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ALMOST EVERYWHERE CONVERGENCE OF SEQUENCES OF TWO-DIMENSIONAL WALSH–FEJÉR MEANS OF INTEGRABLE FUNCTIONS

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Abstract. The aim of this paper is to prove the a.e. convergence of sequences of the Fejér means of the Walsh–Fourier series of bivariate integrable functions. That is, let $a = (a_1, a_2) : \mathbb{N} \to \mathbb{N}^2$ such that $a_j(n+1) \geqq \delta \sup_{k \leqq n} a_j(n)$ ($j = 1, 2$, $n \in \mathbb{N}$) for some $\delta > 0$ and $a_1(+\infty) = a_2(+\infty) = +\infty$. Then for each integrable function $f \in L^1(I^2)$ we have the a.e. relation $\lim_{n\to\infty} \sigma_{a_1(n),a_2(n)} f = f$. It will be a straightforward and easy consequence of this result the cone restricted a.e. convergence of the two-dimensional Walsh–Fejér means of integrable functions which was proved earlier by the author and Weisz [3,8].

First, we give a brief introduction to the theory of Walsh–Fourier series.

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$, and $I := [0, 1)$. For any set E let E^2 be the Cartesian product $E \times E$. Thus, $\mathbb{N}^2$ is the set of integral lattice points in the first quadrant and I^2 is the unit square. Let $E^1 = E$ and fix $j = 1$ or 2. Denote the j-dimensional Lebesgue measure of any set $E \subset I^j$ by $\mu(E)$. Denote the $L^p(I^j)$ norm of any function f by $\|f\|_p$ $(1 \leqq p \leqq \infty)$.

Denote the dyadic expansion of $n \in \mathbb{N}$ and $x \in I$ by $n = \sum_{j=0}^{\infty} n_j 2^j$ and $x = \sum_{j=0}^{\infty} x_j 2^{-j-1}$ (in the case of $x = \frac{k}{2^m}$ $k, m \in \mathbb{N}$ choose the expansion which terminates in zeros). n_i, x_i are the i-th coordinates of n, x, respectively. Set $e_i := 1/2^{i+1} \in I$, the ith coordinate of e_i is 1, the rest are zeros

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($i \in \mathbb{N}$). Define the dyadic addition $+$ as

$$x + y = \sum_{j=0}^{\infty} |x_j - y_j| 2^{-j-1}.$$

The sets $I_n(x) := \{y \in I : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$ for $x \in I$, $I_n := I_n(0)$ for $n \in \mathbb{P}$ and $I_0(x) := I$ are the dyadic intervals of I. The set of dyadic intervals on I is denoted by $\mathcal{I} := \{I_n(x) : x \in I, n \in \mathbb{N}\}$. Denote by $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)$ $(x \in I)$ and let E_n be the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbb{N})$. C denotes a constant which may be different from line to line.

For $t = (t_1, t_2) \in I^2$, $b = (b_1, b_2) \in \mathbb{N}^2$ let $I_b(t) := I_{b_1}(t_1) \times I_{b_2}(t_2)$ be the two-dimensional dyadic rectangle, i.e. the two-dimensional dyadic interval on I^2. We also use the notation $I_b(t) := I_b(t_1) \times I_b(t_2)$ for $b \in \mathbb{N}$, $t = (t_1, t_2)$. For $n = (n_1, n_2) \in \mathbb{N}^2$ denote by $E_n = E_{n_1,n_2}$ the two-dimensional expectation operator with respect to the σ algebra $\mathcal{A}_n = \mathcal{A}_{n_1,n_2}$ generated by the two-dimensional rectangles $I_{n_1}(x_1) \times I_{n_2}(x_2)$ $\left(x = (x_1, x_2) \in I^2\right)$. For $n \in \mathbb{N}$ denote $|n| := \max(j \in \mathbb{N} : n_j \neq 0)$, that is, $2^{|n|} \leqq n < 2^{|n|+1}$. The Rademacher functions on I are defined as

$$r_n(x) := (-1)^{x_n} \quad (x \in I,\ n \in \mathbb{N}).$$

The Walsh–Paley system (on I) is defined as the sequence of the Walsh–Paley functions:

$$\omega_n(x) := \prod_{k=0}^{\infty} \left(r_k(x)\right)^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k} \quad (x \in I,\ n \in \mathbb{N}).$$

That is, $\omega := (\omega_n,\ n \in \mathbb{N})$. (For details see Fine [2] and [7].)

Consider the Dirichlet and the Fejér kernel functions:

$$D_n := \sum_{k=0}^{n-1} \omega_k, \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k, \quad D_0, K_0 := 0.$$

