Representative product systems

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The complete product of finite groups

We deal with Fourier analysis on a generalization of Walsh-Paley and Vilenkin systems.

The group $\left(G := \mathop{\times}\limits_{k=0}^{\infty} G_k\right)$

Denote by G the complete product of arbitrary finite groups of order m_k ($m_k \geq 2$, $k \in \mathbf{N}$), with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. G has the product topology and measure. (Haar measure)

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Orthonormal systems on finite groups

The dual object (Σ_k) of the finite group G_k ($k \in \mathbf{N}$)

is the set of all continuous irreducible unitary representations of the group G_k which are not equivalents.

The Coordinate functions

For any $\sigma \in \Sigma_k$, let $\{\xi_1, \dots, \xi_{d_\sigma}\}$ be a fixed basis of the representation space of a representation $U^{(\sigma)}$ in the class σ having the dimension d_σ. The Coordinate functions:

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j \rangle, \quad i, j \in \{1, ..., d_\sigma\}, \sigma \in \Sigma_k$$

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Orthonormal systems on finite groups

The system φ_k

We order the all normalized coordinate functions of the finite group G_k ($\varphi_k^0(x) = 1$) to obtain exactly m_k number of functions.

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_k,\ s = 0, \ldots, m_k - 1),$$

where $\sigma \in \Sigma_k$, $i, j \in \{1, ..., d_\sigma\}$.

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Example 1: Cyclic groups of order m, $\mathcal{Z}_m$

The generalized Rademacher functions

$$\varphi^s(x) = \exp(2\pi\imath s x/m) \qquad (s \in \{0, \dots m-1\},\ x \in \mathcal{Z}_m,\ \imath^2 = -1)$$

- All of the members of system φ are characters.
- $|\varphi^s(x)| = 1$ for all $x \in \mathcal{Z}_m$ and $s \in \{0, \dots m-1\}$.
- $\|\varphi^s\|_1 = 1, \quad \|\varphi^s\|_\infty = 1.$

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Example 2: The permutation group of 3 elements, $\mathcal{S}_3$

	e	(12)	(13)	(23)	(123)	(132)	$\lVert\varphi^s\rVert_1$	$\lVert\varphi^s\rVert_\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
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φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

- The system has 2 characters and a representation of dimension 2.

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$$\max_{s=0..5} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3}.$$

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Representative product systems

The m-adic expansion of n: $(n_0, n_1, \ldots)$

Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, $(k \in \mathbf{N})$. Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \leq n_k < m_k).$$

A representative product systems G. Gát and R. Toledo

is the product system of φ:

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G).$$

Representative product systems

The m-adic expansion of n: $(n_0, n_1, \dots)$

Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, $(k \in \mathbf{N})$. Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \leq n_k < m_k).$$

A representative product systems G. Gát and R. Toledo

is the product system of φ:

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Theorem

A representative product system is an orthonormal and complete system on $L^2(G)$.

Characteristics of the system ψ for noncommutative cases:

- It is not uniformly bounded.
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Convergence in L^p-norm of Fourier series

The n-th partial sums of Fourier series

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \qquad (n \in \mathbf{N}), \qquad \text{where } \widehat{f}_k := \int_G f \overline{\psi}_k d\mu.$$

Theorem (P. Simon, F. Schipp and W. S. Young)

Let G be a Vilenkin group and $1 < p < \infty$. *Then for all function* $f \in L^p(G)$ *the sequence of partial sums* $S_n f$ *of the Fourier series of* f *converges to the function* f *in* L^p*-norm.*

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Convergence in L^p-norm of Fourier series

The sequence Ψ

$$\Psi_k = \prod_{i=0}^{k-1} \max_{s<m_i} \|\varphi_i^s\|_1 \|\varphi_i^s\|_\infty \qquad (k \in \mathbf{N}).$$

Theorem (R. Toledo)

If G is a bounded group with unbounded sequence Ψ, then for all $p \neq 2$, $1 < p < \infty$ there exists a function $f \in L^p(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^p-norm.

