POINTWISE CONVERGENCE OF CONE-LIKE RESTRICTED TWO-DIMENSIONAL FEJÉR MEANS OF WALSH-FOURIER SERIES

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ABSTRACT. For the two-dimensional Walsh system Gát and Weisz proved the a.e. convergence of Fejér means $\sigma_n f$ of integrable functions, where the set of indices is inside a positive cone around the identical function, that is, $\beta^{-1} \leq n_1/n_2 \leq \beta$ is provided with some fixed parameter $\beta \geq 1$. In this paper we generalize the result of Gát and Weisz. We do not only generalize this theorem, but give a necessary and sufficient condition for cone-like sets in order to preserve this convergence property.

1. INTRODUCTION

For double trigonometric Fourier series Marcinkiewicz and Zygmund [8] proved the a.e. convergence of Fejér means $\sigma_n f$ of integrable functions, where the set of indices is inside a positive cone around the identical function, that is $\beta^{-1} \leq n_1/n_2 \leq \beta$ is provided with some fixed parameter $\beta \geq 1$. This was proved also in the book [12]. We mention that Jessen, Marcinkiewicz and Zygmund [7] also proved the a.e. convergence $\sigma_n f \to f$ without any restriction on the indices, but for functions in $L \log^+ L$.

In the paper [4] the first author gave a common generalization of the two more than 60 year old result of Marcinkiewicz and Zygmund and the result of Jessen, Marcinkiewicz and Zygmund above. He not only generalized these theorems, but gave a necessary and sufficient condition for cone-like sets in order to preserve this convergence property.

For double Walsh-Fourier series, Móricz, Schipp and Wade [9] proved that $\sigma_n f$ converge to f a.e. in the Pringsheim sense (that is, no restriction on the indices other than $\min(n_1, n_2) \to \infty$) for all functions $f \in L \log^+ L$. In the paper [3] Gát proved that the theorem of Móricz, Schipp and Wade can not be sharpened. Namely, the following was proved. Let $\delta : [0, +\infty) \to [0, +\infty)$ be a measurable function with property $\lim_{+\infty} \delta = 0$, then there exists a function $f \in L \log^+ L\delta(L)$ such that $\sigma_n f$ does not converge to f a.e. as $\min(n_1, n_2) \to \infty$.

For double Walsh system the result of Marcinkiewicz and Zygmund was proved by Gát [2] and Weisz [11].

In this article, we would like to give a common generalization of results of Gát, Weisz and the result of Móricz, Schipp, Wade (with respect to Walsh system) in the same direction and way as Gát did in [4] with respect to the trigonometric system. Moreover, we would like to give a necessary and sufficient condition for cone-like sets in order to preserve the convergence property.

Now, we give a brief introduction to the dyadic harmonic analysis, for more details see [1, 10].

Denote G the Walsh group, and μ the normalized Haar measure on G. Dyadic intervalls are defined by

$$I_0(x) := G,\ I_n(x) := \{y \in G : y = (x_0, x_1, ..., x_{n-1}, y_n, y_{n+1}, ...)\}$$

for $x \in G, n \in \mathbb{P}$. Let $0 := (0 : i \in \mathbb{N})$ be the nullelement of G and $I_n := I_n(0)$ for $n \in \mathbb{N}$.

The Rademacher functions are defined by

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbb{N}).$$

Each natural number n can be uniquely expressed as $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0, 1\} (i \in \mathbb{N})$. Define the Walsh-Paley functions by

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k}.$$

Let us consider the Dirichlet and Fejér kernel functions

$$D_n := \sum_{k=0}^{n-1} \omega_k, \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k, \quad D_0 = K_0 := 0.$$

The Fourier coefficients, the nth partial sum of Fourier series and the Fejér means are defined in the usual way for $f \in L^1$.

Define the two-dimensional Dirichlet and Fejér kernel functions as the Kronecker product of one-dimensional functions

$$D_n(x) := D_{n_1}(x_1) D_{n_2}(x_2), \quad K_n(x) := K_{n_1}(x_1) K_{n_2}(x_2),$$

where $x = (x_1, x_2) \in G^2$ and $n = (n_1, n_2) \in \mathbb{N}^2$.

2. Definitions and notations

Let $\alpha : [1, +\infty) \to [1, +\infty)$ be a strictly monotone increasing continuous function with property $\lim_{+\infty} \alpha = +\infty$, $\alpha(1) = 1$, and β :

$[1, +\infty) \to [1, +\infty)$ be a monotone increasing function with property $\beta(1) > 1$.