The Fourier coefficients, the n-th partial sum of the Fourier series, the n-th $(C, 1)$ mean of $f \in L^1(I)$ are

$$\hat{f}(n) := \int_I f(x) \omega_n(x)\, d\mu(x) \quad (n \in \mathbb{N}),$$

$$S_n f(y) := \sum_{k=0}^{n-1} \hat{f}(k) \omega_k(y) = \int_I f(x + y) D_n(x)\, d\mu(x),$$

$$\sigma_n f(y) := \frac{1}{n}\sum_{k=1}^{n} S_k f(y) = \int_I f(x+y)K_n(x)\,d\mu(x) \quad (n \in \mathbb{P},\ y \in I),$$

resp. The two-dimensional Walsh–Paley functions, Dirichlet and Fejér kernels, resp. are defined as follows:

$$\omega_n(x) = \omega_{n_1}(x_1)\omega_{n_2}(x_2), \quad D_n(x) = D_{n_1}(x_1)D_{n_2}(x_2),$$

$$K_n(x) = K_{n_1}(x_1)K_{n_2}(x_2),$$

where $n \in \mathbb{N}^2$, $x \in I^2$. Moreover, the two-dimensional Fourier coefficients, the $n \in \mathbb{N}^2$-th rectangular partial sum of the Fourier series, the $n \in \mathbb{P}^2$-th $(C,1)$ mean of $f \in L^1(I^2)$ are

$$\hat{f}(n) := \int_{I^2} f(x)\omega_n(x)\,d\mu(x) \quad (n \in \mathbb{N}^2),$$

$$S_n f(y) := \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} \hat{f}(k_1,k_2)\omega_{(k_1,k_2)}(y) = \int_{I^2} f(x+y)D_n(x)\,d\mu(x),$$

$$\sigma_n f(y) := \frac{1}{n_1 n_2}\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} S_k f(y) = \int_{I^2} f(x+y)K_n(x)\,d\mu(x)$$

$(n \in \mathbb{P}^2, y \in I^2)$, resp. For double trigonometric Fourier series Marcinkiewicz and Zygmund [5] proved the a.e. convergence of Fejér means of integrable functions, where the set of indices is inside a positive cone around the identical function, that is $\beta^{-1} \leqq n_1/n_2 \leqq \beta$ holds with some fixed parameter $\beta > 1$. We mention that Jessen, Marcinkiewicz and Zygmund [1] also proved the a.e. convergence $\sigma_n f \to f$ without any restriction on the indices (other than $\min\{n_1,n_2\} \to \infty$), but for functions in $L\log^+ L$. For double Walsh–Fourier series, Móricz, Schipp and Wade [6] proved that $\sigma_n f$ converges to f a.e. in the Pringsheim sense (that is, no restriction on the indices other than $\min\{n_1,n_2\} \to \infty$ for all functions $f \in L\log^+ L$). In [4] Gát proved that the theorem of Móricz, Schipp and Wade can not be improved. Namely, the following was proved. Let $\gamma : [0,+\infty) \to [0,+\infty)$ be a measurable function with property $\gamma(+\infty) = 0$, then there exists a function $f \in L\log^+ L\gamma(L)$ such that $\sigma_n f$ does not converge to f a.e. as $\min\{n_1,n_2\} \to +\infty$. For double Walsh systems the result of Marcinkiewicz and Zygmund was proved by Gát [3] and Weisz [8]. In this paper we prove the following a.e. convergence theorem with respect to the Fejér means of the Walsh–Fourier series of integrable functions.

THEOREM 1. *Let* $a=(a_1,a_2):\mathbb{N}\to\mathbb{N}^2$ *be a sequence with property* $a_j(+\infty)=+\infty$ $(j=1,2)$. *Suppose that there exists a* $\delta>0$ *such that* $a_j(n+1)\geqq\delta\sup_{k\leqq n}a_j(n)$ $(j=1,2,\ n\in\mathbb{N})$. *Then for each integrable function* $f\in L^1(I^2)$ *we have the a.e. relation*

$$\lim_{n\to\infty}\sigma_{a(n)}f=f.$$

This theorem, which is the main result of this paper is an easy consequence of the following lemma.

LEMMA 2. *Let* $a=(a_1,a_2):\mathbb{N}\to\mathbb{N}^2$ *be a sequence with property* $a_j(+\infty)=+\infty$ $(j=1,2)$. *Suppose that* $\lfloor\log_2 a_j\rfloor$ *is monotone increasing* $(j=1,2)$. *Then for each integrable function* $f\in L^1(I^2)$ *we have the a.e. relation*

$$\lim_{n\to\infty}\sigma_{a(n)}f=f.$$

A straightforward and easy consequence of Lemma 2 is the result of Gát and Weisz [3,8] with respect to the “cone restricted” almost everywhere convergence of two-dimensional Walsh–Fejér means of integrable functions.

COROLLARY 3. *Let* $\beta>1$ *and* $f\in L^1(I^2)$. *Then we have the a.e. relation*

$$\lim_{\substack{n_1,n_2\to\infty\\ 1/\beta\leqq n_1/n_2\leqq\beta}}\sigma_{n_1,n_2}f=f.$$

