

On the maximal value of Dirichlet kernels with respect to representative product systems

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Abstract. Dirichlet kernels play a prominent role in the study of Fourier series. For Walsh and Vilenkin systems the maximal value of the n -th Dirichlet kernel is exactly n . This situation is very different if we generalize these systems on the complete product of arbitrary groups, not necessarily commutative groups.

Mathematical Subject Classification (2010). 42C10

Keywords and phrases. Vilenkin systems, Walsh-Paley system, representative product system, Fourier series, Dirichlet kernels

Representative product systems are systems given by the finite products of normalized coordinate functions considering the complete product of arbitrary groups, not necessarily commutative groups. These systems are ordered as the order introduced by Paley for the Walsh system.

In Section 1 we introduce basic concepts in the study of representative product systems and Fourier analysis. We also give some examples of these systems showing the properties of them. Representative product systems were discussed in [1], [2], [6] and [7]. Section 2 contains the results for the maximal value of the n -th Dirichlet kernel, denoted by D_n . We prove that

$$n \leq D_n \leq M_{A+1},$$

if $M_A \leq n \leq M_{A+1}$. Then we study the condition by which $D_n = n$ holds, like in the commutative cases, and also we find the first number n such that $D_n = M_{A+1}$ holds.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Throughout this work denote by \mathbf{N} , \mathbf{P} , \mathbf{C} the set of nonnegative, positive integers and complex numbers, respectively. In order to simplicity we always use the multiplication to denote the group operation and use the symbol e to denote the identity of the groups. We use the notation of Hewitt and Ross [3].

1 Representative product systems

Let $m := (m_k, k \in \mathbf{N})$ be a sequence of positive integers such that $m_k \geq 2$ and G_k a finite group with order m_k , ($k \in \mathbf{N}$). Suppose that each group has discrete topology and normalized Haar measure μ_k . Let G be the compact group formed by the complete direct product of G_k with the product of the topologies, operations and measures (μ). Thus each $x \in G$ are sequences $x := (x_0, x_1, \dots)$, where $x_k \in G_k$, ($k \in \mathbf{N}$). We call this sequence the *expansion* of x . The compact totally disconnected group G is called a *bounded group* if the sequence m is bounded.

If $M_0 := 1$ and $M_{k+1} := m_k M_k$, $k \in \mathbf{N}$, then every $n \in \mathbf{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \leq n_k < m_k$, $n_k \in \mathbf{N}$. This allows us to say that the sequence (n_0, n_1, \dots) is the expansion of n with respect to m . We often use the following notations: let $|n| := \max\{k \in \mathbf{N} : n_k \neq 0\}$ and

$$n_{(k)} := \sum_{j=0}^{k-1} n_j M_j, \quad n^{(k)} := \sum_{j=k}^{\infty} n_j M_j.$$

Denote by Σ_k the dual object of the finite group G_k ($k \in \mathbf{N}$). Thus each $\sigma \in \Sigma_k$ is a set of continuous irreducible unitary representations of G_k which are equivalent to some fixed representation $U^{(\sigma)}$. Let d_σ be the dimension of its representation space and let $\{\zeta_1, \zeta_2, \dots, \zeta_{d_\sigma}\}$ be a fixed but arbitrary orthonormal basis in the representation space. The functions

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \zeta_i, \zeta_j \rangle \quad (i, j \in \{1, \dots, d_\sigma\}, x \in G_k)$$

are called the coordinate functions for $U^{(\sigma)}$ and the basis $\{\zeta_1, \zeta_2, \dots, \zeta_{d_\sigma}\}$. In this manner for each $\sigma \in \Sigma_k$ we obtain d_σ^2 number of coordinate functions, in total m_k number of functions for the whole dual object of G_k . The L^2 -norm of these functions is $1/\sqrt{d_\sigma}$.

Let $\{\varphi_k^s : 0 \leq s < m_k\}$ be a system of all *normalized coordinate functions* of the group G_k . Now, we do not decide the order of the system φ , only suppose that φ_k^0 is always the character 1. Thus for every $0 \leq s < m_k$

there exists a $\sigma \in \Sigma_k$, $i, j \in \{1, \dots, d_\sigma\}$ such that

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \quad (x \in G_k).$$

Let ψ be the product system of φ_k^s , namely

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \quad (x \in G),$$

where n is of the form $n = \sum_{k=0}^{\infty} n_k M_k$ and $x = (x_0, x_1, \dots)$. Thus we say that ψ is the *representative product system* of φ . The Weyl-Peter's theorem (see [3]) gives that the system ψ is orthonormal and complete on $L^2(G)$.

