APPROXIMATION BY NÖRLUND MEANS OF QUADRATICAL PARTIAL SUMS OF DOUBLE WALSH-FOURIER SERIES

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Dedicated to Professor Ferenc Móricz on the occasion of his seventieth birthday.

AMS Classification: 42C10.

Keywords and phrases: Walsh group, Walsh system, Walsh-Fourier series, Nörlund mean, two-dimensional system, approximation .

Abstract. In this article we discuss the Nörlund means of cubical partial sums of Walsh-Fourier series of a function in L^p $(1 \leq p \leq \infty)$. We investigate the rate of the approximation by this means, in particular, in $\mathrm{Lip}(\alpha, p)$, where $\alpha > 0$ and $1 \leq p \leq \infty$. In case $p = \infty$ by L^p we mean C_W, the collection of the uniformly W-continuous functions. Our main theorems state that the approximation behavior of the two-dimensional Walsh-Nörlund means is so good as the approximation behavior of the one-dimensional Walsh-Nörlund means.

As special case we get the Nörlund logarithmic means of cubical partial sums of Walsh-Fourier series discussed recently by Gát and Goginava in 2004 [5] and the (C, β)-means of Marcinkiewicz type with respect to double Walsh-Fourier series discussed by Goginava [10].

Earlier results on one-dimensional Nörlund means of the Walsh-Fourier series was given by Móricz and Siddiqi [14].

1. Introduction

Now, we give a brief introduction to the Walsh-Fourier analysis [15, 1].

Let denote by $\mathbf{Z}_2$ the discrete cyclic group of order 2, the group operation is the modulo 2 addition and every subset is open. The normalized Haar measure on $\mathbf{Z}_2$ is given in the way that the measure of a singleton is 1/2. Let $G := \mathop{\times}\limits_{k=0}^{\infty} \mathbf{Z}_2$, G is called the Walsh group. The elements of G are sequences $x = (x_0, x_1, ..., x_k, ...)$ with $x_k \in \{0, 1\}$ $(k \in \mathbf{N})$.

The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure (denoted by μ) and the topology are the product measure and topology. Dyadic intervalls are defined by

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y = (x_0, ..., x_{n-1}, y_n, y_{n+1} ...)\}$$

for $x \in G, n \in \mathbf{P}$. They form a base for the neighborhoods of G. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \mathbf{N}$. Set $e_i := (0, ..., 0, 1, 0, ...)$, where the ith coordinate is 1 and the rest are 0 $(i \in \mathbf{N})$.

Let L^p denote the usual Lebesgue spaces on G (with the corresponding norm $\|.\|_p$). For the sake of brevity in notation, we agree to write L^∞ instead of C_W and set $\|f\|_\infty := \sup\{|f(x)| : x \in G\}$.

Next, we define the modulus of continuity in $L^p, 1 \leq p \leq \infty$, of a function $f \in L^p$ by

$$\omega_p(\delta, f) := \sup_{|t|<\delta} \|f(. + t) - f(.)\|_p, \quad \delta > 0.$$

The Lipschitz classes in L^p for each $\alpha > 0$ are defined by

$$\mathrm{Lip}(\alpha, p) := \{f \in L^p : \omega_p(\delta, f) = O(\delta^\alpha) \text{ as } \delta \to 0\}.$$

The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbf{N}).$$

Let the Walsh-Paley functions be the product functions of the Rademacher functions. Namely, each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0, 1\} \ (i \in \mathbf{N}),$$

where only a finite number of n_i's different from zero. Let the order of $n > 0$ be denoted by $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$. Walsh-Paley functions are $w_0 = 1$ and for $n \geq 1$

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k}.$$

The Dirichlet kernels are defined by

$$D_n^w := \sum_{k=0}^{n-1} w_k,$$

where $n \in \mathbf{P}$, $D_0^w := 0$. The 2^nth Dirichlet kernels have a closed form (see e.g. [15])

$$D_{2^n}^w = D_{2^n}(x) = \begin{cases} 2^n, & x \in I_n, \\ 0, & \text{otherwise } (n \in \mathbf{N}). \end{cases} \tag{1}$$

The nth Fejér mean and the Fejér kernel of the Fourier series of a function f [6] is defined by

$$\sigma_n^w(f; x) := \frac{1}{n} \sum_{i=0}^{n} S_i^w(f; x), \quad K_n^w(x) := \frac{1}{n} \sum_{k=0}^{n} D_k^w(x) \quad (x \in G),$$

and $K_0^w = 0$.

On G^2 we consider the two-dimensional system as $\{w_{n^1}(x^1) \times w_{n^2}(x^2) : (n^1, n^2) \in \mathbf{N}^2\}$. The two-dimenional Fourier coefficients, the rectangular partial sums of the Fourier series and Dirichlet kernels are defined in the usual way. Define the n-th Marcinkiewicz kernel $\mathcal{K}_n^w$ by

$$\mathcal{K}_n^w(x^1, x^2) := \frac{1}{n} \sum_{k=0}^{n} D_k^w(x^1) D_k^w(x^2) \quad (x = (x^1, x^2) \in G^2).$$

For $x \in G$ we define $|x|$ by $|x| := \sum_{j=0}^{\infty} x_j 2^{-j-1}$, for $x = (x^1, x^2) \in G^2$ by $|x|^2 := (x^1)^2 + (x^2)^2$. Thus, for $f \in L^p(G^2)$ $(1 \leq p \leq \infty)$ the modulus of continuity $\omega_p(\delta, f)$ is well defined for $\delta > 0$. We define the mixed modulus of continuity as follows

$$\omega_{1,2}^p(\delta_1, \delta_2, f) := \sup\{\|f(.+x^1, .+x^2) - f(.+x^1, .) - f(., .+x^2) + f(., .)\|_p : |x^1| \leq \delta_1, |x^2| \leq \delta_2\},$$

where $\delta_1, \delta_2 > 0$.

2. Nörlund means

Let $\{q_k : k \geq 1\}$ be a sequence of nonnegative numbers. The Nörlund means t_n^w and kernels L_n^w for the Walsh-Fourier series are defined by

$$t_n^w(f,x) := \frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k}S_k^w(f,x), \quad L_n^w(x) := \frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k}D_k^w(x),$$

where $Q_n := \sum_{k=1}^{n-1} q_k$ $(n \geq 1)$.