Define the cone-like restriction sets of $\mathbb{N}^2$ as follows:

$$\begin{aligned} \mathbb{N}_{\alpha,\beta,1} &:= \left\{ n \in \mathbb{N}^2 : \frac{\alpha(n_1)}{\beta(n_1)} \le n_2 \le \alpha(n_1)\beta(n_1) \right\}, \\ \mathbb{N}_{\alpha,\beta,2} &:= \left\{ n \in \mathbb{N}^2 : \frac{\alpha^{-1}(n_2)}{\beta(n_2)} \le n_1 \le \alpha^{-1}(n_2)\beta(n_2) \right\}. \end{aligned}$$

For $\alpha(x) := x$, $\beta(x) := \beta$ ($\beta \in (1, \infty)$) we get the restriction set $\mathbb{N}_{\alpha,\beta,1} = \mathbb{N}_{\alpha,\beta,2} = \left\{ n \in \mathbb{N}^2 : \frac{1}{\beta} \le \frac{n_2}{n_1} \le \beta \right\}$ used in [2, 8, 11].

Let $\beta(x) = \beta$ be a constant function. It is natural that $\mathbb{N}_{\alpha,\beta_1,1} \subset \mathbb{N}_{\alpha,\beta_2,1}$ and $\mathbb{N}_{\alpha,\beta_1,2} \subset \mathbb{N}_{\alpha,\beta_2,2}$ for any $\beta_1 \le \beta_2$.

For $i = 1, 2$ set

$$\mathbb{N}_{\alpha,i} := \{\mathbb{N}_{\alpha,\beta,i} : \beta > 1\}.$$

For a fixed $i \in \{1, 2\}$, we say that $\mathbb{N}_{\alpha,i}$ is weaker than $\mathbb{N}_{\alpha,3-i}$, if for all $L \in \mathbb{N}_{\alpha,i}$, there exists an $\tilde{L} \in \mathbb{N}_{\alpha,3-i}$ such that $L \subset \tilde{L}$. This will be denoted by $\mathbb{N}_{\alpha,i} \prec \mathbb{N}_{\alpha,3-i}$.

If $\mathbb{N}_{\alpha,1} \prec \mathbb{N}_{\alpha,2}$ and $\mathbb{N}_{\alpha,2} \prec \mathbb{N}_{\alpha,1}$, then we say that $\mathbb{N}_{\alpha,1}$ and $\mathbb{N}_{\alpha,2}$ are equivalent and denote this by $\mathbb{N}_{\alpha,1} \sim \mathbb{N}_{\alpha,2}$.

We say that the function α is a cone-like restriction function (CRF), if $\mathbb{N}_{\alpha,1} \sim \mathbb{N}_{\alpha,2}$.

Set $\mathbb{N}_\alpha := \mathbb{N}_{\alpha,1} \cup \mathbb{N}_{\alpha,2}$. We say that the cone-like set $L \in \mathbb{N}_\alpha$ is based on the function α.

We study the a.e. convergence of $(C, 1)$ means $\sigma_n f$ of integrable functions $f \in L^1$, where the convergence is restricted by $n \in L$, $L \in \mathbb{N}_\alpha$ and α is CRF, while $\wedge n \to +\infty$. Analogue question could be asked for the Nörlund logarithmic means (for more details see [6]) or for other means.

The properties of a CRF is given in the following theorem [4]:

Theorem 1. *Function α is a CRF if and only if there exist $\zeta, \gamma_1, \gamma_2 > 1$ such that*

$$\gamma_1 \alpha(x) \le \alpha(\zeta x) \le \gamma_2 \alpha(x)$$

hold for each $x \ge 1$.

In other words, the condition $\gamma_1 \alpha(x) \le \alpha(\zeta x) \le \gamma_2 \alpha(x)$ is very natural, since it is necessary and sufficient in order to have that for all restriction set $L \in \mathbb{N}_{\alpha,1}$ there exists an restriction set $\tilde{L} \in \mathbb{N}_{\alpha,2}$ such that $L \subset \tilde{L}$, and in the same way backwards also.

3. The convergence theorem

Define the maximal operator

$$\sigma_L^* := \sup_{n\in L} |\sigma_n f|.$$

For the maximal operator we prove the following theorem.

Theorem 2. *Let α be CRF, $L \in \mathbb{N}_\alpha$. Then the operator σ_L^* is of weak type* $(1,1)$.

By standard argument we have

Theorem 3. *Let α be CRF, $L \in \mathbb{N}_\alpha$. Then for any $f \in L^1$ the equality*

$$\lim_{\substack{\wedge n\to\infty\\ n\in L}} \sigma_n f = f$$

holds a.e.

We immediately have the theorem of Weisz and Gát [2, 11] as a corollary.

Corollary 1. *Let $f \in L^1$ and $\beta \geq 1$ be a fixed parameter. Then*

$$\lim_{\substack{\wedge n\to\infty\\ \beta^{-1}\leq n_1/n_2\leq\beta}} \sigma_n f = f$$

holds a.e.

