ON THE FEJÉR MEANS OF DOUBLE FOURIER SERIES WITH RESPECT TO THE WALSH-KACZMARZ SYSTEM

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ABSTRACT. The main aim of this paper is to prove that the maximal operator $\sigma_0^{\kappa *} := \sup_n |\sigma_{n,n}^{\kappa}|$ of the Fejér means of double Fourier series with respect to the Kaczmarz system is not bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.

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In 1948 Šneider [8] introduced the Walsh-Kaczmarz system and showed that the inequality

$$\limsup_{n\to\infty} \frac{D_n^{\kappa}(x)}{\log n} \geq C > 0$$

holds a.e. In 1974 Schipp [4] and Young [9] proved that the Walsh-Kaczmarz system is a convergence system. Skvortsov in 1981 [7] showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to f for any continuous functions f. Gát [1] proved, for any integrable functions, that the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere to the function and Gát proved that $\|\sigma^{\kappa *}\|_1 \leq C\||f|\|_{H_1}$. Gát's result was extended to the Hardy space by Simon [5], who proved that $\sigma^{\kappa *}$ is of type (H_p, L_p) for $p > 1/2$. Weisz [13] showed that in endpoint case $p = 1/2$ the maximal operator is of weak type $\left(H_{1/2}, L_{1/2}\right)$.

For the two-dimensional Walsh-Kaczmarz-Fourier series Simon proved [6] that the restricted maximal operator $\sigma_{\lambda}^{\kappa *}$ is bounded from the Hardy space H_p to the space L_p for all $p > 1/2$. The main aim of this paper is to prove that the assumption $p > 1/2$ is essential. Moreover, the maximal operator $\sigma_0^{\kappa *} := \sup_n |\sigma_{n,n}^{\kappa}|$ of the Fejér means of double Fourier series with respect to the Walsh-Kaczmarz system is not bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.

Let denote by $\mathbf{Z}_2$ the discrete cyclic group of order 2, the group operation is the modulo 2 addition and every subset is open. The normalized Haar measure on $\mathbf{Z}_2$ is given in the way that the measure of a singleton is 1/2. Let $G := \mathop{\times}\limits_{k=0}^{\infty} \mathbf{Z}_2$, G be called the Walsh group. The elements of G are sequences $x = (x_0, x_1, ..., x_k, ...)$ with $x_k \in \{0,1\}$ $(k \in \mathbf{N})$.

The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure (denoted by μ) and the topology are the product measure and topology. Dyadic intervalls are defined by

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y = (x_0, ..., x_{n-1}, y_n, y_{n+1} ...)\}$$

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for $x \in G, n \in \mathbf{P}$. They form a base for the neighborhoods of G. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \mathbf{N}$.

Let $L_p(G)$ denote the usual Lebesgue spaces on G (with the corresponding norm $\|.\|_p$). The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbf{N}).$$

Let the Walsh-Paley functions be the product functions of the Rademacher functions. Namely, each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0,1\} \ (i \in \mathbf{N}),$$

where only a finite number of n_i's different from zero. Let the order of $n > 0$ be denoted by $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$. Walsh-Paley functions are $\omega_0 = 1$ and for $n \geq 1$

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k}.$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 = 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The set of Walsh-Kaczmarz functions and the set of Walsh-Paley functions is the same in dyadic blocks. Namely,

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{\omega_k : 2^k \leq n < 2^{k+1}\}$$

for all $k \in \mathbf{P}$ and $\kappa_0 = \omega_0$.

V. A. Skvortsov (see [7]) gave a relation between the Walsh-Kaczmarz functions and the Walsh-Paley functions by the transformation $\tau_A : G \to G$ defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, ..., x_1, x_0, x_A, x_{A+1}, ...)$$

for $A \in \mathbf{N}$. By the definition of τ_A, we have

$$\kappa_n(x) = r_{|n|}(x)\omega_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbf{N}, x \in G).$$

The Dirichlet kernels are defined by

$$D_n^{\alpha} := \sum_{k=0}^{n-1} \alpha_k,$$

where $\alpha_n = \omega_n$ or κ_n $(n \in \mathbf{P})$, $D_0^{\alpha} := 0$. The 2^nth Dirichlet kernels have a closed form (see e.g. [3])

$$D_{2^n}^{\omega}(x) = D_{2^n}^{\kappa}(x) = D_{2^n}(x) = \begin{cases} 0, & \text{if } x \notin I_n \\ 2^n, & \text{if } x \in I_n. \end{cases}$$

Next, we introduce some notation with respect to the theory of two-dimensional system. Let the two-dimensional Walsh group be $G \times G$ and the two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, Dirichlet kernels be defined by

$$\begin{aligned}
\hat{f}^{\alpha}(i,j) &:= \int_{G\times G} f\alpha_i\alpha_j d\mu,\\
S^{\alpha}_{M,N}(f;x^1,x^2) &:= \sum_{k=0}^{M-1}\sum_{l=0}^{N-1}\hat{f}^{\alpha}(k,l)\alpha_k(x^1)\alpha_l(x^2),\\
D^{\alpha}_{M,N}(x^1,x^2) &:= D^{\alpha}_M(x^1)D^{\alpha}_N(x^2),
\end{aligned}$$

where $\alpha_n = \omega_n$ or κ_n $(n \in \mathbf{P})$.