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Dirichlet kernels

The Dirichlet kernels

$$D_n(x,y) := \sum_{k=0}^{n-1} \psi_k(x)\overline{\psi}_k(y) \qquad (n \in \mathbf{P})$$

$$S_n f(x) = \int_G f(y) D_n(x,y) d\mu(y)$$

The maximal value of Dirichlet kernels

$$D_n := \sup_{x,y \in G} |D_n(x,y)| \qquad (n \in \mathbf{P})$$

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If G is commutative then $D_n = n$, but for an arbitrary group G

Theorem (R. Toledo)

If $n \in \mathbf{P}$ *and* $A := \max\{k \in \mathbf{N} : n_k \neq 0\}$, *then*

$$n \leq D_n \leq M_{A+1}.$$

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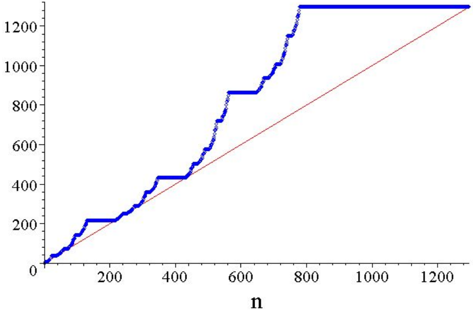

D_n ($n \leq 6^4$) on the complete product of $\mathcal{S}_3$

$D_n = n$

Denote the indexes A and B by

$$A := \max\{k \in \mathbf{N} : n_k \neq 0\}, \quad \text{and} \quad B := \min\{k \in \mathbf{N} : n_k \neq 0, k \leq A\}.$$

It is not difficult to see that $D_n = n$ if n satisfies the following two properties

(i) $\displaystyle\sum_{s=0}^{n_B-1} |\varphi_B^s(x_B)|^2 = n_B$ for all $x_B \in G_B$,

(ii) $B = A$ or all of $\varphi_i^{n_i}$ are characters for all $B < i \leq A$ and $x_i \in G_i$.

$D_n = n$

The properties (i) and (ii) are not necessary for $D_n = n$. For instance, take the alternating group $\mathfrak{A}_4$ and $\alpha = \exp(2\pi\imath/3)$.

	e	(12)(34)	(13)(24)	(14)(23)	(123)	(142)	(134)	(234)	(124)	(132)	(234)	(143)
φ^0	1	1	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	α	α	α	α	α^2	α^2	α^2	α^2
φ^2	1	1	1	1	α^2	α^2	α^2	α^2	α	α	α	α
φ^3	$\sqrt{3}$	$-\sqrt{3}$	0	0	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
φ^4	$\sqrt{3}$	$-\sqrt{3}$	0	0	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
φ^5	$\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$-\sqrt{3}$	0	0	0	0	0	0	0	0
φ^6	0	0	$\sqrt{3}$	$-\sqrt{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
φ^7	0	0	$\sqrt{3}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
φ^8	0	0	0	0	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$
φ^9	0	0	0	0	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$
φ^{10}	0	0	0	0	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$
φ^{11}	0	0	0	0	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$

Take a new order: $\{\varphi^0, \varphi^8, \varphi^5, \varphi^1, \varphi^2, \varphi^3, \varphi^4, \varphi^6, \varphi^7, \varphi^9, \varphi^{10}, \varphi^{11}\}$.

Let $G = \mathfrak{Z}_2 \times \mathfrak{A}_4 \times \mathfrak{Z}_2 \times \mathfrak{Z}_2 \times \dots$, thus $D_5 = 5$, but property (ii) is not true because $5 = (1, 2, 0, 0, \dots)$ and $\varphi_1^2 = \varphi^5$ is not a character.

$D_n = n$

The properties (i) and (ii) are not necessary for $D_n = n$. For instance, take the alternating group $\mathfrak{U}_4$ and $\alpha = \exp(2\pi\imath/3)$.

	e	(12)(34)	(13)(24)	(14)(23)	(123)	(142)	(134)	(234)	(124)	(132)	(234)	(143)
φ^0	1	1	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	α	α	α	α	α^2	α^2	α^2	α^2
φ^2	1	1	1	1	α^2	α^2	α^2	α^2	α	α	α	α
φ^3	$\sqrt{3}$	$-\sqrt{3}$	0	0	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
φ^4	$\sqrt{3}$	$-\sqrt{3}$	0	0	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
φ^5	$\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$-\sqrt{3}$	0	0	0	0	0	0	0	0
φ^6	0	0	$\sqrt{3}$	$-\sqrt{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
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φ^{11}	0	0	0	0	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$

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φ^3	$\sqrt{3}$	$-\sqrt{3}$	0	0	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
φ^4	$\sqrt{3}$	$-\sqrt{3}$	0	0	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
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φ^7	0	0	$\sqrt{3}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
φ^8	0	0	0	0	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$
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Property (ii*) $B = A$ or there exist $r_i < d_i^{(n_i)}$ nonnegative integers such that

$$(d_i^{(n_i)} - 1)n_{(i)} = r_i M_i$$

and

$$n_i - \sum_{s=0}^{n_i-1} |\varphi_i^s(x_i)|^2 = \begin{cases} 0 & \text{if } d_i^{(n_i)} = 1 \\ \frac{r_i}{d_i^{(n_i)}-1}\left(|\varphi_i^{n_i}(x_i)|^2 - 1\right) & \text{if } d_i^{(n_i)} > 1 \end{cases}$$

for all $B < i \leq A$ and $x_i \in G_i$.