Theorem 4. *Let α be CRF, $\beta : [1,+\infty) \to [1,+\infty)$ be a monotone increasing function with property $\lim_{+\infty}\beta = +\infty$, and $\delta : [1,+\infty) \to [0,+\infty)$ be a measurable function with property $\lim_{+\infty}\delta = 0$. Let $L := \mathbb{N}_{\alpha,\beta,1}$ or $L := \mathbb{N}_{\alpha,\beta,2}$. Then there exists a function $f \in L^1 \log^+ L\delta(L)$ such that*

$$\limsup_{\substack{\wedge n\to\infty\\ n\in L}} \sigma_n f = +\infty$$

holds a.e.

Corollary 2. *Let α be CRF, $\beta : [1,+\infty) \to [1,+\infty)$ be a monotone increasing function with property $\beta(1) > 1$, and $L := \mathbb{N}_{\alpha,\beta,1}$ or $L := \mathbb{N}_{\alpha,\beta,2}$. Then*

$$\limsup_{\substack{\wedge n\to\infty\\ n\in L}} \sigma_n f = +\infty$$

holds a.e. for all $f \in L^1$ if and only if the function β is bounded.

For $\alpha(x) = x$ the "divergence part" of this corollary for two-dimensional Walsh-Paley system can be find in [3] and the "convergence part" in [2, 11].

In other words, only two cases are possible:

A cone-like restriction and we have the whole L^1 as convergence space for the two-dimensional Fejér means, or no restriction at all, and we have $L\log^+ L$ as maximal convergence space for the two-dimensional Fejér means. That is, there is no "interim space" between L^1 and $L\log^+ L$.

To prove Theorem 2 we need the following decomposition Lemma of Calderon and Zygmund proved in [4].

Lemma 1. *Let the function* $\phi_j : [1,+\infty) \to [1,+\infty)$ *be monotone increasing and continuous with property* $\lim_{+\infty} \phi_j = +\infty$ $(j = 1,2)$. *Set* $\psi_j := \lfloor \phi_j \rfloor$ $(j = 1,2)$ *(*$\lfloor x \rfloor$ *denotes the lower integral part of* x*).*

Let $f \in L^1$ *and* $\lambda > \|f\|_1$. *Then there exists a sequence of integrable functions* (f_i) *such that*

$$f = \sum_{i=0}^{\infty} f_i,$$

where $\|f_0\|_\infty \le C\lambda$, $\|f_0\|_1 \le C\|f\|_1$ *and* $\operatorname{supp} f_i \subset I_{k^{i,1}}(x_1^i) \times I_{k^{i,2}}(x_2^i) =: J_1^i \times J_2^i$ $(x_1^i, x_2^i \in G)$ *with measures*

$$\mu(I_{k^{i,1}}(x_1^i)) = 2^{-\psi_1(s_i)} \text{ and } \mu(I_{k^{i,2}}(x_2^i)) = 2^{-\psi_2(s_i)}$$

for some $s_i \ge 1$. *Moreover,* $\int_{G^2} f_i = 0$ $(i \ge 1)$, *the dyadic rectangles* $J_1^i \times J_2^i$ *are disjoint, and with the definition* $F := \bigcup_{i=1}^{\infty} (I_{k^{i,1}}(x_1^i) \times I_{k^{i,2}}(x_2^i))$ *we have* $\mu(F) \le C\|f\|_1/\lambda$.

We will also use the lemma of Gát for the one-dimensional Fejér kernel [2].

Lemma 2. *Let* $\tau, A \in \mathbb{N}$. *Then*

$$\int_{I_\tau \setminus I_{\tau+1}} \sup_{n \ge 2^A} |K_n| \le c\sqrt{\frac{2^\tau}{2^A}}.$$

During the proofs C and c will denote constants which may depend only on $\zeta, \gamma_1, \gamma_2$ and could vary at different occurrences.

Proof of Theorem 2: First, we apply Lemma 1 for functions $\psi_1(s) := \lfloor \log_2(s) \rfloor$ and $\psi_2(s) := \lfloor \log_2(\alpha(s)) \rfloor$, where α is CRF. Let $L \in \mathbb{N}_\alpha$. Without loss of generality, $L = \mathbb{N}_{\alpha,\beta,1}$ can be supposed for some $\beta > 1$.

Set $f \in L^1$ and $\operatorname{supp} f \subset J_1 \times J_2$ with measure $\mu(J_i) = 2^{-\psi_i(s)}$ for some $s \ge 1$ $(i = 1,2)$. We can also suppose that the centre of J_1 and J_2 is 0.

Set $k^j := \psi_j(s)$ for $j = 1,2$, that is $J_1 = I_{k^1}$ and $J_2 = I_{k^2}$.