We always assume that $q_1 > 0$ and

$$\lim_{n\to\infty} Q_n = \infty. \tag{2}$$

In this case, the summability method generated by $\{q_k\}$ is regular (see [14]) if and only if

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0. \tag{3}$$

In particular case t_n^w are the Fejér means (for all k set $q_k = 1$) and t_n^w are the (C,β)-means ($q_k := A_k^\beta := \binom{\beta+k}{k}$ for $k \geq 1$ and $\beta \neq -1,-2,...$).

In the paper [14] the rate of the approximation by Nörlund means for Walsh-Fourier series of a function in L^p (in particular, in $\mathrm{Lip}(\alpha,p)$, where $\alpha > 0$ and $1 \leq p \leq \infty$) was studied. In case $p = \infty$, by L^p we mean C_W, the collection of the uniform W-continuous functions. As special cases Móricz and Siddiqi obtained the earlier results given by Yano [18], Jastrebova [11] and Skvortsov [16] on the rate of the approximation by Cesàro means. The approximation properties of the Cesàro means of negativ order was studied by Goginava in 2002 [9].

The case when $q_k = \frac{1}{k}$ is not discussed in the paper of Móricz and Siddiqi, in this case t_n^w are called the Nörlund logarithmic means. The Nörlund logarithmic means for the Walsh-Fourier series was discussed by Gát, Goginava and Tkebuchava earlier [4, 8], for unbounded Vilenkin system by Blahota and Gát [2].

In 2004 Gát and Goginava [5] discussed the uniform and L-convergence of the Nörlund logarithmic means of cubical partial sums of the two-dimensional Walsh-Fourier series.

Motivated by the work of Gát and Goginava we investigate the two-dimensional Nörlund means of cubical partial sums of the two-dimensional Walsh-Fourier series. Define the means and kernels by the usual way

$$\mathbf{t}_n^w(f,x^1,x^2) := \frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k}S_{k,k}^w(f,x^1,x^2),$$

$$\mathcal{L}_n^w(x^1,x^2) := \frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k}D_k^w(x^1)D_k^w(x^2).$$

The means $\mathbf{t}_n^w$ could be called Walsh-Nörlund means of Marcinkiewicz type. We mention that the case that $q_k := \frac{1}{k}$ is not included in this paper. For Walsh system this case is discussed by Gát and Goginava in [5]. If we choose $q_k := A_k^\beta = \binom{\beta+k}{k}$ (for $k \geq 1$ and $\beta \neq -1,-2,...$), then we get the (C,β)-means of Marcinkiewicz-type which was discussed by Goginava [10] with respect to double Walsh-Fourier series.

3. The rate of the approximation

In the following lemma we give a decomposition of the Walsh-Nörlund kernels of Marcinkiewicz type. This lemma is the two-dimensional analogue of the Lemma proved by Móricz and Siddiqi in [14, Lemma 3].

Lemma 1. *Let $|n| = A \geq 1$ and $\{q_k\}$ be a sequence of nonnegative numbers then*

$$\begin{aligned}
Q_n\mathcal{L}_n^w(x^1,x^2) &= Q_{n-2^A+1}D_{2^A}(x^1)D_{2^A}(x^2) + D_{2^A}(x^1)r_A(x^2)Q_{n-2^A}L^w_{n-2^A}(x^2)\\
&+ D_{2^A}(x^2)r_A(x^1)Q_{n-2^A}L^w_{n-2^A}(x^1) + r_A(x^1)r_A(x^2)Q_{n-2^A}\mathcal{L}^w_{n-2^A}(x^1,x^2)\\
&+ \sum_{j=0}^{A-1}(Q_{n-2^j+1} - Q_{n-2^{j+1}+1})D_{2^j}(x^1)D_{2^j}(x^2)\\
&+ \sum_{j=0}^{A-1} D_{2^j}(x^2)r_j(x^1)\sum_{i=1}^{2^j-2}(q_{n-2^j-i} - q_{n-2^j-i-1})iK_i^w(x^1)\\
&+ \sum_{j=0}^{A-1} D_{2^j}(x^2)r_j(x^1)q_{n-2^{j+1}+1}(2^j-1)K^w_{2^j-1}(x^1)\\
&+ \sum_{j=0}^{A-1} D_{2^j}(x^1)r_j(x^2)\sum_{i=1}^{2^j-2}(q_{n-2^j-i} - q_{n-2^j-i-1})iK_i^w(x^2)\\
&+ \sum_{j=0}^{A-1} D_{2^j}(x^1)r_j(x^2)q_{n-2^{j+1}+1}(2^j-1)K^w_{2^j-1}(x^2)\\
&+ \sum_{j=0}^{A-1} r_j(x^1)r_j(x^2)\sum_{i=1}^{2^j-2}(q_{n-2^j-i} - q_{n-2^j-i-1})i\mathcal{K}_i^w(x^1,x^2)\\
&+ \sum_{j=0}^{A-1} r_j(x^1)r_j(x^2)q_{n-2^{j+1}+1}(2^j-1)\mathcal{K}^w_{2^j-1}(x^1,x^2).
\end{aligned}$$

Proof. During the proof of Lemma 1 we use the following equations:

$$D^w_{2^A+j}(x) = D_{2^A}(x) + r_A(x)D^w_j(x), \quad j = 0,1,...,2^A-1. \tag{4}$$

Let $|n| = A$, then we could write

$$Q_n\mathcal{L}_n^w(x^1,x^2) = \sum_{k=1}^{2^A-1} q_{n-k}D_k^w(x^1)D_k^w(x^2) + \sum_{k=2^A}^{n-1} q_{n-k}D_k^w(x^1)D_k^w(x^2) =: I + II.$$

By the help of (4), we decompose II.

$$\begin{aligned}
II &= \sum_{j=0}^{n-2^A-1} q_{n-2^A-j} D^w_{2^A+j}(x^1) D^w_{2^A+j}(x^2) \\
&= D_{2^A}(x^1)D_{2^A}(x^2) \sum_{j=0}^{n-2^A-1} q_{n-2^A-j} + D_{2^A}(x^1) r_A(x^2) \sum_{j=0}^{n-2^A-1} q_{n-2^A-j} D^w_j(x^2) \\
&+ D_{2^A}(x^2) r_A(x^1) \sum_{j=0}^{n-2^A-1} q_{n-2^A-j} D^w_j(x^1) + r_A(x^1) r_A(x^2) Q_{n-2^A} \mathcal{L}^w_{n-2^A}(x^1, x^2) \\
&= Q_{n-2^A+1} D_{2^A}(x^1) D_{2^A}(x^2) + D_{2^A}(x^1) r_A(x^2) Q_{n-2^A} L^w_{n-2^A}(x^2) \\
&+ D_{2^A}(x^2) r_A(x^1) Q_{n-2^A} L^w_{n-2^A}(x^1) + r_A(x^1) r_A(x^2) Q_{n-2^A} \mathcal{L}^w_{n-2^A}(x^1, x^2).
\end{aligned}$$

