(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^p-norm.*

Corollary. *If G is the complete product of $\mathcal{S}_3$, then for all $p \neq 2$ there exists a function $f \in L^p(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^p-norm.*

G. Gát and R. Toledo, Anal. Math., 1996.

Theorem. *If* G *is a bounded group and* $f \in L^p(G)$, $1 < p < \infty$, *then* $\sigma_n f \to f$ *in* L^p*-norm.*