Now, we will show the inequality

$$\int_{\overline{I_{k^1} \times I_{k^2}}} \sup_{n \in L} |\sigma_n f| \le c\|f\|_1. \tag{1}$$

Set $\delta := \zeta^{\log_{\gamma_1} 2\beta+1}$. If $n_1 \leq 2^{\psi_1(s)}/\delta$, then

$$\begin{aligned} n_2 &\leq \beta\alpha(n_1) \leq \beta\alpha(2^{\psi_1(s)}\zeta^{-\log_{\gamma_1} 2\beta-1}) \\ &\leq \beta\frac{1}{\gamma_1^{\log_{\gamma_1} 2\beta+1}}\alpha(2^{\psi_1(s)}) \leq \frac{\alpha(2^{\psi_1(s)})}{2} < 2^{\psi_2(s)}. \end{aligned}$$

$\zeta, \gamma_1, \gamma_2 > 1$ implies $n_1 < 2^{k^1}$ and $n_2 < 2^{k^2}$. In this case the (n,m)-th Fourier coefficients are zeros for $n \leq n_1, m \leq n_2$. More exactly,

$$\hat{f}(n,m) = \int_{G\times G} f(\omega_n \times \omega_m) = \int_{I_{k^1}\times I_{k^2}} f(\omega_n \times \omega_m) = (\omega_n \times \omega_m)\int_{I_{k^1}\times I_{k^2}} f, = 0.$$

This gives $\sigma_n f = 0$.

That is, we could suppose that $n_1 > 2^{\psi_1(s)}/\delta$. This yields that

$$n_2 \geq \frac{\alpha(n_1)}{\beta} \geq \frac{\alpha(2^{\psi_1(s)}/\delta)}{\beta} \geq \frac{1}{\beta\gamma_2^{\log_{\gamma_1} 2\beta+1}}\alpha(2^{\psi_1(s)}) \geq \frac{2^{\psi_2(s)}}{\delta'}.$$

First, we discuss the integral $\int_{\overline{I_{k^1}}\times\overline{I_{k^2}}} \sup_{n\in L} |\sigma_n f|$.

We decompose the sets $\overline{I_{k^i}}$ $(i = 1,2)$ in the following way:

$$\overline{I_{k^i}} = \bigcup_{a=0}^{k^i-1}(I_a \backslash I_{a+1}).$$

We introduce the notation

$$J^{a,b} := (I_a\backslash I_{a+1}) \times (I_b\backslash I_{b+1}) \quad (a = 0,1,..,k^1-1,\ b = 0,1,...,k^2-1).$$

Cosequently,

$$\overline{I_{k^1}} \times \overline{I_{k^2}} = \bigcup_{a=0}^{k^1-1}\bigcup_{b=0}^{k^2-1} J^{a,b}.$$

By the theorem of Fubini and Lemma 2 we have

$$\begin{aligned} \int_{J^{a,b}} \sup_{n\in L} |\int_{J^1\times J^2} f*(K_{n_1}\times K_{n_2})| &\leq \int_{J^1\times J^2} |f| \int_{J^{a,b}} \sup_{n\in L} |K_{n_2}\times K_{n_2}| \\ &\leq \|f\|_1 \int_{J^{a,b}} \sup_{n_1\geq 2^{k^1}/\delta} |K_{n_1}| \sup_{n_2\geq 2^{k^2}/\delta'} |K_{n_2}| \\ &\leq c\|f\|_1\sqrt{\frac{2^{a+b}}{2^{k^1+k^2}}} \end{aligned}$$

and

$$\int_{\overline{I_{k^1}}\times\overline{I_{k^2}}} \sup_{n\in L} |\sigma_n f| \leq \sum_{a=0}^{k^1-1}\sum_{b=0}^{k^2-1}\int_{J^{a,b}} \sup_{n\in L} |\sigma_n f| \leq c\|f\|_1.$$

Second, we discuss the integral $\int_{I_{k^1}\times\overline{I_{k^2}}}\sup_{n\in L}|\sigma_n f|$.

For $r\geq k^1$ set $\epsilon:=\sum_{i=k^1}^{r}\epsilon_i e_i$, where $\epsilon_i\in\{0,1\}$ $(i=k^1,...,r)$ and $e_i:=(0,...,0,1,0,...)$, where the ith coordinate is 1 and the rest are zeros. Then

$$I_{k^1}=\bigcup_{\substack{\epsilon_i\in\{0,1\}\\ i=k^1,...,r}} I_{r+1}(\epsilon).$$

Define the sets $J_\epsilon^{a,b}$ and J_ϵ^{b} for an arbitrary $\epsilon=\sum_{i=k^1}^{r}\epsilon_i e_i$, for each $a=k^1,...,r$ and $b=0,1,...,k^2-1$ by

$$J_\epsilon^{a,b}:=(I_a(\epsilon)\backslash I_{a+1}(\epsilon))\times(I_b\backslash I_{b+1})$$

and

$$J_\epsilon^{b}:=I_{r+1}(\epsilon)\times(I_b\backslash I_{b+1}).$$

We have the following disjoint union

$$I_{k^1}\times\overline{I_{k^2}}=\bigcup_{a=k^1}^{r}\bigcup_{b=0}^{k^2-1}J_\epsilon^{a,b}\bigcup\bigcup_{b=0}^{k^2-1}J_\epsilon^{b}.$$