We use (4) for the discussion of I

$$\begin{aligned}
I &= \sum_{k=1}^{2^A-1} q_{n-k} D^w_k(x^1) D^w_k(x^2) \\
&= \sum_{j=0}^{A-1} \sum_{i=0}^{2^j-1} q_{n-2^j-i} D^w_{2^j+i}(x^1) D^w_{2^j+i}(x^2) \\
&= \sum_{j=0}^{A-1} (Q_{n-2^j+1} - Q_{n-2^{j+1}+1}) D_{2^j}(x^1) D_{2^j}(x^2) + \sum_{j=0}^{A-1} D_{2^j}(x^2) r_j(x^1) \sum_{i=0}^{2^j-1} q_{n-2^j-i} D^w_i(x^1) \\
&+ \sum_{j=0}^{A-1} D_{2^j}(x^1) r_j(x^2) \sum_{i=0}^{2^j-1} q_{n-2^j-i} D^w_i(x^2) + \sum_{j=0}^{A-1} r_j(x^1) r_j(x^2) \sum_{i=0}^{2^j-1} q_{n-2^j-i} D^w_i(x^1) D^w_i(x^2) \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We use Abel's transformation to study I_2, I_3, I_4.

$$I_2 = \sum_{j=0}^{A-1} D_{2^j}(x^2) r_j(x^1) \left(\sum_{i=0}^{2^j-2} (q_{n-2^j-i} - q_{n-2^j-i-1}) i K^w_i(x^1) + q_{n-2^{j+1}+1} (2^j - 1) K^w_{2^j-1}(x^1) \right).$$

(I_3 goes analogously.)

$$I_4 = \sum_{j=0}^{A-1} r_j(x^1) r_j(x^2) \left(\sum_{i=0}^{2^j-2} (q_{n-2^j-i} - q_{n-2^j-i-1}) i \mathcal{K}^w_i(x^1, x^2) + q_{n-2^{j+1}+1} (2^j - 1) \mathcal{K}^w_{2^j-1}(x^1, x^2) \right).$$

For sequence $q_k \downarrow$ we would like to reach the same result as for sequence $q_k \uparrow$, to do this we have to decompose the expression I in another way into two parts. But, we do not write our result to the statement of Lemma 1.

Let $n \in \mathbf{N}$ be fixed and set $|n| = A$. We write for I that

$$\begin{aligned}
I &= \sum_{j=0}^{A-2} \sum_{i=0}^{2^j-1} q_{n-2^j-i} D^w_{2^j+i}(x^1) D^w_{2^j+i}(x^2) + \sum_{i=0}^{2^{A-1}-1} q_{n-2^{A-1}-i} D^w_{2^{A-1}+i}(x^1) D^w_{2^{A-1}+i}(x^2) \\
&=: I^1 + I^2.
\end{aligned}$$

I^1 is studied in Lemma 1 too. To decompose I^2 we will use the following formula in [14]

$$D^w_{2^j+i} - D^w_{2^{j+1}} = -w_{2^{j+1}-1} D^w_{2^j-i} \quad (0 \le i < 2^j). \tag{5}$$

Now, we write for I^2 that

$$\begin{aligned}
I^2 &= \sum_{i=0}^{2^{A-1}-1} q_{n-2^{A-1}-i} (D^w_{2^{A-1}+i}(x^1) - D_{2^A}(x^1)) D^w_{2^{A-1}+i}(x^2) \\
&+ D_{2^A}(x^1) \sum_{i=0}^{2^{A-1}-1} q_{n-2^{A-1}-i} D^w_{2^{A-1}+i}(x^2) \\
&= \sum_{i=0}^{2^{A-1}-1} q_{n-2^{A-1}-i} (D^w_{2^{A-1}+i}(x^1) - D_{2^A}(x^1))(D^w_{2^{A-1}+i}(x^2) - D_{2^A}(x^2)) \\
&+ D_{2^A}(x^2) \sum_{i=0}^{2^{A-1}-1} q_{n-2^{A-1}-i} (D^w_{2^{A-1}+i}(x^1) - D_{2^A}(x^1)) \\
&+ D_{2^A}(x^1) \sum_{i=0}^{2^{A-1}-1} q_{n-2^{A-1}-i} (D^w_{2^{A-1}+i}(x^2) - D_{2^A}(x^2)) \\
&+ D_{2^A}(x^1) D_{2^A}(x^2) \sum_{i=0}^{2^{A-1}-1} q_{n-2^{A-1}-i}
\end{aligned}$$

Substituting the formula (5) into I^2, using an Abel's transformation we get the decomposition

$$\begin{aligned}
I^2 &= (Q_{n-2^{A-1}+1} - Q_{n-2^A+1}) D_{2^A}(x^1) D_{2^A}(x^2) \\
&- D_{2^A}(x^2) w_{2^A-1}(x^1) \sum_{l=1}^{2^{A-1}-1} (q_{n-2^A+l} - q_{n-2^A+l+1}) l K^w_l(x^1) \\
&- D_{2^A}(x^1) w_{2^A-1}(x^2) \sum_{l=1}^{2^{A-1}-1} (q_{n-2^A+l} - q_{n-2^A+l+1}) l K^w_l(x^2) \\
&+ D_{2^A}(x^2) w_{2^A-1}(x^1) q_{n-2^{A-1}} 2^{A-1} K^w_{2^{A-1}}(x^1) \\
&+ D_{2^A}(x^1) w_{2^A-1}(x^2) q_{n-2^{A-1}} 2^{A-1} K^w_{2^{A-1}}(x^2) \\
&+ w_{2^A-1}(x^1 + x^2) \sum_{l=1}^{2^{A-1}-1} (q_{n-2^A+l} - q_{n-2^A+l+1}) l \mathcal{K}^w_l(x^1, x^2) \\
&+ w_{2^A-1}(x^1 + x^2) q_{n-2^{A-1}} 2^{A-1} \mathcal{K}^w_{2^{A-1}}(x^1, x^2).
\end{aligned}$$

This completes the proof of Lemma 1. □

By the help of this lemma we have our main theorem which states that the approximation behavior of the two-dimensional Walsh-Nörlund means of Marcinkiewicz type is so good as the approximation behavior of the one-dimensional Walsh-Nörlund means. The last one was investigated by Móricz and Siddiqi [14]. Recently, Fridli, Manchanda and Siddiqi generalized the result of Móricz and Siddiqi for homogeneous Banach spaces and dyadic Hardy spaces [3].

