(iii) $(X : Y \cap Z) = (X : Y) + (X : Z)$ for any finitely generated submodules Y, Z and arbitrary submodule X.

Theorem 12. Let R be a semi-local ring. If M is a distributive R-module, then it would be a Bézot module.

Theorem 13. Let $(R, \mathfrak{m})$ be a quasi-local ring. For an R-module M the following statements are equivalent:

(i) M is uniserial;

(ii) If N is a completely irreducible submodules of M, then there exists a maximal ideal $\mathfrak{m}$ and a nonzero element $x \in M$ such that $N = \mathfrak{m}x$;

(iii) For each nonzero element $x \in M, \mathfrak{m}x$ is an irreducible submodule of M.

Theorem 14. Let M be a module over a ring R. Then, M is distributive iff its completely irreducible submodules is the set

$$\{\mathfrak{m}Rx_{(\mathfrak{m})} \;:\; \mathfrak{m} \in Max(R),\ x \in M \text{ and } \mathfrak{m} \in \mathrm{Supp}(Rx)\}.$$

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Proof of Cantor's Hypothesis

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Abstract

In this talk shall attempt to prove Cantors Hypothesis, a special case of which is known as the Continuum Hypothesis, namely that the cardinality of the power set of an infinite set is the consecutive cardinality. An ordered field of cardinality $\aleph_\alpha$ with interval topology of weight $\aleph_\alpha$ is constructed, where $\aleph_\alpha$ is an uncountable isolated cardinal.

We list here some results:

Theorem 1. Cantor's Hypothesis: Cardinality of the power set of an infinite set is the consecutive cardinality.
Let ω and Ω be the first countably infinite and the first uncountable ordinal of cardinalities $\aleph_0$ and $\aleph_1$, respectively. The well-known special case:

Theorem 2. Continuum Hypothesis: Cardinality of the power set of ω is the cardinality $\aleph_1$.
Construct a number system "of base ω" for countable ordinals. For any countable ordinal α, let $\tau_0 = 1$, and

$$\tau_\alpha = \sup\{\tau_\beta * j \mid \beta < \alpha,\, j \in \omega\}.$$

Theorem 3. Any nonzero countable ordinal γ can be written uniquely as $\gamma = \bigoplus_{j=1}^{n} \tau_{\alpha_j} * k_j$ with $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, the $k_j \in \omega$ are all nonzero.
Let ${}_{\mathbb{Z}}\Omega$ be the free left module over the ring of integers $\mathbb{Z}$ with well-ordered basis $\{\tau_\mu \mid \mu \in \Omega\}$. The set Ω of countable ordinals has a natural embedding $f : \Omega \to {}_{\mathbb{Z}}\Omega$, $0 \mapsto 0$, $\bigoplus_{i=1}^{n} \tau_{\mu_i} * k_i \mapsto \sum_{i=1}^{n} k_i \tau_{\mu_i}$. The well-order of the basis induces an order on the module ${}_{\mathbb{Z}}\Omega$. Let the positivity domain ${}_{\mathbb{Z}}\Omega^+$ be the set of all elements with positive leading coefficients k_1.

Theorem 4. The pair $({}_{\mathbb{Z}}\Omega, +)$ becomes an ordered free abelian group of cardinality $\aleph_1$.

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Define multiplication of basis elements of the module by the rule $\tau_\mu \tau_\nu = \tau_{f^{-1}(f(\mu)+f(\nu))}$, where f is the natural embedding. Extend the multiplication by distributivity.

Theorem 5. $({}_{\mathbb{Z}}\Omega, +, \cdot)$ becomes an ordered integral domain.
Let Q_Ω be the quotient field of the integral domain with positivity domain Q_Ω^+ the set of all fractions with numerator and denominator both either positive or negative.

Theorem 6. Q_Ω is a (linearly) ordered field of cardinality $\aleph_1$, endowed with the interval topology.

Theorem 7. A strictly increasing sequence $\{u_i\}_{i\in\omega^+}$ in the unit interval $[0, 1]$ of the field Q_Ω has an upper bound h so that $h - \varepsilon$ is not an upper bound for an arbitrarily small $\varepsilon \in Q_\Omega^+$, called an ε-least upper bound.
A nondegenerate interval of the field Q_Ω contains uncountably many elements.

Theorem 8. In the field Q_Ω the intersection of a strictly decreasing sequence of closed intervals within the unit interval contains a nondegenerate closed interval.
Apply the construction of Cantor's triadic set in the field Q_Ω. It follows that Q_Ω contains a subset of continuum cardinality. Consequently $\aleph_1$ is the cardinality of the power set of a set of cardinality $\aleph_0$.
Proof of the general Hypothesis is completely analogous.