It is easy to see that, $c\alpha(2^r)\leq n_2\leq C\alpha(2^r)$ for $n\in L$ and $2^r\leq n_1\leq c2^r$. The theorem of Fubini and the decomposition above imply

$$\int_{I_{k^1}\times\overline{I_{k^2}}}\sup_{n\in L}|\sigma_n f|\leq\sum_{r=k^1}^{\infty}\int_{I_{k^1}\times\overline{I_{k^2}}}\sup_{\substack{2^{r-c}\leq n_1\leq 2^{r+c}\\ n\in L}}\left|\int_{I_{k^1}\times I_{k^2}}f*(K_{n_1}\times K_{n_2})\right|$$

$$\leq\sum_{r=k^1}^{\infty}\sum_{\epsilon}\int_{I_{k^1}\times\overline{I_{k^2}}}\sup_{\substack{2^{r-c}\leq n_1\leq 2^{r+c}\\ n\in L}}\left|\int_{I_{r+1}(\epsilon)\times I_{k^2}}f*(K_{n_1}\times K_{n_2})\right|$$

$$\leq\sum_{r=k^1}^{\infty}\sum_{\epsilon}\sum_{a=k^1}^{r}\sum_{b=0}^{k^2-1}\int_{J_\epsilon^{a,b}}\sup_{\substack{2^{r-c}\leq n_1\leq 2^{r+c}\\ n\in L}}\left|\int_{I_{r+1}(\epsilon)\times I_{k^2}}f*(K_{n_1}\times K_{n_2})\right|$$

$$+\sum_{r=k^1}^{\infty}\sum_{\epsilon}\sum_{b=0}^{k^2-1}\int_{J_\epsilon^{b}}\sup_{\substack{2^{r-c}\leq n_1\leq 2^{r+c}\\ n\in L}}\left|\int_{I_{r+1}(\epsilon)\times I_{k^2}}f*(K_{n_1}\times K_{n_2})\right|$$

$$\leq\sum_{r=k^1}^{\infty}\sum_{\epsilon}\sum_{a=k^1}^{r}\sum_{b=0}^{k^2-1}\int_{I_{r+1}(\epsilon)\times I_{k^2}}|f|\int_{J_\epsilon^{a,b}}\sup_{2^{r-c}\leq n_1}|K_{n_1}|\sup_{c\alpha(2^r)\leq n_2}|K_{n_2}|$$

$$+\sum_{r=k^1}^{\infty}\sum_{\epsilon}\sum_{b=0}^{k^2-1}\int_{I_{r+1}(\epsilon)\times I_{k^2}}|f|\int_{J_\epsilon^{b}}\sup_{n_1\leq 2^{r+c}}|K_{n_1}|\sup_{c\alpha(2^r)\leq n_2}|K_{n_2}|.$$

From Lemma 2 we get

$$\begin{aligned}\int_{I_{k^1}\times\overline{I_{k^2}}}\sup_{n\in L}|\sigma_n f| &\le c\sum_{r=k^1}^{\infty}\sum_{\epsilon}\sum_{a=k^1}^{r}\sum_{b=0}^{k^2-1}\int_{I_{r+1}(\epsilon)\times I_{k^2}}|f|\sqrt{\frac{2^{a+b}}{2^r\alpha(2^r)}}\\ &+ c\sum_{r=k^1}^{\infty}\sum_{\epsilon}\sum_{b=0}^{k^2-1}\int_{I_{r+1}(\epsilon)\times I_{k^2}}|f|\sqrt{\frac{2^{b}}{\alpha(2^r)}}\\ &\le c\|f\|_1\sum_{r=k^1}^{\infty}\sqrt{\frac{2^{k^2}}{\alpha(2^r)}}.\end{aligned}$$

Now, we will show that $\sum_{r=k^1}^{\infty}\sqrt{\frac{2^{k^2}}{\alpha(2^r)}}\le c$. To do this, we write for an arbitrary A (we will give more details about A later)

$$\sum_{r=k^1}^{\infty}\sqrt{\frac{1}{\alpha(2^r)}} = \sum_{j=0}^{A-1}\sum_{i=0}^{\infty}\sqrt{\frac{1}{\alpha(2^{k^1+Ai+j})}}.$$

Now, we choose A such that the inequality

$$\sqrt{\alpha(2^{k^1+Ai+j+A})}\ge 2\sqrt{\alpha(2^{k^1+Ai+j})}$$

holds. (We could choose such an A because α is CRF.) By this we have

$$\sum_{r=k^1}^{\infty}\sqrt{\frac{1}{\alpha(2^r)}}\le c\sum_{j=0}^{A-1}\sqrt{\frac{1}{\alpha(2^{k^1+j})}}\le c\sqrt{\frac{1}{\alpha(2^{k^1})}}.$$

By simple consideration $2^{k^2}\le 2\alpha(s)$, $\alpha(2^{k^1})=\alpha(2^{\lfloor\log_2 s\rfloor})\ge\alpha(s/2)\ge c\alpha(s)$ and

$$\sum_{r=k^1}^{\infty}\sqrt{\frac{2^{k^2}}{\alpha(2^r)}}\le c\sqrt{\frac{2^{k^2}}{\alpha(2^{k^1})}}\le c.$$

Third, we discuss the integral $\int_{\overline{I_{k^1}}\times I_{k^2}}\sup_{n\in L}|\sigma_n f|$.