Theorem 1. *Let $f \in L^p$, $1 \le p \le \infty$, and let $|n| = A \ge 1$ and let $\{q_k : k \ge 1\}$ be a sequence of nonnegative numbers.*

If $\{q_k\}$ is nondecreasing, in sign $\uparrow$, then

$$\|\mathbf{t}_n^w(f) - f\|_p \le \frac{c}{Q_n} \sum_{l=0}^{A-1} q_{n-2^l} 2^l \omega_p(2^{-l}, f) + O(\omega_p(2^{-A}, f)).$$

If $\{q_k\}$ is nonincreasing, in sign $\downarrow$, such that

$$\frac{n}{Q_n^2} \sum_{k=1}^{n-1} q_k^2 = O(1) \tag{6}$$

then

$$\|\mathbf{t}_n^w(f) - f\|_p \le \frac{c}{Q_n} \sum_{l=0}^{A-1} q_{n-2^l} 2^l \omega_p(2^{-l}, f) + O(\omega_p(2^{-A}, f)).$$

To prove our theorem we need the following lemmas given by Schipp, Móricz [13], Yano [17], and Glukhov [7].

Lemma 2. *(Schipp and Móricz) If the condition (6) is satisfied, then there exists a constant C such that*

$$\|L_n^w\|_1 \le C, \quad (n \ge 1).$$

Lemma 3. *(Yano) Let $n \ge 1$, then*

$$\|K_n^w\|_1 \le 2.$$

Lemma 4. *(Glukhov) Let $\alpha_1, ..., \alpha_n$ be real numbers. Then*

$$\frac{1}{n} \left\| \sum_{k=1}^{n} \alpha_k D_k^w D_k^w \right\|_1 \le \frac{c}{\sqrt{n}} \left(\sum_{k=1}^{n} \alpha_k^2 \right)^{1/2},$$

where c is an absolute constant.

As corollary of the Lemma of Glukhov, we get that

$$\|\mathcal{K}_n^w\|_1 \le C \quad (n \ge 1) \tag{7}$$

where C is an absolute constant and the fact that condition (6) implies

$$\|\mathcal{L}_n^w\|_1 \le C \quad (n \ge 1), \tag{8}$$

where C is an absolute constant.

Proof of Theorem 1: Clearly, condition (6) implies the regularity of the summability method. We make the proof for $1 \le p < \infty$, for $p = \infty$ the proof goes in a similar way (where $L^\infty = C_W$).

For sequence $q_k \uparrow$ we use the decomposition Lemma 1, for sequence $q_k \downarrow$ we use the decomposition Lemma 1 and the decomposition of I^1 and I^2 in the proof of Lemma 1.

Let $n \in \mathbf{N}$ be fixed and set $|n| = A$. By Lemma 1 and Minkowski inequality we may write that

$$
\begin{aligned}
& Q_n\|\mathbf{t}_n^w(f)-f\|_p \\
\leq\ & Q_{n-2^A+1}\left\|\int_{G^2}(f(.+x)-f(.))D_{2^A}(x^1)D_{2^A}(x^2)d\mu(x)\right\|_p \\
+\ & Q_{n-2^A}\left\|\int_{G^2}(f(.+x)-f(.))D_{2^A}(x^1)r_A(x^2)L^w_{n-2^A}(x^2)d\mu(x)\right\|_p \\
+\ & Q_{n-2^A}\left\|\int_{G^2}(f(.+x)-f(.))D_{2^A}(x^2)r_A(x^1)L^w_{n-2^A}(x^1)d\mu(x)\right\|_p \\
+\ & Q_{n-2^A}\left\|\int_{G^2}(f(.+x)-f(.))r_A(x^1)r_A(x^2)\mathcal{L}^w_{n-2^A}(x^1,x^2)d\mu(x)\right\|_p \\
+\ & \sum_{j=0}^{A-1}(Q_{n-2^j+1}-Q_{n-2^{j+1}+1})\left\|\int_{G^2}(f(.+x)-f(.))D_{2^j}(x^1)D_{2^j}(x^2)d\mu(x)\right\|_p \\
+\ & \sum_{j=0}^{A-1}\sum_{i=1}^{2^j-2}|q_{n-2^j-i}-q_{n-2^j-i-1}|i\left\|\int_{G^2}(f(.+x)-f(.))D_{2^j}(x^2)r_j(x^1)K^w_i(x^1)d\mu(x)\right\|_p \\
+\ & \sum_{j=0}^{A-1}\sum_{i=1}^{2^j-2}|q_{n-2^j-i}-q_{n-2^j-i-1}|i\left\|\int_{G^2}(f(.+x)-f(.))D_{2^j}(x^1)r_j(x^2)K^w_i(x^2)d\mu(x)\right\|_p \\
+\ & \sum_{j=0}^{A-1}q_{n-2^{j+1}+1}(2^j-1)\left\|\int_{G^2}(f(.+x)-f(.))D_{2^j}(x^2)r_j(x^1)K^w_{2^j-1}(x^1)d\mu(x)\right\|_p \\
+\ & \sum_{j=0}^{A-1}q_{n-2^{j+1}+1}(2^j-1)\left\|\int_{G^2}(f(.+x)-f(.))D_{2^j}(x^1)r_j(x^2)K^w_{2^j-1}(x^2)d\mu(x)\right\|_p \\
+\ & \sum_{j=0}^{A-1}\sum_{i=1}^{2^j-2}|q_{n-2^j-i}-q_{n-2^j-i-1}|i\left\|\int_{G^2}(f(.+x)-f(.))r_j(x^1)r_j(x^2)\mathcal{K}^w_i(x^1,x^2)d\mu(x)\right\|_p \\
+\ & \sum_{j=0}^{A-1}q_{n-2^{j+1}+1}(2^j-1)\left\|\int_{G^2}(f(.+x)-f(.))r_j(x^1)r_j(x^2)\mathcal{K}^w_{2^j-1}(x^1,x^2)d\mu(x)\right\|_p \\
=:\ & \sum_{i=1}^{11}A_{n,i}.
\end{aligned}
$$

By the above written for sequence $q_k\downarrow$, in the expression $A_{n,5},...,A_{n,11}$ the sum goes upto $A-2$ and we have 7 extra expression $A_{n,12},...,A_{n,18}$ from the expression I^2, but we will discuss them at the *second part* of this proof.