The functions ψ_n ($n \in \mathbf{N}$) are not necessary uniformly bounded, so define

$$\Psi_k := \max_{n < M_k} \|\psi_n\|_1 \|\psi_n\|_\infty \quad (k \in \mathbf{N}).$$

Ψ_k is the multiplication of the greatest product of L^∞ -norm and L^1 -norm of the functions φ_j appeared in all group G_{m_j} for $0 \leq j < k$. Namely

$$\Psi_k = \prod_{i=0}^{k-1} \max_{s < m_i} \|\varphi_i^s\|_1 \|\varphi_i^s\|_\infty \quad (k \in \mathbf{N}).$$

It seems that the boundedness of the sequence Ψ plays an important role in the norm convergence of Fourier series.

For an integrable complex function f defined in G we define the Fourier coefficients and partial sums by

$$\widehat{f}_k := \int_{G_m} f \bar{\psi}_k d\mu \quad (k \in \mathbf{N}), \quad S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \quad (n \in \mathbf{P}).$$

According to the theorem of Banach-Steinhaus, $S_n f \rightarrow f$ as $n \rightarrow \infty$ in L^p norm for all $f \in L^p(G)$ if and only if there exists a $C_p > 0$ such that

$$\|S_n f\|_p \leq C_p \|f\|_p, \quad (f \in L^p(G)).$$

Thus, we say that the operator S_n is of *type* (p, p) . Since system ψ forms an orthonormal base in the Hilbert space $L^2(G)$, it is obvious that S_n is of type $(2, 2)$.

The *Dirichlet kernels* are defined as follows:

$$D_n(x, y) := \sum_{k=0}^{n-1} \psi_k(x) \bar{\psi}_k(y) \quad (n \in \mathbf{P}).$$

It is easy to see that

$$S_n f(x) = \int_G f(y) D_n(x, y) d\mu(y). \quad (1.1)$$

This shows the importance of the Dirichlet kernels in the study of the convergence of Fourier series. The following lemmas were proved in [2].

Lemma 1.1 *If $n \in \mathbf{N}$ and $x, y \in G$, then*

$$D_n(x, y) = \sum_{k=0}^{\infty} D_{M_k}(x, y) \left(\sum_{s=0}^{n_k-1} \varphi_k^s(x_k) \overline{\varphi_k^s(y_k)} \right) \psi_{n(k+1)}(x) \overline{\psi_{n(k+1)}(y)},$$

where (n_0, n_1, \dots) is the expansion of n and $x = (x_0, x_1, \dots)$, $y = (y_0, y_1, \dots)$.

Lemma 1.2 (Paley lemma) *If $n \in \mathbf{N}$ and $x, y \in G$, then*

$$D_{M_n}(x, y) = \begin{cases} M_n & \text{for } x \in I_n(y), \\ 0 & \text{for } x \notin I_n(y) \end{cases}$$

The Paley lemma is used to prove that the $S_{M_n} f$ partial sequence of Fourier sums converge to f in L^p -norm and a.e., if $f \in L^p(G)$, $p \geq 1$. Indeed, the

$$S_{M_n} f(x) = \int_G f(y) D_{M_n}(x, y) d\mu(y) = \frac{1}{\mu(I_n(x))} \int_{I_n(x)} f d\mu$$

operator is the conditional expectation operator with respect to the σ -algebra generated by the sets $I_n(x)$, $x \in G$. Thus, according to the martingale convergence theorem, for each $f \in L^p(G)$, $p \geq 1$, $S_{M_n} f$ converge to f in L^p -norm, and a.e.

The representative product systems are the generalization of the well-known Walsh-Paley and Vilenkin systems. Indeed, we obtain the Walsh-Paley system if $m_k = 2$ and $G_k := \mathcal{Z}_2$, the cyclic group of order 2 for all $k \in \mathbf{N}$. The characters of \mathcal{Z}_2 are the Rademacher functions:

$$\varphi^s(x) = (-1)^{sx} \quad (s \in \{0, 1\}, x \in \mathcal{Z}_2).$$

Moreover, we obtain the Vilenkin systems if the sequence m is an arbitrary sequence of integers greater than 1 and $G_k := \mathcal{Z}_{m_k}$, the cyclic group of order m_k for all $k \in \mathbf{N}$. The characters of \mathcal{Z}_{m_k} are the generalized Rademacher functions:

$$\varphi_k^s(x) = \exp(2\pi i s x / m_k) \quad (s \in \{0, \dots, m_k - 1\}, x \in \mathcal{Z}_{m_k}, i^2 = -1).$$

Now, we consider the complete product of \mathcal{S}_3 where \mathcal{S}_3 is the symmetric group on 3 elements. Thus $m_k = 6$ for all $k \in \mathbf{N}$. \mathcal{S}_3 has two characters and a 2-dimensional representation. The values of the system φ obtained from the 2-dimensional representation depend on the chosen basis. Table 1 contains the values of a possible system φ .

	e	(12)	(13)	(23)	(123)	(132)	$\ \varphi^s\ _1$	$\ \varphi^s\ _\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

Table 1: The system φ for \mathcal{S}_3

Notice that the functions φ_k^s can take the value 0, and the product system of φ is not uniformly bounded. This facts encumber the study of this systems. On the other hand, $\max_{0 \leq s < 6} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3}$, thus $\Psi_k = \left(\frac{4}{3}\right)^k \rightarrow \infty$ if $k \rightarrow \infty$.