Using the substitutions $t=\alpha(s)$ and $s=\alpha^{-1}(t)$, we write

$$\overline{I_{\psi_1(s)}}\times I_{\psi_2(s)}=\overline{I_{\lfloor\log_2 s\rfloor}}\times I_{\lfloor\log_2\alpha(s)\rfloor}=\overline{I_{\lfloor\log_2\alpha^{-1}(t)\rfloor}}\times I_{\lfloor\log_2 t\rfloor}=\overline{I_{\tilde\psi_2(t)}}\times I_{\tilde\psi_1(t)}.$$

If α is CRF, then α^{-1} is CRF too. To show this we use Theorem 1.

$$\alpha^{-1}(\gamma_1\alpha(x))\le\zeta x\le\alpha^{-1}(\gamma_2\alpha(x)),$$

substituting $y=\alpha(x)$ we have for z big enough that

$$\zeta\alpha^{-1}(y)\le\alpha^{-1}(\gamma_2 y)\le\alpha^{-1}(\gamma_1^z y)\le\zeta^z\alpha^{-1}(y).$$

Set $\tilde\gamma_1:=\zeta$, $\tilde\zeta:=\gamma_1^z$ and $\tilde\gamma_2:=\zeta^z$. If $L\in\mathbb{N}_\alpha$, then $L\subset\tilde L$, where $\tilde L\in\mathbb{N}_{\alpha^{-1}}$.

$$\int_{\overline{I_{\psi_1(s)}\times I_{\psi_2(s)}}} \sup_{n\in L} |\sigma_n f| \le \int_{\overline{I_{\tilde\psi_2(t)}\times I_{\tilde\psi_1(t)}}} \sup_{n\in \tilde L} |\sigma_n f|.$$

The *second* step above gives that

$$\int_{\overline{I_{k^1}\times I_{k^2}}} \sup_{n\in L} |\sigma_n f| \le c\|f\|_1.$$

The operator σ_L^* is of type (∞,∞) it follows from $\|K_n\|_1 \le c$ for all $n\in\mathbb{N}$.

This, inequality (1) and Lemma 1 give by standard argument our theorem. □

By the interpolation lemma of Marcinkiewicz [10] and the fact that the operator σ_L^* is sublinear we immediately have the following corollary.

Corollary 3. *Let α be CRF and $L\in\mathbb{N}_\alpha$. Then the operator σ_L^* is of type (p,p) for all $1<p\le\infty$.*

4. The divergence theorem

The main aim of this section is to prove the divergence Theorem 4.

Let α,β,δ be such a function as given in Theorem 4, and let $L := \mathbb{N}_{\alpha,\beta,1}$ (the case $L := \mathbb{N}_{\alpha,\beta,2}$ can be discussed in the same way, therefore it is left to the reader). We will construct a function $f\in L^1\log^+ L\delta(L)$, such that $\sup_{n\in L}\sigma_n f = +\infty$ almost everywhere. We will use the counterexample function f defined in [4, page 96].

The construction: For $n,a\in\mathbb{N}^2$ we define a set of dyadic rectangles $\mathcal{I}_{n,a}(x)$ $(x\in G^2)$ by

$$\mathcal{I}_{n,a}(x) := \{I_{n_1+j}(x_1)\times I_{n_2+a_2-j}(x_2) :\ j=0,1,...,\wedge a\}.$$

It is easy to calculate that $\bigcap \mathcal{I}_{n,a}(x) = I_{n_1+\wedge a}(x_1)\times I_{n_2+a_2}(x_2)$.

Let $b\in\mathbb{N}^2$ be given, but discussed later. Let a_1 be so large that

$$\gamma_1^{\lfloor a_1 \log_\zeta 2\rfloor} \ge 4$$

and define the sequence $(d_{k,2})$ (with $d_{0,2}=0$) by

$$d_{k,2} := \frac{\lceil \log_2(\alpha(2^{b_1+ka_1}))\rceil}{k},$$

where $\lceil x\rceil$ denotes the upper integral part of x. The sequence $(kd_{k,2})$ is monotone increasing and satisfies the inequality

$$Ca_1 \le kd_{k,2} - (k-1)d_{k-1,2} \le \tilde C a_1$$

with some positive constants $C,\tilde C$ depending on ζ,γ_1 and γ_2 (for more details see [4]).