Now, we discuss $A_{n,1}$. By (1) we find that

$$\left\|\int_{G^2}(f(.+x)-f(.))D_{2^A}(x^1)D_{2^A}(x^2)d\mu(x)\right\|_p \leq$$

$$\leq \int_{I_A^2} D_{2^A}(x^1)D_{2^A}(x^2)\left(\int_{G^2}|f(y+x)-f(y)|^p d\mu(y)\right)^{1/p} d\mu(x)$$

$$\leq c\omega_p(2^{-A},f). \tag{9}$$

Thus, we immediately have

$$A_{n,1} \leq cQ_{n-2^A+1}\omega_p(2^{-A},f)$$

and

$$A_{n,5} \leq c\sum_{j=0}^{A-1}(Q_{n-2^j+1}-Q_{n-2^{j+1}+1})\omega_p(2^{-j},f).$$

If $q_k \uparrow$, we get that $(Q_{n-2^j+1}-Q_{n-2^{j+1}+1}) \leq 2^j q_{n-2^j}$ and

$$A_{n,5} \leq c\sum_{j=0}^{A-1} 2^j q_{n-2^j}\omega_p(2^{-j},f).$$

If $q_k \downarrow$, we get that $(Q_{n-2^j+1}-Q_{n-2^{j+1}+1}) \leq 2^j q_{n-2^{j+1}}$ and

$$A_{n,5} \leq c\sum_{j=0}^{A-2} 2^j q_{n-2^{j+1}}\omega_p(2^{-j},f) \leq c\sum_{l=1}^{A-1} 2^l q_{n-2^l}\omega_p(2^{-l+1},f).$$

To discuss $A_{n,2}, A_{n,3}, A_{n,6}, A_{n,7}, A_{n,8}, A_{n,9}$ we write the following for any $\varepsilon \in G, y \in G^2$ and $A \in \mathbf{P}$

$$\left|\int_{I_A(\varepsilon)\times I_A}(f(y+x)-f(y))r_A(x^1)d\mu(x)\right| = \left|\int_{I_A(\varepsilon)\times I_A} f(y+x)r_A(x^1)d\mu(x)\right|$$

$$= \left|\int_{I_{A+1}(\varepsilon)\times I_A} f(y+x)r_A(x^1)d\mu(x) + \int_{I_{A+1}(\varepsilon+e_A)\times I_A} f(y+x)r_A(x^1)d\mu(x)\right|$$

$$= \left|\int_{I_{A+1}(\varepsilon)\times I_A} f(y+x)-f(y+x+e_A^1)d\mu(x)\right|$$

$$\leq \int_{I_{A+1}(\varepsilon)\times I_A} |f(y+x)-f(y+x+e_A^1)|d\mu(x), \tag{10}$$

where $e_A^1 := (e_A, 0)$ (and $e_A^2 := (0, e_A)$ we will use it later too).

To discuss $A_{n,6}$, we write for any $|j| \leq k$ that

$$B_j^k := \left\|\int_{G^2}(f(.+x)-f(.))D_{2^k}(x^2)r_k(x^1)K_j^w(x^1)d\mu(x)\right\|_p$$

$$= \left\|\sum_{\substack{\varepsilon_i=0 \\ i\in\{0,1,\ldots,k-1\} \\ \varepsilon_l=0,\ l\geq k}}^{1}\int_{I_k(\varepsilon)\times I_k}(f(.+x)-f(.))D_{2^k}(x^2)r_k(x^1)K_j^w(x^1)d\mu(x)\right\|_p.$$

The function $K_j^w(x^1)$ is constant on the sets $I_k(\varepsilon) \quad (\varepsilon \in G,\ |j| \le k)$. Thus, (10) and Lemma 3 imply

$$\begin{aligned}
B_j^k &= \left\| \sum_{\substack{\varepsilon_i=0\\ i\in\{0,1,\dots,k-1\}\\ \varepsilon_l=0,\ l\ge k}}^{1} 2^k K_j^w(\varepsilon) \int_{I_k(\varepsilon)\times I_k} (f(.+x)-f(.))r_k(x^1)d\mu(x) \right\|_p \\
&\le \sum_{\substack{\varepsilon_i=0\\ i\in\{0,1,\dots,k-1\}\\ \varepsilon_l=0,\ l\ge k}}^{1} 2^k |K_j^w(\varepsilon)| \left(\int_{G^2} \left| \int_{I_k(\varepsilon)\times I_k} (f(y+x)-f(y))r_k(x^1)d\mu(x) \right|^p d\mu(y) \right)^{1/p} \\
&\le \sum_{\substack{\varepsilon_i=0\\ i\in\{0,1,\dots,k-1\}\\ \varepsilon_l=0,\ l\ge k}}^{1} 2^k |K_j^w(\varepsilon)| \left(\int_{G^2} \left(\int_{I_{k+1}(\varepsilon)\times I_k} |f(y+x)-f(y+x+e_k^1)| d\mu(x) \right)^p d\mu(y) \right)^{1/p} \\
&\le \sum_{\substack{\varepsilon_i=0\\ i\in\{0,1,\dots,k-1\}\\ \varepsilon_l=0,\ l\ge k}}^{1} 2^k |K_j^w(\varepsilon)| \int_{I_{k+1}(\varepsilon)\times I_k} \left(\int_{G^2} |f(y+x)-f(y+x+e_k^1)|^p d\mu(y) \right)^{1/p} d\mu(x) \\
&\le c \sum_{\substack{\varepsilon_i=0\\ i\in\{0,1,\dots,k-1\}\\ \varepsilon_l=0,\ l\ge k}}^{1} 2^k |K_j^w(\varepsilon)| \omega_p(2^{-k},f) \int_{I_{k+1}(\varepsilon)\times I_k} d\mu(x) \\
&\le c\omega_p(2^{-k},f)\|K_j^w\|_1 \\
&\le c\omega_p(2^{-k},f).
\end{aligned}$$

That is,

$$B_j^k := \left\| \int_{G^2} (f(.+x)-f(.))D_{2^k}(x^2)r_k(x^1)K_j^w(x^1)d\mu(x) \right\|_p \le c\omega_p(2^{-k},f) \tag{11}$$

for any $|j| \le k$. This implies that

$$\begin{aligned}
A_{n,6} &= \sum_{j=0}^{A-1} \sum_{i=0}^{2^j-2} |q_{n-2^j-i} - q_{n-2^j-i-1}| i B_i^j \\
&\le c \sum_{j=0}^{A-1} \sum_{i=0}^{2^j-2} |q_{n-2^j-i} - q_{n-2^j-i-1}| i\omega_p(2^{-j},f)
\end{aligned}$$

($A_{n,7}$ goes similarly to $A_{n,6}$).