Let $m_k = 8$ for all $k \in \mathbf{N}$ and \mathcal{Q}_2 be the the *quaternion group* of order 8, i.e.

$$\mathcal{Q}_2 := \{[a, b] : a^4 = e, b^2 = a^2, bab^{-1} = a^3\}.$$

Let $G_k = \mathcal{Q}_2$ for all $k \in \mathbf{N}$. \mathcal{Q}_2 has four characters and a 2-dimensional representation ($8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$). Table 2 contains the values of a possible system φ .

Notice that $|\varphi^s|$ can only take the values 0 or the square root of the corresponding dimension because the representation is monomial. Hence, $\Psi_k = 1$ for $k \in \mathbf{N}$, but the group G is not abelian.

Finally, we write the values of a possible system φ for the alternating group \mathcal{U}_4 . These values appear in Table 3, where $\alpha = \exp(2\pi i/3)$. We use this system in the results of Section 2.

	e	a	a^2	a^3	b	ab	a^2b	a^3b	$\ \varphi^s\ _1$	$\ \varphi^s\ _\infty$
φ^0	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1	1	1
φ^2	1	-1	1	-1	1	-1	1	-1	1	1
φ^3	1	-1	1	-1	-1	1	-1	1	1	1
φ^4	$\sqrt{2}$	$\sqrt{2}i$	$-\sqrt{2}$	$-\sqrt{2}i$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}i$	$-\sqrt{2}$	$\sqrt{2}i$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}i$	$-\sqrt{2}$	$\sqrt{2}i$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}i$	$\sqrt{2}$	$\sqrt{2}i$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$

Table 2: The system φ for \mathcal{Q}_2

2 The maximal value of Dirichlet kernels

Define by

$$D_n := \sup_{x, y \in G} |D_n(x, y)| \quad (n \in \mathbf{P})$$

the maximal value of the Dirichlet kernel D_n . Using the inequality of Cauchy-Bunyakovskij we have

$$\begin{aligned} \left| \sum_{k=0}^{n-1} \psi_k(x) \bar{\psi}_k(y) \right|^2 &\leq \sum_{k=0}^{n-1} |\psi_k(x)|^2 \sum_{k=0}^{n-1} |\psi_k(y)|^2 \\ &\leq \max^2 \left\{ \sum_{k=0}^{n-1} |\psi_k(x)|^2, \sum_{k=0}^{n-1} |\psi_k(y)|^2 \right\} \quad (x, y \in G). \end{aligned}$$

Moreover,

$$D_n \geq \sup_{x \in G} |D_n(x, x)| = \sup_{x \in G} \sum_{k=0}^{n-1} |\psi_k(x)|^2.$$

Consequently,

$$D_n = \sup_{x \in G} D_n(x, x) = \sup_{x \in G} \sum_{k=0}^{n-1} |\psi_k(x)|^2 \quad (n \in \mathbf{P}) \quad (2.1)$$

	e	(12)(34)	(13)(24)	(14)(23)	(123)	(142)	(134)	(234)	(124)	(132)	(234)	(143)
φ^0	1	1	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	α	α	α	α	α^2	α^2	α^2	α^2
φ^2	1	1	1	1	α^2	α^2	α^2	α^2	α	α	α	α
φ^3	$\sqrt{3}$	$-\sqrt{3}$	0	0	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
φ^4	$\sqrt{3}$	$-\sqrt{3}$	0	0	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
φ^5	$\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$-\sqrt{3}$	0	0	0	0	0	0	0	0
φ^6	0	0	$\sqrt{3}$	$-\sqrt{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
φ^7	0	0	$\sqrt{3}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
φ^8	0	0	0	0	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$
φ^9	0	0	0	0	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$
φ^{10}	0	0	0	0	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$
φ^{11}	0	0	0	0	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$

Table 3: The system φ for \mathcal{U}_4

from which we have that D_n is monotone increasing sequence.

For the commutative case $D_n = n$ for all $n \in \mathbf{P}$, but the general case is a bit more different.