For $t \in G^2, k \in \mathbb{N}$, we define $d_k = (d_{k,1}, d_{k,2}) := (a_1, d_{k,2})$,

$$\Delta_k = (\Delta_{k,1}, \Delta_{k,2}) := (a_1, kd_{k,2} - (k-1)d_{k-1,2})$$

and we define the sets $J^k_{b,d}(t)$, $\Omega^k_{b,d}(t)$ recursively:

$$J^0_{b,d}(t) := \{t\}, \quad \Omega^0_{b,d}(t) := \bigcup \mathcal{I}_{b,\Delta_1}(t).$$

If the sets $J^j_{b,d}(t)$, $\Omega^j_{b,d}(t)$ are defined for $j < k$, then we consider the set

$$(I_{b_1}(t_1) \times I_{b_2}(t_2)) \backslash \bigcup_{j=0}^{k-1} \Omega^j_{b,d}(t)$$

as the disjoint union of dyadic rectangles of the form $I_{b_1+ka_1}(x_1) \times I_{b_2+kd_{k,2}}(x_2)$. We take from each rectangle an element x as a representative, and the set of this elements be denoted by $J^k_{b,d}(t)$. This means that

$$(I_{b_1}(t_1) \times I_{b_2}(t_2)) \backslash \bigcup_{j=0}^{k-1} \Omega^j_{b,d}(t) = \bigcup_{x \in J^k_{b,d}(t)} (I_{b_1+ka_1}(x_1) \times I_{b_2+kd_{k,2}}(x_2)).$$

Let

$$\Omega^k_{b,d}(t) := \bigcup_{x \in J^k_{b,d}(t)} \bigcup \mathcal{I}_{b+kd_k,\Delta_{k+1}}(x).$$

The construction implies the a.e. equality $I_b(t) = \bigcup_{j=0}^{\infty} \Omega^j_{b,d}(t)$ (for more details see [4, page 98]). By this we get

$$G^2 = \bigcup_{\substack{t_{i,l} \in \{0,1\} \\ i=0,1,\ldots,b_l-1 \\ l=1,2}} \bigcup_{j=0}^{\infty} \Omega^j_{b,d}(t),$$

where $t = (t_1, t_2) = (t_{0,1}e_0 + \ldots + t_{b_1-1,1}e_{b_1-1}, t_{0,2}e_0 + \ldots + t_{b_1-1,2}e_{b_2-1})$. Define the function $f_{b,d} : G^2 \to [0, +\infty)$ by

$$f_{b,d} := \sum_{\substack{t_{i,l} \in \{0,1\} \\ i=0,1,\ldots,b_l-1 \\ l=1,2}} \sum_{k=0}^{\infty} \sum_{y \in J^k_{b,d}(t)} 2^{\wedge \Delta_{k+1}} 1_{I_{b_1+ka_1+\wedge\Delta_{k+1}} \times I_{b_2+(k+1)d_{k+1,2}}(y)}(x).$$

In the paper [4] it is proved that $f_{b,d} \in L^1 \log^+ L$. To prove our main theorem we need the following lemma of Gát [5]:

Lemma 3. *Let $A, t \in \mathbb{N}$, $A > t$. Suppose that $x \in I_t \backslash I_{t+1}$, then*

$$K_{2^A}(x) = \begin{cases} 0, & \text{if } x - x_t e_t \notin I_A, \\ 2^{t-1}, & \text{if } x - x_t e_t \in I_A. \end{cases}$$

If $x \in I_A$*, then* $K_{2^A}(x) = 2^{A-1} + 1/2$.

By the help of this Lemma we will prove the following Lemma.

Lemma 4. *Let* b, d *as given above. Then*

$$\sup_{n \in L} \sigma_n f_{b,d}(y) \geq \frac{1}{64}$$

for almost every $y \in G^2$.

Proof. From the construction it follows that for almost all $y \in G^2$ there exist a unique $t \in G^2$ and $k \in \mathbb{N}$ such that $y \in \Omega^k_{b,d}(t)$ and a unique $u \in J^k_{b,d}(t)$ with $y \in \bigcup \mathcal{I}_{b+kd_k,\Delta_{k+1}}(u)$. Then $y \in I_{b_1+ka_1+j}(u_1) \times I_{b_2+(k+1)d_{k+1,2}-j}(u_2)$ for a $j \in \{0, 1, ..., \wedge\Delta_{k+1}\}$. By the nonnegativity of $f_{b,d}$ and the 2^Ath Fejér kernels (for all $A \in \mathbb{P}$ see Lemma 3) we give the lower bound of $\sup_{n \in L} \sigma_n f_{b,d}(y)$.