Moreover,

$$A_{n,8} \le c \sum_{j=0}^{A-1} q_{n-2^{j+1}+1} 2^j B_{2^j-1}^j \le c \sum_{j=0}^{A-1} q_{n-2^{j+1}+1} 2^j \omega_p(2^{-j},f)$$

($A_{n,9}$ goes similarly to $A_{n,8}$).

If $q_k \uparrow$, we get that

$$\begin{aligned}\sum_{i=0}^{2^j-2}|q_{n-2^j-i}-q_{n-2^j-i-1}|i &\le \sum_{i=1}^{2^j-2}q_{n-2^j-i}-(2^j-2)q_{n-2^{j+1}+1}\\ &\le \sum_{i=1}^{2^j-2}q_{n-2^j-i}\le 2^jq_{n-2^j}\end{aligned}\tag{12}$$

and

$$A_{n,6}\le c\sum_{j=0}^{A-1}2^jq_{n-2^j}\omega_p(2^{-j},f).$$

Moreover,

$$A_{n,8}\le c\sum_{j=0}^{A-1}q_{n-2^j}2^j\omega_p(2^{-j},f).$$

While, in the case when $q_k\downarrow$

$$\begin{aligned}\sum_{i=0}^{2^j-2}|q_{n-2^j-i}-q_{n-2^j-i-1}|i &\le (2^j-2)q_{n-2^{j+1}+1}-\sum_{i=1}^{2^j-2}q_{n-2^j-i}\\ &\le 2^jq_{n-2^{j+1}}\end{aligned}\tag{13}$$

and

$$A_{n,6}\ \le\ c\sum_{j=0}^{A-2}2^jq_{n-2^{j+1}}\omega_p(2^{-j},f).$$

Moreover,

$$A_{n,8}\le c\sum_{j=0}^{A-2}q_{n-2^{j+1}}2^j\omega_p(2^{-j},f).$$

Now, we introduce the notation $\tilde{B}_j^A$ to discuss $A_{n,2},A_{n,3}$.

First, let $q_k\downarrow$

$$\tilde{B}_j^A:=\left\|\int_{G^2}(f(.+x)-f(.))D_{2^A}(x^2)r_A(x^1)L_j^w(x^1)d\mu(x)\right\|_p.$$

The method above (see (11)), the condition (6) and Lemma 2 imply

$$\tilde{B}_{n-2^A}^A\le c\omega_p(2^{-A},f)\|L_{n-2^A}^w\|_1\le c\omega_p(2^{-A},f)$$

(we note that $|n-2^A|\le A-1$) and

$$A_{n,3}\le cQ_{n-2^A}\omega_p(2^{-A},f).$$

Now, let $q_k\uparrow$. We use Abel's transformation for $Q_{n-2^A}L_{n-2^A}^w$.

$$Q_{n-2^A}L_{n-2^A}^w=\sum_{j=1}^{n-2^A-2}(q_{n-2^A-j}-q_{n-2^A-j-1})jK_j^w+q_1(n-2^A-1)K_{n-2^A-1}^w$$

and the definition of B_j^A and (11) imply

$$\begin{aligned} A_{n,3} &\leq c\omega_p(2^{-A},f)\left(\sum_{j=1}^{n-2^A-2}|q_{n-2^A-j}-q_{n-2^A-j-1}|j+q_1(n-2^A-1)\right)\\ &\leq c\omega_p(2^{-A},f)\left(\sum_{j=1}^{n-2^A-2}q_{n-2^A-j}+q_1(n-2^A-1)\right)\\ &\leq c\omega_p(2^{-A},f)(Q_{n-2^A}+q_1(n-2^A-1)). \end{aligned}$$

(We note that $Q_n\geq (n-1)q_1$ for increasing sequence q_n. $A_{n,2}$ goes similarly.)

At last, we discuss $A_{n,4}, A_{n,10}, A_{n,11}$. First, we investigate $A_{n,4}$ and the others goes similarly, but we will write some words about it.

First, let $q_k\downarrow$. We note that $\mathcal{L}_j^w(x^1,x^2)$ is constant on the sets $I_A(\varepsilon)\times I_A(\rho)$ $(\varepsilon,\rho\in G)$. This and the generalized Minkowski inequality give

$$\begin{aligned} F_j^A &:= \left\|\int_{G^2}(f(.+x)-f(.))r_A(x^1)r_A(x^2)\mathcal{L}_j^w(x^1,x^2)d\mu(x)\right\|_p\\ &= \left\|\sum_{\substack{\varepsilon_i=0\\ i\in\{0,\dots,A-1\}}}^{1}\sum_{\substack{\rho_j=0\\ j\in\{0,\dots,A-1\}}}^{1}\int_{I_A(\varepsilon)\times I_A(\rho)}(f(.+x)-f(.))r_A(x^1)r_A(x^2)\mathcal{L}_j^w(x^1,x^2)d\mu(x)\right\|_p\\ &\leq \sum_{\substack{\varepsilon_i=0\\ i\in\{0,\dots,A-1\}}}^{1}\sum_{\substack{\rho_j=0\\ j\in\{0,\dots,A-1\}}}^{1}|\mathcal{L}_j^w(\varepsilon,\rho)|\left\|\int_{I_A(\varepsilon)\times I_A(\rho)}(f(.+x)-f(.))r_A(x^1)r_A(x^2)d\mu(x)\right\|_p\\ &\leq \sum_{\substack{\varepsilon_i=0\\ i\in\{0,\dots,A-1\}}}^{1}\sum_{\substack{\rho_j=0\\ j\in\{0,\dots,A-1\}}}^{1}|\mathcal{L}_j^w(\varepsilon,\rho)|\times\\ &\qquad\times\left(\int_{G^2}\left|\int_{I_A(\varepsilon)\times I_A(\rho)}(f(y+x)-f(y))r_A(x^1)r_A(x^2)d\mu(x)\right|^p d\mu(y)\right)^{1/p} \end{aligned}$$

for $|j|\leq A$. In the way of (10) we easily get that

$$\left|\int_{I_A(\varepsilon)\times I_A(\rho)}(f(y+x)-f(y))r_A(x^1)r_A(x^2)d\mu(x)\right|\leq\int_{I_{A+1}(\varepsilon)\times I_{A+1}(\rho)}\Delta_A f(x,y)d\mu(x), \tag{14}$$