Theorem 2.1 *If $n \in \mathbf{P}$ and $A := \max\{k \in \mathbf{N} : n_k \neq 0\}$, then*

$$n \leq D_n \leq M_{A+1}.$$

Proof. Let G_k , $k \in \mathbf{N}$, be an arbitrary finite group appeared in the product to obtain G . The unitary property of the representations implies (see [6])

$$\sum_{s=0}^j |\varphi_k^s(x_k)|^2 \leq m_k \quad (j < m_k, x_k \in G_k). \quad (2.2)$$

Moreover, let $n \in \mathbf{N}$, $x, y \in G$. Thus, we have

$$D_n(x, y) = D_{M_A}(x, y) \left(\sum_{s=0}^{n_A-1} \varphi_A^s(x_A) \overline{\varphi_A^s(y_A)} \right) + \varphi_A^{n_A}(x_A) \overline{\varphi_A^{n_A}(y_A)} D_{n_{(A)}}(x, y).$$

By Paley lemma and (2.1) we obtain

$$D_n(x, x) = M_A \left(\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 \right) + |\varphi_A^{n_A}(x_A)|^2 D_{n_{(A)}}(x, x). \quad (2.3)$$

Observe, $n = \sum_{k=0}^A n_k M_k$ and $n_{(A)} = \sum_{k=0}^{A-1} n_k M_k$. By induction on A we prove $D_n(x, x) \leq M_{A+1}$ for all $x \in G$. Indeed, by (2.2)

$$D_{n_{(1)}}(x, x) = \sum_{s=0}^{n_0-1} |\varphi_0^s(x_0)|^2 \leq m_0 = M_1$$

and supposing $D_{n_{(A)}}(x, x) \leq M_A$, by (2.2) and (2.3) we have

$$\begin{aligned} D_{n_{(A+1)}}(x, x) &= D_n(x, x) \leq M_A \left(\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 \right) + |\varphi_A^{n_A}(x_A)|^2 M_A \\ &= M_A \left(\sum_{s=0}^{n_A} |\varphi_A^s(x_A)|^2 \right) \leq M_A m_A = M_{A+1}, \end{aligned}$$

from which the inequality $D_n \leq M_{|n|+1}$ follows.

On the other hand, by the orthonormality of the system ψ we have

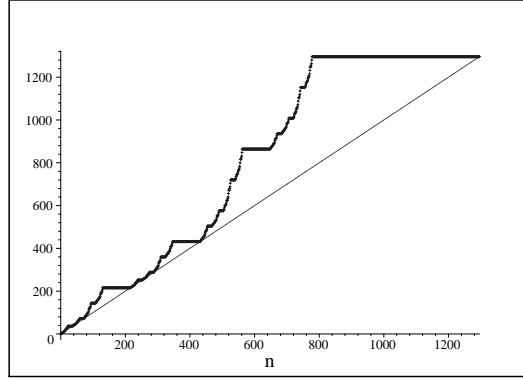
$$\int_G \sum_{k=0}^{n-1} |\psi_k|^2 d\mu = \sum_{k=0}^{n-1} \int_G |\psi_k|^2 d\mu = n$$

and by the $\mu(G) = 1$ property

$$\int_G \sum_{k=0}^{n-1} |\psi_k|^2 d\mu \leq \sup_{x \in G} \sum_{k=0}^{n-1} |\psi_k|^2 = D_n \quad (2.4)$$

from which we obtain the property $D_n \geq n$. This completes the proof of the lemma. \square

Figure 1 illustrates the statements of Theorem 2.1 with respect to the system φ on \mathcal{S}_3 appeared in Table 1.

Figure 1: D_n ($n \leq 6^4$) on the complete product of \mathcal{S}_3

Notice that $D_n = n$ for some values of n . Let $n \in \mathbf{P}$ with expansion (n_0, n_1, \dots) . Denote the indexes A and B by

$$A := \max\{k \in \mathbf{N} : n_k \neq 0\}, \quad \text{and} \quad B := \min\{k \in \mathbf{N} : n_k \neq 0, k \leq A\}.$$

Thus $B = A$ if the expansion of n has only one nonzero coordinate. By Lemma 1.1 it is not hard to see that $D_n = n$ when n satisfies the following two properties

- (i) $\sum_{s=0}^{n_B-1} |\varphi_B^s(x_B)|^2 = n_B$ for all $x_B \in G_B$,
- (ii) $B = A$ or all of $\varphi_i^{n_i}$ are characters such that $\sum_{s=0}^{n_i-1} |\varphi_i^s(x_i)|^2 = n_i$ for all $B < i \leq A$ and $x_i \in G_i$.