PROOF. The proof of this corollary comes directly from Lemma 2. So, let $\gamma:=\lceil\log_2\beta\rceil$. For $k,l\in\mathbb{N}$ set $N_{\gamma,l,k}:=\{(n_1,n_2)\in\mathbb{N}^2:\,2^k\leqq n_1<2^{k+1},\ 2^{k-\gamma+l}\leqq n_2<2^{k-\gamma+l+1}\}$. Let $N_{\gamma,l}$ be the union of the disjoint sets $N_{\gamma,l,k}$. It is easy to give a sequence $a:\mathbb{N}\to\mathbb{N}^2$ such that $\lfloor\log_2 a_1\rfloor$, $\lfloor\log_2 a_2\rfloor$ are monotone increasing (for $n\in N_{\gamma,l,k}$ we have $\lfloor\log_2 n_1\rfloor=k$, $\lfloor\log_2 n_2\rfloor=k-\gamma+l$) and $a(\mathbb{N})=N_{\gamma,l}$. This by Lemma 2 gives that for each integrable function f

$$\sigma_{n_1,n_2}f\to f$$

a.e. provided that $n\in N_{\gamma,l}$ and $n_1,n_2\to\infty$. Hence, we also have this a.e. relation for $n\in\bigcup_{l=0}^{2\gamma}N_{\gamma,l}=:N_\gamma$ and $n_1,n_2\to\infty$. Then, let $n\in\mathbb{N}^2$ be such that $1/\beta\leqq n_1/n_2\leqq\beta$. Denote by k the natural number for which $2^k\leqq n_1<2^{k+1}$. Then, $2^{k-\gamma}\leqq 2^k/\beta\leqq n_2<2^{k+1}\beta\leqq 2^{k+\gamma+1}$. Consequently, $n\in N_\gamma$. □

Let $A=(A_1,A_2):\mathbb{N}\to\mathbb{N}^2$ be a sequence of pairs of natural numbers which are monotone increasing with respect to both indices, and they do not increase too fast. More precisely, suppose that there exists a constant $C>0$ depending only on A such that

$$A_j(n)\leqq A_j(n+1)\leqq A_j(n)+C$$

for $n \in \mathbb{N}$ and $j = 1, 2$.

In order to prove Lemma 2 we need some lemmas. The first is the following Calderon–Zygmund type decomposition lemma. Let $A_j(+\infty) = +\infty$ $(j = 1, 2)$ and

$$C_A = 2^{\sup_n \{A_1(n+1)-A_1(n)+A_2(n+1)-A_2(n)\}}.$$

LEMMA 4. *Let $f \in L^1(I^2)$, $\lambda > 0$ and $A = (A_1, A_2) : \mathbb{N} \to \mathbb{N}^2$ as above. Then there exists a sequence of natural numbers (k_n) and $x_n = (x_{n,1}, x_{n,2}) \in I^2$ $(n \in \mathbb{P}^2)$ such that for the disjoint two-dimensional dyadic rectangles $J_n = I_{A(k_n)}(x_n)$ there exist functions $f_n \in L^1(I^2)$ $(n \in \mathbb{N})$ satisfying the a.e. relation $f = \sum_{n=0}^{\infty} f_n$ and besides,*

$$\|f_0\|_\infty \leqq C_A\lambda, \quad \|f_0\|_1 \leqq \|f\|_1, \quad \operatorname{supp} f_n \subset J_n, \quad \int_{J_n} f_n \, d\mu = 0 \quad (n \in \mathbb{P}).$$

Moreover, for the set $F = \bigcup_{n=1}^{\infty} J_n$ we have $\mu(F) \leqq \|f\|_1/\lambda$.

PROOF. Consider the following sets of two-dimensional dyadic rectangles. (Recall that $I_{A(k_n)}(x_n) = I_{A_1(k_n)}(x_{n,1}) \times I_{A_2(k_n)}(x_{n,2})$.)

$$\Omega_0 := \left\{ I_{A(0)}(x) : \, 2^{A_1(0)+A_2(0)} \int_{I_{A(0)}(x)} |f(y)| \, d\mu(y) > \lambda, \ x \in I^2 \right\},$$

$$\Omega_1 := \Big\{ I_{A(1)}(x) : \, 2^{A_1(1)+A_2(1)} \int_{I_{A(1)}(x)} |f(y)| \, d\mu(y) > \lambda,$$

$$\nexists J \in \Omega_0 : \, I_{A(1)}(x) \subset J, \ x \in I^2 \Big\},$$

$$\vdots$$

$$\Omega_n := \Big\{ I_{A(n)}(x) : \, 2^{A_1(n)+A_2(n)} \int_{I_{A(n)}(x)} |f(y)| \, d\mu(y) > \lambda,$$

$$\nexists J \in \bigcup_{i=0}^{n-1} \Omega_i : \, I_{A(n)}(x) \subset J, \ x \in I^2 \Big\}.$$

Then the elements of Ω_n are disjoint rectangles of measure $2^{-A_1(n)-A_2(n)}$ $(n \in \mathbb{N})$. Moreover, if $i \neq j$, then for all $J \in \Omega_i$, $K \in \Omega_j$ we have $J \cap K = \emptyset$.

If $I_{A(n)}(x) \in \Omega_n$, then since there is no $J \in \bigcup_{i=0}^{n-1} \Omega_i$ such that $I_{A(n)}(x) \subset J$, then we have

$$2^{A_1(i)+A_2(i)} \int_{I_{A(i)}(x)} |f(y)| \, d\mu(y) \leqq \lambda$$

for $i=0,1,\dots,n-1$. Thus,

$$\lambda < 2^{A_1(n)+A_2(n)}\int_{I_{A(n)}(x)} |f(y)|\,d\mu(y) \leqq 2^{A_1(n)+A_2(n)}\int_{I_{A(n-1)}(x)} |f(y)|\,d\mu(y)$$

$$\leqq C_A 2^{A_1(n-1)+A_2(n-1)}\int_{I_{A(n-1)}(x)} |f(y)|\,d\mu(y) \leqq C_A\lambda.$$