where

$$\Delta_A f(x,y):=|f(x+y)-f(x+y+e_A^2)-f(x+y+e_A^1)+f(x+y+e_A^1+e_A^2)|.$$

(14), condition (6) and Lemma 4 (see equation (8)) imply that

$$\begin{aligned}
F_j^A &\le \sum_{\substack{\varepsilon_i=0\\ i\in\{0,\dots,A-1\}}}^{1} \sum_{\substack{\rho_j=0\\ j\in\{0,\dots,A-1\}}}^{1} |\mathcal{L}_j^w(\varepsilon,\rho)| \left(\int_{G^2}\left(\int_{I_{A+1}(\varepsilon)\times I_{A+1}(\rho)} \Delta_A f(x,y)d\mu(x)\right)^p d\mu(y)\right)^{1/p} \\
&\le \sum_{\substack{\varepsilon_i=0\\ i\in\{0,\dots,A-1\}}}^{1} \sum_{\substack{\rho_j=0\\ j\in\{0,\dots,A-1\}}}^{1} |\mathcal{L}_j^w(\varepsilon,\rho)| \times \\
&\quad\times \int_{I_{A+1}(\varepsilon)\times I_{A+1}(\rho)} \left(\int_{G^2}(\Delta_A f(x,y))^p d\mu(y)\right)^{1/p} d\mu(x) \\
&\le c \sum_{\substack{\varepsilon_i=0\\ i\in\{0,\dots,A-1\}}}^{1} \sum_{\substack{\rho_j=0\\ j\in\{0,\dots,A-1\}}}^{1} \int_{I_{A+1}(\varepsilon)\times I_{A+1}(\rho)} |\mathcal{L}_j^w(\varepsilon,\rho)|\, d\mu(x) \;\; \omega_{1,2}^p(2^{-A},2^{-A},f) \\
&\le c\|\mathcal{L}_j^w\|_1 \omega_{1,2}^p(2^{-A},2^{-A},f) \\
&\le c\omega_{1,2}^p(2^{-A},2^{-A},f).
\end{aligned}$$

That is,

$$F_j^A \le c\|\mathcal{L}_j^w\|_1 \omega_{1,2}^p(2^{-A},2^{-A},f) \le c\omega_{1,2}^p(2^{-A},2^{-A},f) \tag{15}$$

for $|j|\le A$, and

$$A_{n,4} \le Q_{n-2^A} F_{n-2^A}^A \le cQ_{n-2^A}\omega_{1,2}^p(2^{-A},2^{-A},f) \le cQ_{n-2^A}\omega_p(2^{-A},f).$$

Now, we discuss $A_{n,4}$ for sequence $q_k\uparrow$. By Abel's transformation we write

$$Q_{n-2^A}\mathcal{L}_{n-2^A}^w = \sum_{j=1}^{n-2^A-2} (q_{n-2^A-j} - q_{n-2^A-j-1}) j\mathcal{K}_j^w + q_1(n-2^A-1)\mathcal{K}_{n-2^A-1}^w.$$

Set

$$\tilde{F}_j^A := \left\| \int_{G^2} (f(.+x)-f(.)) r_A(x^1) r_A(x^2) \mathcal{K}_j^w(x^1,x^2) d\mu(x) \right\|_p \qquad \text{for } |j|\le A.$$

The method of the discussion of F_j^A (see (15)) and Lemma 4 (see equation (7)) imply that

$$\tilde{F}_j^A \le c\omega_{1,2}^p(2^{-A},2^{-A},f)\|\mathcal{K}_j^w\|_1 \le c\omega_p(2^{-A},f)$$

for $|j|\le A$ and

$$A_{n,4} = c\omega_p(2^{-A},f)\left(Q_{n-2^A} + q_1(n-2^A-1)\right).$$

(For more details see $A_{n,2}, A_{n,3}$, while $q_k\uparrow$.)

We discuss $A_{n,10}, A_{n,11}$.

$$\begin{aligned}
A_{n,10} &= \sum_{j=0}^{A-1}\sum_{i=0}^{2^j-2} |q_{n-2^j-i} - q_{n-2^j-i-1}| i\tilde{F}_i^j \\
&\le c\sum_{j=0}^{A-1}\sum_{i=0}^{2^j-2} |q_{n-2^j-i} - q_{n-2^j-i-1}| i\omega_p(2^{-j},f).
\end{aligned}$$

If $q_k \uparrow$, by (12) we get that

$$A_{n,10} \leq \sum_{j=0}^{A-1} 2^j q_{n-2^j} \omega_p(2^{-j}, f).$$

If $q_k \downarrow$, by (13) we get that

$$A_{n,10} \leq \sum_{j=0}^{A-2} 2^j q_{n-2^{j+1}} \omega_p(2^{-j}, f).$$

At last,

$$A_{n,11} \leq \sum_{j=0}^{A-1} q_{n-2^{j+1}+1} 2^j \tilde{F}^j_{2^j-1} \leq c \sum_{j=0}^{A-1} q_{n-2^{j+1}+1} 2^j \omega_p(2^{-j}, f).$$

If $q_k \uparrow$, then

$$A_{n,11} \leq c \sum_{j=0}^{A-1} q_{n-2^j} 2^j \omega_p(2^{-j}, f),$$

while, if $q_k \downarrow$

$$A_{n,11} \leq c \sum_{j=0}^{A-2} q_{n-2^{j+1}} 2^j \omega_p(2^{-j}, f).$$

Summarising our results on $A_{n,i}$ $(i = 1, ..., 11)$ we could complete the proof of this theorem for sequence $q_k \uparrow$.

The second part of the proof of Theorem 1: Let the sequence q_k be nonincreasing ($q_k \downarrow$). We define $A_{n,i}$ $(i = 12, ..., 18)$ analogously as we do for $A_{n,1}, ..., A_{n,11}$ (see the decomposition of I^2).