Indeed, if the above properties are true we have

$$D_n(x, x) = \sum_{i=B}^A M_i \left(\sum_{s=0}^{n_i-1} |\varphi_i^s(x_i)|^2 \right) |\psi_{n(i+1)}(x)|^2 = \sum_{i=B}^A M_i n_i = n$$

for all $x \in G$. However, property (ii) is too strong. To show this fact we consider the complete product of finite groups where G_0 is the dyadic group and G_1 is the alternating group \mathcal{U}_4 with the system appeared in Table 3 but in different order, i.e. $\{\varphi^0, \varphi^8, \varphi^5, \varphi^1, \varphi^2, \varphi^3, \varphi^4, \varphi^6, \varphi^7, \varphi^9, \varphi^{10}, \varphi^{11}\}$. The rest of finite groups in the product will be arbitrary. Thus $D_5 = 5$, but property (ii) is not true because $5 = (1, 2, 0, 0, \dots)$ and $\varphi_1^2 = \varphi^5$ is not a character.

In order to obtain necessary and sufficient conditions we generalize the property (ii). Denote by $d_i^{(s)}$ the dimension of the representation corresponded to the normalized coordinate function φ_i^s . We say that the positive integer n satisfies the property (ii*) if $B = A$ or there exist $r_i < d_i^{(n_i)}$ nonnegative integers such that

$$(d_i^{(n_i)} - 1)n_{(i)} = r_i M_i \quad (2.5)$$

and

$$n_i - \sum_{s=0}^{n_i-1} |\varphi_i^s(x_i)|^2 = \begin{cases} 0 & \text{if } d_i^{(n_i)} = 1 \\ \frac{r_i}{d_i^{(n_i)} - 1} (|\varphi_i^{n_i}(x_i)|^2 - 1) & \text{if } d_i^{(n_i)} > 1 \end{cases} \quad (2.6)$$

for all $B < i \leq A$ and $x_i \in G_i$. It is clear, if φ_i^s are characters for all $k < i \leq A$, then property (ii) follows from property (ii*).

Lemma 2.2 *Let $n \in \mathbf{P}$. $D_n = n$ if and only if n satisfies properties (i) and (ii*).*

Proof. By (2.4) $D_n = n$ is possible if and only if

$$\sum_{k=0}^{n-1} |\psi_k(x)|^2 = n$$

for all $x \in G$. Thus by (2.3) the sum

$$D_n(x, x) = M_A \left(\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 \right) + |\varphi_A^{n_A}(x_A)|^2 D_{n_{(A)}}(x, x)$$

does not depend on the value of x_A and x . Since there is an $x_A \in G_A$ such that $\varphi_A^{n_A}(x_A) \neq 0$, the value of $D_{n_{(A)}}(x, x)$ does not depend on the value of x , that is

$$D_{n_{(A)}}(x, x) = n_{(A)} \quad (2.7)$$

for all $x \in G$. Since $n = M_A n_A + n_{(A)}$ we also obtain

$$M_A \left(\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 \right) + |\varphi_A^{n_A}(x_A)|^2 n_{(A)} = M_A n_A + n_{(A)} \quad (2.8)$$

for all $x_A \in G_A$. If $B = A$ (in this case $n_{(A)} = 0$) or $\varphi_A^{n_A}$ is a character, then (2.8) is reduced to

$$\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 = n_A \quad (2.9)$$

for all $x_A \in G_A$. Otherwise, denoting by $e = (e_1, e_2, \dots)$ the identity of G we have

$$\sum_{s=0}^{n_A-1} |\varphi_A^s(e_A)|^2 \leq n_A \quad \text{and} \quad |\varphi_A^{n_A}(e_A)|^2 = d_A^{(n_A)}. \quad (2.10)$$

Indeed, the values of $|\varphi_A^s(e_A)|^2$ are integers, so if $\sum_{s=0}^{n_A-1} |\varphi_A^s(e_A)|^2 \geq n_A + 1$ then $D_n(e, e) \geq M_A(n_A + 1) > n$, but $D_n(e, e) = n$. On the other hand, if $|\varphi_A^{n_A}(e_A)|^2 \neq d_A^{(n_A)}$ then $|\varphi_A^{n_A}(e_A)|^2 = 0$. Thus we substitute x_A by e_A in (2.8) to obtain $\sum_{s=0}^{n_A-1} |\varphi_A^s(e_A)|^2 > n_A$ which is not possible.

Denote

$$r_A := n_A - \sum_{s=0}^{n_A-1} |\varphi_A^s(e_A)|^2.$$

Thus r_A is a nonnegative integer. After a simple computation in (2.8) for $x_A = e_A$, we have

$$(d_A^{(n_A)} - 1)n_{(A)} = r_A M_A \quad (2.11)$$

from which we have that $r_A = 0$ if and only if $d_A^{(n_A)} = 1$, and also for $r_A > 0$ we have that $r_A < d_A^{(n_A)} - 1$ and $d_A^{(n_A)} > 2$. In the last case we calculate the value of $n_{(A)}$ in (2.11) and we replace it in (2.8) to obtain

$$\frac{r_A}{d_A^{(n_A)} - 1} (|\varphi_A^{n_A}(x_A)|^2 - 1) = n_A - \sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 \quad (2.12)$$

for all $x_A \in G_A$.