Since Ω_n has a finite number of elements, then we can set the notation

$$\Omega_n = \bigl\{I_{A(n)}(x_{n,i}) : i=1,\dots,l_n\bigr\},\quad F := \bigcup_{n=0}^{\infty}\bigcup_{i=1}^{l_n} I_{A(n)}(x_{n,i}).$$

For the measure of set F we get

$$\mu(F) = \sum_{n=0}^{\infty}\sum_{i=1}^{l_n}\mu\bigl(I_{A(n)}(x_{n,i})\bigr) = \frac{1}{\lambda}\sum_{n=0}^{\infty}\sum_{i=1}^{l_n}\lambda\mu\bigl(I_{A(n)}(x_{n,i})\bigr)$$

$$\leqq \frac{1}{\lambda}\sum_{n=0}^{\infty}\sum_{i=1}^{l_n}\int_{I_{A(n)}(x_{n,i})}|f|\,d\mu \leqq \frac{1}{\lambda}\int_{I^2}|f|\,d\mu = \|f\|_1/\lambda.$$

Let $1_B(x) := \begin{cases} 1, & \text{if } x\in B,\\ 0, & \text{if } x\notin B\end{cases}$ be the characteristic function of the set $B\subset I^2$ $(x\in I^2)$. Then,

$$f = \sum_{n=0}^{\infty}\sum_{i=1}^{l_n} f1_{I_{A(n)}(x_{n,i})} + f1_{I^2\setminus F}$$

$$= \sum_{n=0}^{\infty}\sum_{i=1}^{l_n}\Bigl(f - 2^{A_1(n)+A_2(n)}\int_{I_{A(n)}(x_{n,i})} f\,d\mu\Bigr)1_{I_{A(n)}(x_{n,i})}$$

$$+ \sum_{n=0}^{\infty}\sum_{i=1}^{l_n}\Bigl(2^{A_1(n)+A_2(n)}\int_{I_{A(n)}(x_{n,i})} f\,d\mu\Bigr)1_{I_{A(n)}(x_{n,i})} + f1_{I^2\setminus F}$$

$$=: \sum_{n=0}^{\infty}\sum_{i=1}^{l_n} f_{n,i} + f_{0,1} + f_{0,2} =: \sum_{n=0}^{\infty}\sum_{i=1}^{l_n} f_{n,i} + f_0.$$

Now, for $f_{n,i}$ we have $\operatorname{supp} f_{n,i} \subset I_{A(n)}(x_{n,i})$ and

$$\int_{I_{A(n)}(x_{n,i})} f_{n,i}\,d\mu$$

$$= \int_{I_{A(n)}(x_{n,i})} \Big(f(t) - 2^{A_1(n)+A_2(n)} \int_{I_{A(n)}(x_{n,i})} f(y)\,d\mu(y) \Big)\,d\mu(t) = 0.$$

Moreover,

$$2^{A_1(n)+A_2(n)} \int_{I_{A(n)}(x_{n,i})} |f_{n,i}|\,d\mu$$

$$\leqq 2^{A_1(n)+A_2(n)} \int_{I_{A(n)}(x_{n,i})} |f|\,d\mu + 2^{A_1(n)+A_2(n)}$$

$$\times \int_{I_{A(n)}(x_{n,i})} \Big| 2^{A_1(n)+A_2(n)} \int_{I_{A(n)}(x_{n,i})} f\,d\mu \Big|\,d\mu \leqq 2C_A\lambda.$$

Since the rectangles $I_{A(n)}(x_{n,i})$ are disjoint, we also have $\|f_0\|_1 \leqq \|f\|_1$. The only relation left to prove is the inequality $\|f_0\|_\infty \leqq C_A\lambda$. Recall that

$$\sum_{n=0}^{\infty} \sum_{i=1}^{l_n} \Big(2^{A_1(n)+A_2(n)} \int_{I_{A(n)}(x_{n,i})} f\,d\mu \Big) 1_{I_{A(n)}(x_{n,i})} + f 1_{I^2 \setminus F} =: f_{0,1} + f_{0,2}.$$

First,

$$|f_{0,1}| \leqq \sum_{n=0}^{\infty} \sum_{i=1}^{l_n} C_A\lambda 1_{I_{A(n)}(x_{n,i})} = C_A\lambda 1_{\bigcup_{n=0}^{\infty} \bigcup_{i=1}^{l_n} I_{A(n)}(x_{n,i})} \leqq C_A\lambda.$$

Now consider $f_{0,2}$. If $x \in F$, then $f_{0,2}(x) = f(x)1_{I^2\setminus F}(x) = 0$.