$$\begin{aligned}
&\sup_{n \in L} \sigma_n f_{b,d}(y) \geq \\
\geq\ & \int_{G^2} f_{b,d}(x) K_{2^{b_1+ka_1+j-2}}(y_1 - x_1) K_{2^{b_2+(k+1)d_{k+1,2}-j-2}}(y_2 - x_2) d\mu(x) \\
\geq\ & \int_{I_{b_1+ka_1+\wedge\Delta_{k+1}}(u_1) \times I_{b_2+(k+1)d_{k+1,2}}(u_2)} f_{b,d}(x) K_{2^{b_1+ka_1+j-2}}(y_1 - x_1) K_{2^{b_2+(k+1)d_{k+1,2}-j-2}}(y_2 - x_2) d\mu(x) \\
\geq\ & 2^{\wedge\Delta_{k+1}} \frac{2^{b_1+ka_1+j-3+b_2+(k+1)d_{k+1,2}-j-3}}{2^{b_1+ka_1+\wedge\Delta_{k+1}+b_2+(k+1)d_{k+1,2}}} \geq \frac{1}{64}.
\end{aligned}$$

The fact that

$$n = (n_1, n_2) = (2^{b_1+ka_1+j-2}, 2^{b_2+(k+1)d_{k+1,2}-j-2}) \in L,$$

with the condition on $b \in \mathbb{N}^2$

$$\beta(2^{b_1-2}) \geq 2^{b_2} \gamma_2^{(a_1-2)\log_\zeta 2+1} \qquad \text{and } b_2 \geq 2$$

(see [4, page 100] for more details) completes the proof of this Lemma. □

Proof of Theorem 4: Now, we give the counterexample function f. Let (λ_n) be a strictly monotone increasing sequence of natural numbers, such that $\delta(t) \leq \frac{1}{4^n}$ for all $t \geq \lambda_n, n \in \mathbb{N}$. (We note that $\lim_{+\infty} \delta = 0$. Thus, this can be done.) Set $a_1^{(n)}$ such that the inequality

$$2^{a_1^{(n)} \min(1,C)} \geq 2^{a_1^{(n-1)} \max(1,\tilde{C})}, \lambda_n, 2^n$$

is true with the given constants $C, \tilde{C}$. Let $d^{(n)}, b_1^{(n)}$ and $b_2^{(n)}$ be defined as above. Let f be defined by

$$f := \sum_{n=0}^{\infty} 2^n f_n := \sum_{n=0}^{\infty} 2^n f_{b^{(n)},d^{(n)}}.$$

In the article [4, page 101] it is proved that $f \in L^1 \log^+ L\delta(L)$. By the help of Lemma 4 and the nonnegativity of f and 2^Ath Fejér kernels (for all $A \in \mathbb{P}$) we immediately have

$$\sup_{n\in L} \sigma_n f(y) \geq \sup_{n\in L} \sigma_n 2^k f_{b^{(k)},d^{(k)}}(y) \geq \frac{2^k}{64}$$

for almost all $y \in G^2$ and for all $k \in \mathbb{N}$. This completes the proof of this theorem. □

References

[1] G.H. Agaev, N.Ja. Vilenkin, G.M. Dzhafarli, and A.I. Rubinstein, *Multiplicative systems of functions and harmonic analysis on* 0*-dimensional groups,* Izd. ("ELM"), Baku, (1981), (Russian).

[2] G. Gát, Pointwise convergencve of the Cesàro means of double Walsh series, *Anales Univ. Sci. Budapestiensis, Sect. Comp.* **16** (1996), 173-184.

[3] G. Gát, On the divergence of the $(C, 1)$ means of double Walsh series, *Proc. Amer. Math. Soc.* **128 (6)** (1996), 1711-1720.

[4] G. Gát, Pointwise convergence of cone-like restricted two-dimensional $(C, 1)$ means of trigonometric Fourier series, *Journal of Approx. Theory* **149** (2007) 74-102.

[5] G. Gát, On $(C, 1)$ summability of integrable functions with respect to the Walsh-Kaczmarz system, *Studia Mathematica* **130 (2)** (1998) 135-148.

[6] G. Gát, U. Goginava, Uniform and L-convergence of logarithmic means of Walsh-Fourier series, *Acta Math. Sinica, English Series* **22 (2)** (2006), 497-506.

[7] B. Jessen, J. Marcinkiewicz, A. Zygmund, Note on the differentiability of multiple integrals, *Found. Math.* **32** (1935) 217-234.

[8] J. Marcinkiewicz, A. Zygmund, On the summability of double Fourier series, *Found. Math.* **32** (1939) 112-139.

[9] F. Móricz, F. Schipp, W.R. Wade, Cesàro summability of double Walsh-Fourier series, *Trans. Amer. Math. Soc.* **329 (1)** (1992) 131-140.

[10] F. Schipp, W. R. Wade, P. Simon, and J. Pál, *Walsh Series. An Introduction to Dyadic Harmonic Analysis,* Adam Hilger (Bristol-New York 1990).

[11] F. Weisz, Cesàro summability of two-dimensional Walsh-Fourier series, *Trans. Amer. Math. Soc.* **348** (1996) 2169-2181.

[12] F. Weisz, *Summability of multi-dimensional Fourier series and Hardy spaces,* Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

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