The norm (or quasinorm) of the space $L_p(G\times G)$ is defined by

$$\|f\|_p := \left(\int_{G\times G}|f(x^1,x^2)|^p d\mu(x^1,x^2)\right)^{1/p} \qquad (0<p<\infty).$$

Let the space weak-$L_p(G\times G)$ consits of all measurable functions f for which

$$\|f\|_{\text{weak}-L^p(G\times G)} := \sup_{\lambda>0}\lambda\,\text{mes}(|f|>\lambda)^{1/p}<\infty.$$

The dyadic rectangles are of the form

$$I_{n,n}(x^1,x^2) := I_n(x^1)\times I_n(x^2).$$

The σ-algebra generated by the dyadic rectangles $\{I_{n,n}(x^1,x^2) : (x^1,x^2)\in G\times G\}$ is denoted by $F_{n,n}$ $(n\in\mathbf{N})$. The martingale with respect to $(F_{n,n} : n\in\mathbf{N})$ is denoted by $f=(f^{(n,n)} : n\in\mathbf{N})$ (for details see, e. g. [11, 12]).

The maximal function of a martingale f is defined by

$$f^* := \sup_{n\in\mathbf{N}}|f^{(n,n)}|.$$

In the case $f\in L_1(G\times G)$, the maximal function can be given by

$$f^*(x^1,x^2) = \sup_{n\in\mathbf{N}}\frac{1}{\text{mes}(I_{n,n}(x^1,x^2))}\left|\int_{I_{n,n}(x^1,x^2)} f(u^1,u^2)d\mu(u^1,u^2)\right|,$$

where $(x^1,x^2)\in G\times G$. For $0<p<\infty$ the Hardy martingale spaces $H_p(G\times G)$ consits of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

It is easy to show that the sequence $(S_{2^n,2^n}(f) : n\in\mathbf{N})$ is a martingale for all $f\in L_1(G\times G)$.

For a martingale f the Fourier coefficients must be defined in a little bit different way:

$$\hat{f}^{\alpha}(i,j) := \lim_{k\to\infty}\int_{G\times G} f^{(k,k)}(x^1,x^2)\alpha_i(x^1)\alpha_j(x^2)d\mu(x^1,x^2).$$

The Fejér means of order (n,m) of the double Fourier series of a martingale f is defined by

$$\sigma^{\alpha}_{n,m}(f;x^1,x^2) := \frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}S^{\alpha}_{i,j}(f;x^1,x^2),$$

where $\alpha_n := \omega_n$ or κ_n.

The restricted maximal operator of Fejér means are defined by

$$\sigma^{\alpha*}_{\lambda}f(x^1,x^2) := \sup_{2^{-\lambda}<n/m<2^{\lambda}}|\sigma^{\alpha}_{n,m}(f;x^1,x^2)|.$$

The Fejér kernels of the Fourier series are defined by

$$K_n^\alpha(x) := \frac{1}{n}\sum_{k=0}^{n-1} D_k^\alpha(x) \quad (x \in G)$$

and $K_0^\alpha = 0$. For the two-dimesional Walsh-Paley-Fourier series Weisz proved in 1996 [10] for $p > 1/2$ that the restricted maximal operator $\sigma_\lambda^{\omega *}$ is bounded from the Hardy space H_p to the space L^p. Goginava proved in [2] that the assumption $p > 1/2$ is essential. Moreover, the restricted maximal operator $\sigma_0^{\omega *}$ is not bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.

In order to prove the main theorem we need the following lemma [2].

Lemma 1 (Goginava). *Let* $2 < A \in \mathbf{P}$ *and* $n_A := 2^{2A} + 2^{2A-2} + ... + 2^2 + 2^0$. *Then*

$$n_{A-1}|K^\omega_{n_{A-1}}(x)| \geq 2^{2m+2s-3}$$

for $x \in I_{2A}(0, ..., 0, x_{2m} = 1, 0, ..., 0, x_{2s} = 1, x_{2s+1}, ..., x_{2A-1})$, $m = 0, 1, ..., A-3$, $s = m+2, m+3, ..., A-1$.