Finally, suppose that $x \notin F$. This gives

$$2^{A_1(i)+A_2(i)} \int_{I_{A(i)}(x)} |f(y)|\,d\mu(y) \leqq \lambda$$

for all $i = 0, 1, \dots$. This implies

$$S_{2^{A_1(i)},2^{A_2(i)}} |f_{0,2}(x)| = S_{2^{A_1(i)},2^{A_2(i)}} |f 1_{I^2\setminus F}| \leqq S_{2^{A_1(i)},2^{A_2(i)}} |f(x)| \leqq \lambda.$$

Since the functions A_1 and A_2 are monotone increasing, the partial sum operators $S_{2^{A_1(n)},2^{A_2(n)}}$ form a martingale with respect to the σ-algebras $\mathcal{A}_{A(n)}$

$=\left\{I_{A(n)}(x) : x \in I^2\right\}$ $(n \in \mathbb{N}^2)$. Therefore by the martingale convergence theorem

$$\left| f_{0,2}(x)\right| = \lim S_{2^{A_1(n)},2^{A_2(n)}}\left| f_{0,2}(x)\right| \leqq \limsup S_{2^{A_1(n)},2^{A_2(n)}}\left| f_{0,2}(x)\right| \leqq \lambda \text{ a.e.}$$

That is, $|f_{0,1}| \leqq C_A\lambda, |f_{0,2}| \leqq \lambda$ and consequently $\|f_0\|_\infty \leqq C_A\lambda$. (Recall that $\operatorname{supp} f_{0,1} \subset F$ and $\operatorname{supp} f_{0,2} \subset I^2 \setminus F$.) $\square$

Define the following two-dimensional "shifted partial sum" or "shifted expectation operator" for monotone increasing $A_1, A_2 : \mathbb{N} \to \mathbb{N}$ and $i = (i_1, i_2) \in \mathbb{N}^2$:

$$E_{A(n),i}f(x) := 2^{A_1(n)+A_2(n)} \int_{I_{A(n)}(x+e_{A(n)-i})} f(y)\, d\mu(y)$$

$$= 2^{A_1(n)+A_2(n)} \int_{I_{A_1(n)}(x_1+e_{A_1(n)-i_1})\times I_{A_2(n)}(x_2+e_{A_2(n)-i_2})} f(y)\, d\mu(y)$$

$$= S_{2^{A_1(n)},2^{A_2(n)}} f(x_1 + e_{A_1(n)-i_1}, x_2 + e_{A_2(n)-i_2}).$$

Its maximal operator is

$$E^*_{A,i}f := \sup\left\{\left|E_{A(n),i}f(x)\right| : n \in \mathbb{N},\ A_j(n) \geqq i_j,\ j = 1, 2\right\}.$$

If $i_j > A_j(n)$ for $j = 1$ or $j = 2$, then let $E_{A(n),i}f(x) = 0$. Besides, also for integers $k_1 \leqq 0$ or $k_2 \leqq 0$ set $E_{k,i}f(x) = 0$ for every $i \in \mathbb{N}^2$ and $x \in I^2$. Also suppose $A_j(+\infty) = +\infty$ $(j = 1, 2)$.

The following weak type inequality will play a fundamental role in the proof of the main theorem.

LEMMA 5. *Let $f \in L^1(I^2)$, $\lambda > 0$. Then we have*

$$\mu\{x \in I^2 : E^*_{A,i}f(x) > \lambda\} \leqq \frac{2C_A(i_1 + 1)(i_2 + 1)}{\lambda}\|f\|_1$$

for every $i_1, i_2 \in \mathbb{N}$.

PROOF. In this lemma the sequences A_1, A_2 are monotone increasing, but from other point of view they are arbitrary. That is, they can grow "very fast". Nevertheless, we can suppose that $A : \mathbb{N} \to \mathbb{N}^2$ satisfies the condition of Lemma 4, that is the sequences $A_j(n+1) - A_j(n)$ are nonnegative and bounded $(j = 1, 2)$. This is really not a restriction since we can insert members within the elements of the sequence $E_{A(n),i}f$. For instance this can be demonstrated in the following way:

$$(A_1(n), A_2(n)), (A_1(n), A_2(n) + 1), \ldots, (A_1(n), A_2(n + 1)),$$

$$\left(A_1(n)+1, A_2(n+1)\right), \left(A_1(n)+2, A_2(n+1)\right), \dots, \left(A_1(n+1), A_2(n+1)\right).$$

If we denote by $\tilde{A}$ this modified sequence, then for $\tilde{A}$ we certainly have $0 \leqq \tilde{A}_j(n+1) - \tilde{A}_j(n) \leqq 1$ $(n \in \mathbb{N},\ j = 1, 2)$. Besides, $E^*_{\tilde{A},i} f \geqq E^*_{A,i} f$. This means that if we prove the inequality

$$\mu\{\, E^*_{\tilde{A},i} f > \lambda\} \leqq \frac{2C_A(i_1+1)(i_2+1)}{\lambda} \|f\|_1,$$

then we also have it for A. As a result of this assumption we can suppose that $0 \leqq A_j(n+1) - A_j(n) \leqq \log_2 C_A$ $(n \in \mathbb{N},\ j = 1, 2)$. Apply Lemma 4:

$$F = \bigcup_{n=1}^{\infty} J_n = \bigcup_{n=1}^{\infty} I_{A(k_n)}(x_n) = \bigcup_{n=1}^{\infty} \left(I_{A_1(k_n)}(x_{n,1}) \times I_{A_2(k_n)}(x_{n,2}) \right).$$

Enlarge the rectangle J_n in the following way:

$$J_{n,i} := \bigcup_{l_2=0}^{i_2} \bigcup_{l_1=0}^{i_1} \left(I_{A_1(k_n)}\big(x_{n,1} + e_{A_1(k_n)-l_1}\big) \times I_{A_2(k_n)}\big(x_{n,2} + e_{A_2(k_n)-l_2}\big) \right).$$