Summarizing the above results we obtain that $D_n = n$ is possible if and only if $B = A$ and (2.9) is true or $B < A$, (2.7) is true and

$$(d_i^{(n_A)} - 1)n_{(A)} = r_A M_A$$

and

$$n_A - \sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 = \begin{cases} 0 & \text{if } d_A^{(n_A)} = 1 \\ \frac{r_A}{d_A^{(n_A)} - 1} (|\varphi_A^{n_A}(x_A)|^2 - 1) & \text{if } d_A^{(n_A)} > 1 \end{cases}$$

for all $x_A \in G_A$. By (2.7) we use a similar method to the applied in the proof of Theorem 2.1, so repeating the above method for k from A down to B we prove Lemma 2.2. \square

Under some conditions of the group G the properties (i) and (ii) will be enough. Indeed, by 2.11 we obtain that $d_A^{(n_A)} \neq 2$, because if $B < A$ then $0 < n_{(A)} < r_A M_A$. For this reason we can state the following theorem.

Theorem 2.3 *Let G the complete product of finite groups where the dimensions of the representations appeared in the finite groups do not exceed the value 2. Then $D_n = n$ if and only if n satisfies properties (i) and (ii).*

In some cases the values of the systems φ only permit (2.8) if the properties (ii) is valid. Suppose that the square of the modulus of systems φ can only take integer values, for instance when the finite groups are all monomials (see Table 2). In this case we obtain a similar statement.

Theorem 2.4 *Let G the complete product of finite groups where the square of the modulus of all of systems φ can only take integer values. Then $D_n = n$ if and only if n satisfies properties (i) and (ii).*

Proof. If the square of the modulus of systems φ can only take integer values, then we obtain

$$\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 \leq n_A$$

for all $x_A \in G_A$ in a similar way to prove (2.10) in the proof of Lemma 2.2. Thus the orthonormality of the systems φ only allow the equality (2.9) for all $x_A \in G_A$ and by (2.8) we obtain

$$|\varphi_A^{n_A}(x_A)|^2 n_{(A)} = n_{(A)}.$$

for all $x_A \in G_A$. Then $B = A$, so $n_{(A)} = 0$ or $B < A$ and $|\varphi_A^{n_A}(x_A)| = 1$ for all $x_A \in G_A$, so $\varphi_A^{n_A}$ is a character. This prove that the property (ii) is necessary and this complete the proof of the theorem. \square

In another cases the order of the systems φ is determinant. Suppose we order the systems φ such that the positive values of the identity are at the beginning of the systems. Thus it is easy to see that $D_n = D_n(e, e)$, so we have our statement from the fact that the values of $|\varphi_A^s(e_A)|^2$ are integers. Furthermore, if the characters are at the beginning of all of the systems φ ,

like the examples in Table 1, 2 and 3, the equation (2.9) is only possible if and only if all of the functions $\varphi_i^{n_i}$ are characters for all $i \in \mathbf{N}$. Thus we obtain the following statement.

Theorem 2.5 *Suppose we order all of the systems φ such that the positive values of the identity are at the beginning of the systems. Then $D_n = n$ if and only if n satisfies properties (i) and (ii).*

Furthermore, suppose also that the characters are at the beginning of all of the systems φ . Then $D_n = n$ if and only if all of the functions $\varphi_i^{n_i}$ are characters for all $i \in \mathbf{N}$.

On the other hand, it is interesting to study the upper estimation in the inequality of Theorem 2.1, so we found all of positive integers n for which $D_n = M_{A+1}$ holds. In this regard suppose there is a finite group G_k such that

$$\varphi_k^{m_k-1}(x_k) = 0, \text{ for some } x_k \in G_k. \quad (2.13)$$

It is meaning that the value 0 appears at least one time at the last row of the table containing the values of the system φ_k . In fact property (2.13) only depend on the last member of the system φ_k^s , $0 \leq s < m_k$, but G_k can not be a commutative group (characters can not take the value 0). Several finite groups G_k can have property (2.13), so denote by N the smallest index of them, that is G_N is the first group having property (2.13).