Theorem 1. *The maximal operator* $\sigma_0^{\kappa *}$ *is not bounded from the Hardy space* $H_{1/2}$ *to the space weak-*$L^{1/2}$.

Proof. Let $10 < A \in \mathbf{P}$ and

$$f_A(x^1, x^2) := (D_{2^{2A+1}}(x^1) - D_{2^{2A}}(x^1))(D_{2^{2A+1}}(x^2) - D_{2^{2A}}(x^2)).$$

It is simple to calculate

$$\hat{f}^\kappa(i,k) = \begin{cases} 1, & \text{if } i,k = 2^{2A}, ..., 2^{2A+1}-1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$S^\kappa_{i,j}(f; x^1, x^2) = \begin{cases} (D_i^\kappa(x^1) - D_{2^{2A}}(x^1))(D_j^\kappa(x^2) - D_{2^{2A}}(x^2)), & \text{if } i,j = 2^{2A}+1, ..., 2^{2A+1}-1, \\ f_A(x^1, x^2), & \text{if } i,j \geq 2^{2A+1}, \\ 0, & \text{otherwise.} \end{cases}$$

We can write the nth Dirichlet kernel with respect to the Walsh-Kaczmarz system in the following form:

$$\begin{aligned} D_n^\kappa(x) &= D_{2^{|n|}}(x) + \sum_{k=2^{|n|}}^{n-1} r_{|k|}(x)\omega_{k-2^{|n|}}(\tau_{|k|}(x)) \\ &= D_{2^{|n|}}(x) + r_{|n|}(x) D^\omega_{n-2^{|n|}}(\tau_{|n|}(x)). \end{aligned}$$

Thus, we have

$$\begin{aligned}
\sigma_0^{\kappa *} f_A(x^1,x^2) &= \sup_{n\in\mathbf{N}} |\sigma^\kappa_{n,n}(f_A;x^1,x^2)| \geq |\sigma^\kappa_{n_A,n_A}(f_A;x^1,x^2)| \\
&= \frac{1}{n_A^2}\left|\sum_{i=0}^{n_A-1}\sum_{j=0}^{n_A-1} S^\kappa_{i,j}(f_A;x^1,x^2)\right| \\
&= \frac{1}{n_A^2}\left|\sum_{i=2^{2A}+1}^{n_A-1}\sum_{j=2^{2A}+1}^{n_A-1} (D_i^\kappa(x^1)-D_{2^{2A}}(x^1))(D_j^\kappa(x^2)-D_{2^{2A}}(x^2))\right| \\
&= \frac{1}{n_A^2}\left|\sum_{i=1}^{n_{A-1}-1}\sum_{j=1}^{n_{A-1}-1} (D_{i+2^{2A}}^\kappa(x^1)-D_{2^{2A}}(x^1))(D_{j+2^{2A}}^\kappa(x^2)-D_{2^{2A}}(x^2))\right| \\
&= \frac{1}{n_A^2}\left|r_{2A}(x^1+x^2)\sum_{i=1}^{n_{A-1}-1}\sum_{j=1}^{n_{A-1}-1} D_i^\omega(\tau_{2A}(x^1))D_j^\omega(\tau_{2A}(x^2))\right| \\
&= \frac{1}{n_A^2}\left|\sum_{i=1}^{n_{A-1}-1} D_i^\omega(\tau_{2A}(x^1))\sum_{j=1}^{n_{A-1}-1} D_j^\omega(\tau_{2A}(x^2))\right| \\
&= \frac{n_{A-1}^2}{n_A^2}|K_{n_{A-1}}(\tau_{2A}(x^1))||K_{n_{A-1}}(\tau_{2A}(x^2))|.
\end{aligned}$$

Define the set $J_{2A}^{m,s}(x)$ by

$$J_{2A}^{m,s}(x) := I_{2A}(x_0,x_1,...,x_{2A-2s-2},x_{2A-2s-1}=1,0,...,0,x_{2A-2m-1}=1,0,...,0).$$