Also set $F_i := \bigcup_{n=1}^{\infty} J_{n,i}$. The shift invariancy of measure μ gives

$$\mu(F_i) \leqq (i_1+1)(i_2+1)\mu(F) \leqq (i_1+1)(i_2+1)\|f\|_1/\lambda.$$

In the sequel, we prove that $E^*_{A,i} f_n = 0$ on the set $I^2 \setminus J_{n,i}$. That is, let $z \in I^2 \setminus J_{n,i}$. If $u \leqq k_n$, then

$$E_{A(u),i} f_n(z) = S_{2^{A_1(u)},2^{A_2(u)}} f_n\big(z_1 + e_{A_1(u)-i_1}, z_2 + e_{A_2(u)-i_2}\big)$$

$$= \int_{I^2} f_n(y_1, y_2) D_{2^{A_1(u)}}\big(z_1 + e_{A_1(u)-i_1} + y_1\big)$$

$$\times D_{2^{A_2(u)}}\big(z_2 + e_{A_2(u)-i_2} + y_2\big)\, d\mu(y)$$

$$= \int_{I_{A_1(k_n)}(x_{n,1}) \times I_{A_2(k_n)}(x_{n,2})} f_n(y_1, y_2) D_{2^{A_1(u)}}\big(z_1 + e_{A_1(u)-i_1} + x_{n,1}\big)$$

$$\times D_{2^{A_2(u)}}\big(z_2 + e_{A_2(u)-i_2} + x_{n,2}\big)\, d\mu(y) = 0,$$

since

$$\int_{I_{A_1(k_n)}(x_{n,1}) \times I_{A_2(k_n)}(x_{n,2})} f_n(y_1, y_2)\, d\mu(y) = 0.$$

On the other hand, if $u > k_n$, then $y_j \in I_{A_j(k_n)}(x_{n,j})$ and $D_{2^{A_j(u)}}\big(z_j + e_{A_j(u)-i_j} + y_j\big) \neq 0$ implies

$$z_j \in I_{A_j(u)}\big(y_j + e_{A_j(u)-i_j}\big) \subset I_{A_j(k_n)}\big(y_j + e_{A_j(u)-i_j}\big)$$

$$= I_{A_j(k_n)}\big(x_{n,j} + e_{A_j(u)-i_j}\big) \subset \bigcup_{l_j=0}^{i_j} I_{A_j(k_n)}\big(x_{n,j} + e_{A_j(k_n)-l_j}\big) \quad (j = 1, 2).$$

The relation $\subset$ in the last line above is implied by the fact that for $A_j(u) - i_j \geqq A_j(k_n)$ we have $I_{A_j(k_n)}(x_{n,j} + e_{A_j(u)-i_j}) = I_{A_j(k_n)}(x_{n,j}) = I_{A_j(k_n)}\big(x_{n,j} + e_{A_j(k_n)}\big)$ and for $A_j(u) - i_j < A_j(k_n)$ by $u > k_n$ we get $A_j(k_n) - i_j \leqq A_j(u) - i_j < A_j(k_n)$ and thus there exists an $l_j \in \{1, \ldots, i_j\}$ such that

$$I_{A_j(k_n)}\big(x_{n,j} + e_{A_j(u)-i_j}\big) = I_{A_j(k_n)}\big(x_{n,j} + e_{A_j(k_n)-l_j}\big).$$

That is, in every case $D_{2^{A_j(u)}}\big(z_j + e_{A_j(u)-i_j} + y_j\big) \neq 0$ would give the relation

$$z_j \in \bigcup_{l_j=0}^{i_j} I_{A_j(k_n)}\big(x_{n,j} + e_{A_j(k_n)-l_j}\big) \quad (j = 1, 2).$$

Thus, $z = (z_1, z_2)$ would be an element of $J_{n,i}$. This contradiction implies that $E_{A(n),i} f_n(z) = 0$ for all $n \in \mathbb{N}$ and $z \in I^2 \setminus J_{n,i}$. This gives $E^*_{A,i} f_n = 0$ on the set $z \in I^2 \setminus J_{n,i}$.

Next we prove that the operator $E^*_{A,i}$ is of type (∞, ∞). More precisely, we prove $\| E^*_{A,i} g\|_\infty \leqq \|g\|_\infty$ for each $g \in L^\infty(I^2)$. This is quite simple to verify. This property of $E^*_{A,i}$ by the inequality $\|f_0\|_\infty \leqq C_A\lambda$ gives $\mu\{ E^*_{A,i} f_0 > C_A\lambda\} = 0$. Consequently, by the sublinearity of operator $E^*_{A,i}$ we have

$$\mu\{ E^*_{A,i} f > 2C_A\lambda\} \leqq \mu\{ E^*_{A,i} f_0 > C_A\lambda\} + \mu\left\{ E^*_{A,i}\Big(\sum_{n=1}^\infty f_n\Big) > C_A\lambda\right\}$$

$$\leqq \mu(F_i) + \mu\left\{ x \in I^2 \setminus F_i : E^*_{A,i}\Big(\sum_{n=1}^\infty f_n\Big)(x) > C_A\lambda\right\}$$