By (9)

$$A_{n,12} \leq c(Q_{n-2^{A-1}+1} - Q_{n-2^A+1}) \omega_p(2^{-A}, f).$$

To study $A_{n,13}, A_{n,14}, A_{n,15}, A_{n,16}$ we define $\tilde{\tilde{B}}^{A-1}_j$ by

$$\tilde{\tilde{B}}^{A-1}_j := \left\| (f(. + x) - f(.)) D_{2^A}(x^2) r_{A-1}(x^1) \omega_{2^{A-1}-1}(x^1) K^w_j(x^1) d\mu(x) \right\|_p$$

for $|j| \leq A - 1$. The method of the discussion B^A_j (see (11)) and Lemma 3 imply that

$$\tilde{\tilde{B}}^{A-1}_j \leq c\omega_p(2^{-(A-1)}, f),$$

$$A_{n,15}, A_{n,16} \leq c q_{n-2^{A-1}} 2^{A-1} \omega_p(2^{-(A-1)}, f)$$

and

$$\begin{aligned}
A_{n,13}, A_{n,14} &\leq c \sum_{l=1}^{2^{A-1}-1} |q_{n-2^A+l} - q_{n-2^A+l+1}| l \tilde{\tilde{B}}^{A-1}_l \\
&\leq c \sum_{l=1}^{2^{A-1}-1} |q_{n-2^A+l} - q_{n-2^A+l+1}| l \omega_p(2^{-(A-1)}, f) \\
&\leq c(Q_{n-2^{A-1}} - Q_{n-2^A+1}) \omega_p(2^{-A}, f).
\end{aligned}$$

To discuss $A_{n,17}, A_{n,18}$ we define $\tilde{\tilde{F}}_j^{A-1}$ by

$$\tilde{\tilde{F}}_j^{A-1} := \left\|(f(.+x) - f(.))r_{A-1}(x^1+x^2)\omega_{2^{A-1}-1}(x^1+x^2)\mathcal{K}_j^w(x^1,x^2)d\mu(x)\right\|_p$$

for $|j| \leq A-1$. The method of the discussion F_j^A (see (15)) and Lemma 4 imply that

$$\tilde{\tilde{F}}_j^{A-1} \leq c\omega_{1,2}^p(2^{-(A-1)}, 2^{-(A-1)}, f) \leq c\omega_p(2^{-(A-1)}, f).$$

By this

$$A_{n,18} \leq q_{n-2^{A-1}}2^{A-1}\tilde{\tilde{F}}_{2^{A-1}}^{A-1} \leq cq_{n-2^{A-1}}2^{A-1}\omega_p(2^{-(A-1)}, f)$$

and

$$\begin{aligned} A_{n,17} &\leq c\sum_{l=1}^{2^{A-1}-1}|q_{n-2^A+l} - q_{n-2^A+l+1}|l\tilde{\tilde{F}}_l^{A-1} \\ &\leq c\sum_{l=1}^{2^{A-1}-1}|q_{n-2^A+l} - q_{n-2^A+l+1}|l\omega_p(2^{-(A-1)}, f) \\ &\leq c(Q_{n-2^{A-1}} - Q_{n-2^A+1})\omega_p(2^{-A}, f). \end{aligned}$$

These facts complete the proof of Theorem 1. □

We will discuss the following cases:

a.) the nondecreasing $\{q_k\}$, in sign $q_k \uparrow$, satisfies the condition

$$\frac{nq_{n-1}}{Q_n} = O(1). \tag{16}$$

In particular (16) is true if

$$q_k \asymp k^\beta \text{ or } (\log k)^\beta \text{ for some } \beta > 0.$$

b.) the nonincreasing $\{q_k\}$, in sign $q_k \downarrow$, satisfies
bi.) $q_k \asymp k^{-\beta}$ for some $0 < \beta < 1$, or
bii.) $q_k \asymp (\log k)^{-\beta}$ for some $0 < \beta$.
(We note that the condition (6) is satisfied in these cases.)

For more details see [14].

The one-dimensional analogue of the following theorem was proved by Móricz and Siddiqi in [14]. We mention that as special case (set $q_k := 1$ for all k) we get the so-called Marcinkiewicz means of Walsh-Fourier series. More generally, when $q_k := A_k^\beta := \binom{\beta+k}{k}$ for $k \geq 1$ ($\beta \neq -1, -2, ...$) we have the (C, β) mean of Marcinkiewicz type discussed by Goginava [10] with respect to the double Walsh-Fourier series.

At last, we note that the following theorem states that the approximation behavior of the Nörlund means of quadratical partial sums of double Walsh-Fourier series is so good as the approximation behavior of the one-dimensional Nörlund means of Walsh-Fourier series for Lipschitz functions showed by Móricz and Siddiqi [14].

Theorem 2. *Let $f \in Lip(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p \leq \infty$.*

Let $\{q_k : k \geq 1\}$ be a sequence of nonnegative numbers such that in case $q_k \uparrow$ the condition (16) is satisfied, while in case $q_k \downarrow$ the condition bi) or bii.) is satisfied, then

$$\|\mathbf{t}_n^w(f) - f\|_p = \begin{cases} O(n^{-\alpha}), & \text{if } 0 < \alpha < 1, \\ O(n^{-1}\log n), & \text{if } \alpha = 1, \\ O(n^{-1}), & \text{if } \alpha > 1. \end{cases}$$

Proof. Let $f \in \text{Lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \leq p \leq \infty$.

First, let $q_k \uparrow$, which satisfies the condition (16). Theorem 1 and the method of Móricz and Siddiqi in [14] immediately give our statement.

Second, let $q_k \downarrow$, which satisfies the condition bi. That is $q_k \asymp k^{-\beta}$ for some $0 < \beta < 1$, then $Q_n \asymp n^{1-\beta}$. From Theorem 1 it follows that

$$\|\mathbf{t}_n^w(f) - f\|_p \leq \frac{c}{Q_n}\sum_{l=0}^{|n|-1} q_{n-2^l}2^l 2^{-l\alpha} + O(2^{-|n|\alpha}).$$

For $0 \leq l \leq |n|-1$ we have $2^{|n|-1} \leq n - 2^l$ and $q_{n-2^l} \leq c2^{-\beta(|n|-1)}$. Thus,

$$\begin{aligned} \|\mathbf{t}_n^w(f) - f\|_p &\leq \frac{c}{n^{1-\beta}}\sum_{l=0}^{|n|-1} 2^{-\beta|n|}2^{l(1-\alpha)} + O(2^{-|n|\alpha}) \\ &\leq \frac{c}{n}\sum_{l=0}^{|n|-1} 2^{l(1-\alpha)} + O(2^{-|n|\alpha}) \\ &= \begin{cases} O(\frac{2^{|n|(1-\alpha)}}{n}), & \text{if } 0 < \alpha < 1, \\ O(\frac{|n|}{n}), & \text{if } \alpha = 1, \\ O(\frac{1}{n}), & \text{if } \alpha > 1. \end{cases} \end{aligned}$$

Let the condition bii. be satisfied. That is $q_k \asymp (\log k)^{-\beta}$ for some $0 < \beta$, then $Q_n \asymp n(\log n)^{-\beta}$. The proof goes analogously as we wrote above.

□

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