Notice that we can not always found the index N . For instance we can not obtain it in the commutative cases when we only have characters or the order all systems φ does not allow property (2.13) for any group G_k , $k \in \mathbf{N}$. Although for all $k \in \mathbf{N}$ we can find the first number $r \in \{1, 2, \dots, m_k - 1\}$ such that there exists an $x_k \in G_k$ for which

$$\sum_{s=0}^r |\varphi_k^s(x_k)|^2 = m_k \quad \text{and} \quad \varphi_k^r(x_k) \neq 0$$

and denote this number by k^* .

We only obtain an equality in the upper estimation of Theorem 2.1 in case of the existence of the index N .

Theorem 2.6 *Let $A \in \mathbf{N}$ and $M_A \leq n < M_{A+1}$. If the index N exists then $D_n = M_{A+1}$ if and only if*

$$\sum_{k=N}^A k^* M_k < n < M_{A+1}$$

If the index N does not exist then $D_n < M_{A+1}$.

Proof. Let $A \in \mathbf{N}$ and $M_A \leq n < M_{A+1}$. Then by (2.3) $D_n = M_{A+1}$ if and only if there exists an $x \in G$ such that

$$M_A \left(\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 \right) + |\varphi_A^{n_A}(x_A)|^2 D_{n_{(A)}}(x, x) = M_{A+1} = m_A M_A.$$

Let $k = A$ and write the above equation in the following form.

$$|\varphi_k^{n_k}(x_k)|^2 D_{n_{(k)}}(x, x) = \left(m_k - \sum_{s=0}^{n_k-1} |\varphi_k^s(x_k)|^2 \right) M_k. \quad (2.14)$$

The both sides in (2.14) only can be 0 if $\sum_{s=0}^{n_k-1} |\varphi_k^s(x_k)|^2 = m_k$, so $\varphi_k^s(x_k) = 0$ for all $s \geq n_k$. Otherwise from the

$$|\varphi_k^{n_k}(x_k)|^2 \leq m_k - \sum_{s=0}^{n_k-1} |\varphi_k^s(x_k)|^2 \quad \text{and} \quad D_{n_{(k)}}(x, x) \leq M_k$$

inequalities we obtain that (2.14) is only possible if there exists an $x \in G$ such that

$$\sum_{s=0}^{n_k} |\varphi_k^s(x_k)|^2 = m_k \quad \text{and} \quad D_{n_{(k)}}(x, x) = M_k. \quad (2.15)$$

Thus by the definition of k^* if $n_k > k^*$ then (2.14) holds, if $n_k < k^*$ then (2.14) does not hold and if $n_k = k^*$ then (2.14) holds if and only if $D_{n_{(k)}}(x, x) = M_k$. In the last case we have to repeat the above idea for $k = A - 1$ obtaining the same conclusion for n_{k-1} and so on.

If $A < N$ or the index N does not exist then $k^* = m_k - 1$ for all $0 \leq k \leq A$, so we can not find an index k such that $n_k > k^*$ and then (2.14) holds. Thus $D_n < M_{A+1}$.

If $N \leq A$ we have to stop the process for $k = N$, because $D_{n_{(k)}}(x, x) = M_k$ does not hold for $k < N$. Thus $n_k = k^*$ for all $N < k \leq A$. If we choose $n_N = N^* + 1$ we obtain that

$$1 + \sum_{k=N}^A k^* M_k$$

is the smaller number for which (2.14) holds. This completes the proof of the theorem. \square

The study of the quotients $\frac{D_n}{n}$ is important in order to estimate the Dirichlet kernels. This quotients are 1 in the commutative cases, but they

can be unbounded if the dimensions of the representations appeared in the finite groups are also unbounded. To show this property we order the normalized coordinate functions φ of the finite groups such that let $\varphi_k^0 \equiv 1$ and the representation corresponding to φ_k^1 has the greatest dimension of all representations of the group G_k and denoting this dimension by d_k we also suppose that $\varphi_k^1(e_k) = \sqrt{d_k}$. Thus, it is easy to calculate that

$$\frac{D_n}{n} = \frac{1 + d_k}{2} \quad \text{if } n = 2M_k$$

from which we have that under this order the quotients $\frac{D_n}{n}$ are not bounded for all $n \in \mathbf{P}$ if the dimensions d_k are not bounded. Figure 2 illustrates that the order of the systems φ play an important role. The new order here is $\{\varphi^0, \varphi^1, \varphi^4, \varphi^2, \varphi^5, \varphi^3\}$. of Table 1. Compare it with Figure 1.