$$\leqq \frac{(i_1+1)(i_2+1)}{\lambda}\|f\|_1 + \frac{1}{C_A\lambda}\int_{I^2\setminus F_i} E^*_{A,i}\Big(\sum_{n=1}^\infty f_n\Big)\, d\mu$$

$$\leqq \frac{(i_1+1)(i_2+1)}{\lambda}\|f\|_1 + \frac{1}{C_A\lambda}\sum_{n=1}^{\infty}\int_{I^2\setminus F_i} E^*_{A,i} f_n\, d\mu$$

$$\leqq \frac{(i_1+1)(i_2+1)}{\lambda}\|f\|_1 + \frac{1}{C_A\lambda}\sum_{n=1}^{\infty}\int_{I^2\setminus J_{n,i}} E^*_{A,i} f_n\, d\mu$$

$$= \frac{(i_1+1)(i_2+1)}{\lambda}\|f\|_1. \quad \square$$

PROOF OF LEMMA 2. Let $A_j(n) := \lfloor \log_2 a_j(n) \rfloor$ $(n \in \mathbb{N},\ j = 1, 2)$. That is, $2^{A_j(n)} \leqq a_j(n) < 2^{A_j(n)+1}$. Since the sequences $\lfloor \log_2 A_j \rfloor$ are monotone increasing for both $j = 1, 2$, then the conditions of Lemma 5 are satisfied.

In the book of Schipp, Wade and Simon [7] one can find the first line of the following estimation for the one-dimensional Fejér kernel functions for $2^A \leqq a < 2^{A+1}$:

$$|K_a(x)| \leqq \sum_{i=0}^{A} 2^{i-A-1} \sum_{k=i}^{A} \left(D_{2^k}(x) + D_{2^k}(x+e_i) \right)$$

$$= 2^{-A-1}\sum_{k=0}^{A}\sum_{i=0}^{k} 2^i \left(D_{2^k}(x) + D_{2^k}(x+e_i) \right)$$

$$= 2^{-A-1}\sum_{k=0}^{A}\left(2^{k+1}-1\right) D_{2^k}(x) + 2^{-A-1}\sum_{k=0}^{A}\sum_{i=0}^{k} 2^{k-i} D_{2^k}(x+e_{k-i})$$

$$\leqq 2^{-A+1}\sum_{k=0}^{A}\sum_{i=0}^{k} 2^{k-i} D_{2^k}(x+e_{k-i}) = \sum_{k=0}^{A}\sum_{i=0}^{A-k} 2^{1-k-i} D_{2^{A-k}}(x+e_{A-k-i}),$$

since for $i = 0$ we have $D_{2^k}(x+e_{k-i}) = D_{2^k}(x)$. Therefore, for the Fejér means $\sigma_{a(n)} f$ we have

$$|\sigma_{a_1(n),a_1(n)} f|$$

$$\leqq 4 \sum_{k_1=0}^{A_1(n)} \sum_{k_2=0}^{A_2(n)} \sum_{i_1=0}^{A_1(n)-k_1} \sum_{i_2=0}^{A_2(n)-k_2} 2^{-k_1-k_2-i_1-i_2} E_{(A_1(n)-k_1, A_2(n)-k_2),(i_1,i_2)} |f|$$

$$\leqq 4 \sum_{k_1,k_2\in\mathbb{N}} \sum_{i_1,i_2\in\mathbb{N}} 2^{-k_1-k_2-i_1-i_2} E^*_{A-k,i} |f|,$$

where the sequence $A-k$ is $(A_1(n)-k_1, A_2(n)-k_2)$. We recall the definition of $E_{A,i}$. More precisely, the fact that for $A_1(n)-k_1 \leqq 0$ or $A_2(n)-k_2 \leqq 0$ or $A_1(n)-k_1 < i_1$ or $A_2(n)-k_2 < i_2$ we have $E_{A-k,i}f(x) = 0$ for each $x \in I^2$. Also recall that $C_A = 2^{\sup_n \{A_1(n+1)-A_1(n)+A_2(n+1)-A_2(n)\}}$. ($A_1(0) = A_2(0) = 0$ may be supposed.) That is, for every fixed $k \in \mathbb{N}^2$ the equality $C_{A-k} = C_A$ is fulfilled and by Lemma 5

$$\mu\{E^*_{A-k,i}|f| > \lambda\} \leqq 2C_A(i_1+1)(i_2+1)\big\||f|\big\|_1/\lambda$$

$$= 2C_A(i_1+1)(i_2+1)\|f\|_1/\lambda.$$

Set

$$\sigma^*_A f := \sup_n \left|\sigma_{A_1(n),A_2(n)} f\right|.$$

We get that the operator σ^*_A is of weak type $(1,1)$ in the following way:

$$\mu\{\sigma^*_A f > \lambda\} = \mu\left\{4 \sum_{k_1,k_2\in\mathbb{N}} \sum_{i_1,i_2\in\mathbb{N}} 2^{-k_1-k_2-i_1-i_2} E^*_{A-k,i}|f| > \lambda\right\}$$

$$\leqq \mu\left(\bigcup_{k_1,k_2\in\mathbb{N}} \bigcup_{i_1,i_2\in\mathbb{N}} \left\{2^{-k_1-k_2-i_1-i_2} E^*_{A-k,i}|f| > \frac{C\lambda}{(k_1k_2i_1i_2)^2}\right\}\right)$$