Lemma

Let $n \in \mathbf{P}$. $D_n = n$ if and only if n satisfies properties (i) and (ii).*

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Theorem

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Theorem

Let G the complete product of finite groups where the square of the modulus of all of systems φ can only take integer values. Then $D_n = n$ if and only if n satisfies properties (i) and (ii).

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Suppose we order all of the systems φ such that the positive values of the identity are at the beginning of the systems. Then $D_n = n$ if and only if n satisfies properties (i) and (ii).

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$D_n = M_{A+1}$

The finite group G_k has property (N) if there is an $x_k \in G_k$ such that $\varphi_k^{m_k-1}(x_k) = 0$.

Several finite groups G_k can have property (N), so denote by N the smallest index of them, so G_N is the first group having property (N).

In the commutative case the index N does not exist.

For all $k \in \mathbf{N}$ we can find the smaller number $r \in \{1, 2, \ldots, m_k - 1\}$ such that there exists an $x_k \in G_k$ for which

$$\sum_{s=0}^{r} |\varphi_k^s(x_k)|^2 = m_k \quad \text{and} \quad \varphi_k^r(x_k) \neq 0$$

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Theorem

Let $A \in \mathbf{N}$ *and* $M_A \leq n < M_{A+1}$. *If the index N exists then* $D_n = M_{A+1}$ *if and only if*

$$\sum_{k=N}^{A} k^* M_k < n < M_{A+1}$$

If the index N does not exist then $D_n < M_{A+1}$.

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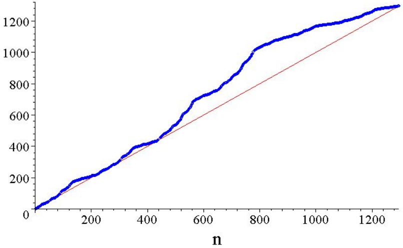

D_n ($n \leq 6^4$) on the complete product of $\mathcal{S}_3$ with new order $\{\varphi^0, \varphi^1, \varphi^4, \varphi^2, \varphi^5, \varphi^3\}$

$D_n = O(n)$

The quotients $\frac{D_n}{n}$ are not always bounded. For instance denote by d_k the dimension corresponding to φ_k^1 and suppose that $\varphi_k^1(e_k) = \sqrt{d_k}$. Thus

$$\frac{D_n}{n} = \frac{1 + d_k}{2} \qquad \text{if } n = 2M_k$$

so it can tend to infinity if $d_k \to \infty$.

Lemma

Let G the complete product of finite groups G_k. The quotients $\frac{D_n}{n}$ are bounded for all $n \in \mathbf{P}$ if and only if the quotients

$$\frac{\sum_{s=0}^{r-1} |\varphi_k^s(x_k)|^2}{r}$$

are bounded for all $k \in \mathbf{N}$, $0 < r \leq m_k$ and $x_k \in G_k$.

$D_n = O(n)$

The quotients $\frac{D_n}{n}$ are not always bounded. For instance denote by d_k the dimension corresponding to φ_k^1 and suppose that $\varphi_k^1(e_k) = \sqrt{d_k}$. Thus

$$\frac{D_n}{n} = \frac{1 + d_k}{2} \qquad \text{if } n = 2M_k$$

so it can tend to infinity if $d_k \to \infty$.

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Applications

Lemma

Let G a bounded group and $xy^{-1} \in G \setminus \{e\}$. Denote by j the index for which $y \in I_j(x) \setminus I_{j+1}(x)$. Then there exists a $c > 0$ such that

$$|D_n(x,y)| \leq c \min \left\{ n, \frac{\left|\psi_{n^{(j+1)}}(x)\overline{\psi}_{n^{(j+1)}}(y)\right|}{|xy^{-1}|} \right\}$$

Theorem (Dini's Test)

Let G the complete product of $\mathcal{Q}_2$ with the product of the system appeared in the Table. Let $x \in G$, $f \in L^1(G)$ and suppose the function

$$g(y) := \frac{f(y) - f(x)}{|xy^{-1}|} \qquad (y \neq x)$$

is integrable. Then $S_n f(x) \to f(x)$ as $n \to \infty$.

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