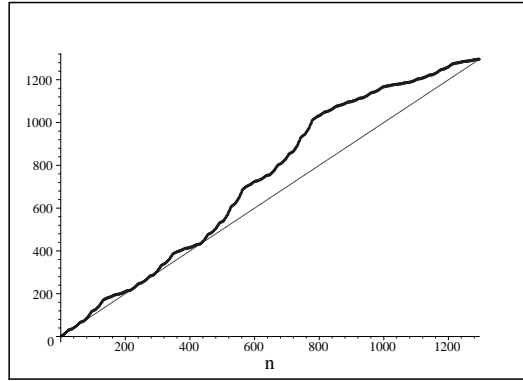


Figure 2: D_n ($n \leq 6^4$) on the complete product of \mathcal{S}_3 using another order of the system φ .

The following lemma states that the boundedness of the quotients $\frac{D_n}{n}$ is given by the boundedness of this quotients on the finite groups.

Lemma 2.7 *Let G the complete product of finite groups G_k . The quotients $\frac{D_n}{n}$ are bounded for all $n \in \mathbf{P}$ if and only if the quotients*

$$\frac{\sum_{s=0}^{r-1} |\varphi_k^s(x_k)|^2}{r} \tag{2.16}$$

are bounded for all $k \in \mathbf{N}$, $0 < r \leq m_k$ and $x_k \in G_k$.

Proof. Let $n = rM_k$. By (2.3) we have

$$\frac{D_n}{n} = \sup_{x \in G_k} \frac{M_k \left(\sum_{s=0}^{r-1} |\varphi_k^s(x_k)|^2 \right)}{rM_k} = \sup_{x \in G_k} \frac{\sum_{s=0}^{r-1} |\varphi_k^s(x_k)|^2}{r}.$$

Thus if the quotients $\frac{D_n}{n}$ are bounded then the quotients (2.16) are bounded for all $k \in \mathbf{N}$, $0 < r \leq m_k$ and $x_k \in G_k$.

To prove the inverse statement let $n = \sum_{k=0}^A n_k M_k$. By Lemma 1.1 and (2.3) there exist an $x \in G$ such that

$$\begin{aligned} D_n &= M_A \left(\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2 \right) + |\varphi_A^{n_A}(x_A)|^2 M_{A-1} \left(\sum_{s=0}^{n_{A-1}-1} |\varphi_{A-1}^s(x_{A-1})|^2 \right) \\ &\quad + |\varphi_A^{n_A}(x_A)|^2 |\varphi_{A-1}^{n_{A-1}}(x_{A-1})|^2 M_{A-1} \left(\sum_{s=0}^{n_{A-2}-1} |\varphi_{A-2}^s(x_{A-2})|^2 \right) + \dots \end{aligned}$$

Let c a positive number for which (2.16) is less than c for all $k \in \mathbf{N}$, $0 < r \leq m_k$ and $x_k \in G_k$. Thus

$$\begin{aligned} \frac{D_n}{n} &\leq \frac{D_n}{n_A M_A} = \frac{\sum_{s=0}^{n_A-1} |\varphi_A^s(x_A)|^2}{n_A} + \frac{|\varphi_A^{n_A}(x_A)|^2 \left(\sum_{s=0}^{n_{A-1}-1} |\varphi_{A-1}^s(x_{A-1})|^2 \right)}{n_A m_{A-1}} \\ &\quad + \frac{|\varphi_A^{n_A}(x_A)|^2 |\varphi_{A-1}^{n_{A-1}}(x_{A-1})|^2 \left(\sum_{s=0}^{n_{A-2}-1} |\varphi_{A-2}^s(x_{A-2})|^2 \right)}{n_A m_{A-1} m_{A-2}} + \dots \end{aligned}$$

From (2.16) we obtain that $\frac{|\varphi_A^{n_A}(x_A)|^2}{n_A} \leq 2c$. Indeed, $\frac{\sum_{s=0}^{n_A} |\varphi_A^s(x_A)|^2}{n_A+1} \leq c$ follows $\frac{|\varphi_A^{n_A}(x_A)|^2}{n_A+1} \leq c$. Thus $\frac{|\varphi_A^{n_A}(x_A)|^2}{n_A} \leq \frac{n_A+1}{n_A} c \leq 2c$. On the other hand

$$\frac{\sum_{s=0}^{n_k-1} |\varphi_k^s(x_k)|^2}{m_k} \leq 1 \quad \text{and} \quad \frac{|\varphi_k^{n_k}(x_k)|^2}{m_k} \leq \frac{1}{2}.$$

Thus

$$\frac{D_n}{n} \leq c + 2c + 2c \frac{1}{2} + 2c \frac{1}{4} + \dots < 5c$$

from which we have that the quotients $\frac{D_n}{n}$ are bounded for all $n \in \mathbf{P}$. This completes the proof of the lemma. \square

Finally, notice that if the group G is bounded or the dimensions of the representations appeared in the finite groups are uniformly bounded

then the conditions of Lemma 2.7 hold. Otherwise we can try to order the systems φ putting the representations with big dimensions at the end of the systems φ .

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