$$\leqq \sum_{k_1,k_2\in\mathbb{N}} \sum_{i_1,i_2\in\mathbb{N}} \mu\left\{2^{-k_1-k_2-i_1-i_2} E^*_{A-k,i}|f| > \frac{C\lambda}{(k_1k_2i_1i_2)^2}\right\}$$

$$\leqq C\cdot C_A \sum_{k_1,k_2\in\mathbb{N}} \sum_{i_1,i_2\in\mathbb{N}} \frac{(k_1k_2i_1i_2)^2(i_1+1)(i_2+1)}{\lambda 2^{k_1+k_2+i_1+i_2}} \|f\|_1 \leqq C\cdot C_A\|f\|_1/\lambda.$$

That is, we proved that the maximal operator σ^*_A is of weak type $(1,1)$. Since for each Walsh polynomial P we have everywhere the relation $\lim_{n\to\infty} \sigma_{a_1(n),a_2(n)} P = P$, then by a standard density argument (see this principle for instance in [7]) the proof of Lemma 2 is complete. □

Finally, we prove Theorem 1. The proof comes from Lemma 2 with some easy calculations.

PROOF OF THEOREM 1. Let L be a positive integer to be determined later. For $l, m = 0, 1, \ldots, L-1$ let some disjoint subsets of $\mathbb{N}$ be defined as:

$$B_{l,m} = \left\{n \in \mathbb{N} :\ (a_1(n), a_2(n)) \in \bigcup_{s,t=0}^{\infty} \left[2^{sL+l}, 2^{sL+l+1}\right) \times \left[2^{tL+m}, 2^{tL+m+1}\right)\right\}.$$

It is clear that these sets are pairwise disjoint and their union is $\mathbb{N}$. Denote the elements of $B_{l,m}$ by $n_1^{l,m} < n_2^{l,m} < \cdots$. We prove that $\lfloor \log_2 a_j(n_k^{l,m}) \rfloor \leqq \lfloor \log_2 a_j(n_{k+1}^{l,m}) \rfloor$ for every $k \in \mathbb{N}$, $l, m \in \{0, 1, \ldots L-1\}$ and $j = 1, 2$. On the contrary, suppose that $\lfloor \log_2 a_j(n_{k+1}^{l,m}) \rfloor < \lfloor \log_2 a_j(n_k^{l,m}) \rfloor$ for some k, l, m and j. Then the definition of $B_{l,m}$ gives that $\lfloor \log_2 a_j(n_{k+1}^{l,m}) \rfloor \leqq \lfloor \log_2 a_j(n_k^{l,m}) \rfloor - L$. Thus,

$$\frac{1}{2} a_j(n_{k+1}^{l,m}) \leqq 2^{\lfloor \log_2 a_j(n_{k+1}^{l,m}) \rfloor} \leqq 2^{\lfloor \log_2 a_j(n_k^{l,m}) \rfloor - L} \leqq \frac{1}{2^L} a_j(n_k^{l,m}).$$

Since, $n_{k+1}^{l,m} > n_k^{l,m}$, then we have $a_j(n_{k+1}^{l,m}) \geqq \delta a_j(n_k^{l,m})$ and consequently, also have $\delta \leqq 2^{1-L}$. This is obviously not possible for an L large enough. That is, we proved that $\lfloor \log_2 a_j(n_k^{l,m}) \rfloor$ is monotone increasing with respect to $k \in \mathbb{N}$. Lemma 2 gives the a.e. convergence

$$\lim_{k\to\infty} \sigma_{a(n_k^{l,m})} f = f$$

for each integrable function f and $l, m = 0, 1, \ldots, L-1$. Merging the L^2 pieces of subsequences of $\sigma_{a(n)} f$, the proof of Theorem 1 is complete. □

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References

[1] A. Zygmund, B. Jessen and J. Marcinkiewicz, Note on the differentiability of multiple integrals, *Fund. Math.*, **32** (1935), 217–234.

[2] N. J. Fine, Cesàro summability of Walsh–Fourier series, *Proc. Nat. Acad. Sci. U.S.A.*, **41** (1955), 558–591.

[3] G. Gát, Pointwise convergence of the Cesàro means of double Walsh series, *Ann. Univ. Sci. Budapest. Rolando Eötvös, Sect. Comput.*, **16** (1996), 173–184.

[4] G. Gát, On the divergence of the $(C,1)$ means of double Walsh–Fourier series, *Proc. Am. Math. Soc.*, **128** (2000), 1711–1720.

[5] J. A. Zygmund and J. Marcinkiewicz, On the summability of double Fourier series, *Fund. Math.*, **32** (1939), 122–132.

[6] F. Móricz, F. Schipp and W. R. Wade, Cesàro summability of double Walsh–Fourier series, *Trans Amer. Math. Soc.*, **329** (1992), 131–140.

[7] F. Schipp, W. R. Wade, P. Simon and J. Pál, *Walsh Series: An Introduction to Dyadic Harmonic Analysis*, Adam Hilger (Bristol and New York, 1990).

[8] F. Weisz, Cesàro summability of two-dimensional Walsh–Fourier series, *Trans. Amer. Math. Soc.*, **348** (1996), 2169–2181.