A simple consideration gives

$$G\backslash I_{2A} \supseteq \bigcup_{m=0}^{A-3}\bigcup_{s=m+2}^{A-1}\bigcup_{x\in G} J_{2A}^{m,s},$$

where on the right side we have disjoint union. Set $(x^1,x^2)\in J_{2A}^{m^1,s^1}(x^1)\times J_{2A}^{m^2,s^2}(x^2)$ for $m^1=0,1,...,A-3$, $s^1=m^1+2,m^1+3,...,A-1$, $m^2=0,1,...,m^1$, $s^2=m^2+2,m^2+3,...,A-1$. Then, by Lemma 1 we have

$$\sigma_0^{\kappa *} f_A(x^1,x^2) \geq 2^{2m^1+2s^1+2m^2+2s^2-4A-10}.$$

Thus, we can write

$$\begin{aligned}
& \operatorname{mes}\{(x^1,x^2)\in G\times G : |\sigma_0^{\kappa *} f_A(x^1,x^2)| > 2^{2A-10}\} \\
={}& \operatorname{mes}\{(x^1,x^2)\in (G\backslash I_{2A})\times(G\backslash I_{2A}) : |\sigma_0^{\kappa *} f_A(x^1,x^2)| > 2^{2A-10}\} \\
\geq{}& c\sum_{m^1=[3A/4]}^{A-3}\sum_{s^1=m^1+2}^{A-1}\sum_{m^2=2A-m^1-s^1+1}^{3A/2-m^1/2-s^1/2-1}\sum_{s^2=3A-m^1-s^1-m^2}^{A-1} \\
& \sum_{x_0^1=0}^{1}\cdots\sum_{x^1_{2A-2s^1-2}=0}^{1}\sum_{x_0^2=0}^{1}\cdots\sum_{x^2_{2A-2s^2-2}=0}^{1} \operatorname{mes}\left(J_{2A}^{m^1,s^1}(x^1)\times J_{2A}^{m^2,s^2}(x^2)\right) \\
\geq{}& c\sum_{m^1=[3A/4]}^{A-3}\sum_{s^1=m^1+2}^{A-1}\sum_{m^2=2A-m^1-s^1+1}^{3A/2-m^1/2-s^1/2-1}\sum_{s^2=3A-m^1-s^1-m^2}^{A-1} \frac{2^{2A-2s^1}2^{2A-2s^2}}{2^{4A}} \geq \frac{cA}{2^{3A}}.
\end{aligned}$$

Since, we have

$$f_A^*(x^1,x^2)=\sup_{n\in\mathbf{N}}|S_{2^n,2^n}(f_A;x^1,x^2)|=|f_A(x^1,x^2)|$$

and

$$\|f_A\|_{H_p}=\|f_A^*\|_p=\|D_{2^{2A}}\|_p^2=2^{4A(1-1/p)},$$

we obtain

$$\frac{2^{2A}(\text{mes}\{(x^1,x^2)\in G\times G:|\sigma_0^{\kappa *}f_A(x^1,x^2)|>2^{2A-10}\})^2}{\|f_A\|_{H_{1/2}}}\geq c\frac{2^{2A}A^2}{2^{6A}2^{4A(1-2)}}\geq cA^2\to\infty \text{ as } A\to\infty.$$

This completes the proof of the main theorem. □

References

[1] G. Gát, On $(C,1)$ summability of integrable functions with respect to the Walsh-Kaczmarz system, *Studia Math.* **130,** (1998), 135-148.

[2] U. Goginava, Maximal operators of Fejér means of double Walsh-Fourier series, (2005), (submitted).

[3] F. Schipp, W.R. Wade, P. Simon, and J. Pál, *Walsh Series. An Introduction to Dyadic Harmonic Analysis,* Adam Hilger (Bristol-New York 1990).

[4] F. Schipp, Pointwise convergence of expansions with respect to certain product systems, *Analysis Math.* **2,** (1976), 63-75.

[5] P. Simon, On the Cesàro summability with respect to the Walsh-Kaczmarz system, J. Approx. Theory, 106(2000), 249-261.

[6] P. Simon, Cesàro summability with respect to two-parameter Walsh system , *Monatsh. Math.* **131,** (2000), 321-334.

[7] V.A. Skvortsov, On Fourier series with respect to the Walsh-Kaczmarz system, *Analysis Math.* **7,** (1981), 141-150.

[8] A.A. Šneider, On series with respect to the Walsh functions with monotone coefficients, *Izv. Akad. Nauk SSSR Ser. Math.* **12,** (1948), 179-192.

[9] W.S. Young, On the a.e. convergence of Walsh-Kaczmarz-Fourier series, *Proc. Amer. Math. Soc.* **44,** (1974), 353-358.

[10] F. Weisz, Cesàro summability of two-dimensional Walsh-Fourier series, *Trans. Amer. Math. Soc.* **348,** (1996), 2169-2181.

[11] F. Weisz, *Martingale Hardy spaces and their applications in Fourier analysis,* Springer-Verlang, Berlin, (1994).

[12] F. Weisz, *Summability of multi-dimensional Fourier series and Hardy space*, Kluwer Academic, Dordrecht, (2002).

[13] F. Weisz, ϑ-summability of Fourier series, emph Acta Math. Hungar. **103(1-2),** (2004)), 139-176.

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