On weighted $(0, \ldots, r-2, r)$-interpolation

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Margit Lénárd

Department of Mathematics & Computer Science
Kuwait
E-mail: lenardkuc01.kuniv.edu.kw

1. Preliminaries

- Lagrange interpolation, Hermite interpolation
- Lacunary interpolation, Birkhoff interpolation
  - (0,2)-interpolation

    P. Turán (1955) initiated the study of this problem in order to get an approximate solution to the differential equation

    $$y'' + f \cdot y = 0.$$

    J. Surányi, P. Turán (1955): for even n on the zeros of the integrated Legendre polynomial the problem is regular.

    $$-n(n-1)\int_{-1}^{x} P_{n-1}(t)dt = (1-x^2)P'_{n-1}(x)$$

    J. Balázs, P. Turán (1958): explicit formulae and convergence.

- Weighted (0,2)-interpolation

J. Balázs (1961): For $n \in \mathbb{N}$ let $x_1, \dots, x_n \in [a, b]$ be distinct points (nodes) and let $w \in C^2(a, b)$ be a given function (weight function).

Find a polynomial R_n of minimal degree such that

$$R_n(x_k) = y_k, \quad (wR_n)''(x_k) = y_k'', \qquad (k = 1, \dots, n),$$

where y_k, y_k'' are arbitrary given real numbers.

Choose the nodal points and the weight function w so that the problem is regular (it has a unique solution). In regular case find simple explicit form of R_n in order to prove convergence.

$\Rightarrow$ On $[-1, 1]$, on the zeros of $P_n^{(\alpha)}(x)$, $w(x) = (1 - x^2)^{(\alpha+1)/2}$, $(\alpha > -1)$.

- No polynomial of degree $\leq 2n - 1$ satisfies the requirements.
- If n is even, then under the condition $R_n(0) = \sum_{k=1}^{n} y_k l_k^2(0)$ the problem is regular (if n is odd, the uniqueness fails)
- Explicit form of this polynomial, convergence theorem.

Weighted $(0,2)$-interpolation with additional Balázs-type condition on the zeros of the classical orthogonal polynomials

$\Rightarrow$ J. Prasad (1967), L. Szili (1985)
$(-\infty,\infty)$, Hermite polynomials $H_n(x)$, $w(x)=e^{-x^2/2}$

$\Rightarrow$ J. Prasad (1967)
$[0,\infty)$, Laguerre polynomials $L_n^{(\alpha)}(x)$, $w(x)=e^{-x/2}x^{(\alpha+1)/2}$

$\Rightarrow$ J. Prasad (1970)
$[-1,1]$, Jacobi polynomials $P_n^{(\alpha,-\alpha)}(x)$,
$w(x)=(1-x)^{(\alpha+1)/2}(1+x)^{(1-\alpha)/2}$

$\Rightarrow$ I. Joó, L. Szili (1995)
$[-1,1]$, Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$,
$w(x)=(1-x)^{(\alpha+1/2)}(1+x)^{(\beta+1)/2}$

$\Rightarrow$ L. Szili (1993), (1996)
on the roots of all classical orthogonal polynomials

Weighted (0,2)-interpolation with additional interpolation-type conditions on the zeros of the classical orthogonal polynomials

⇒ S. A. N. Eneduanya (1985)
extended Tchebycheff nodes of first kind

⇒ P. Bajpai (1994)
extended Tchebycheff nodes of second kind

⇒ L. Szili (1994)
extended Jacobi nodes

⇒ J. Balázs (1998)
on the zeros of the classical orthogonal polynomials with initial type condition

⇒ M. L. (2004)
weighted (0,2)-interpolation with additional Hermite-type conditions

• Weighted (0,1,3)-interpolation

For $n \in \mathbb{N}$ let $x_1, \ldots, x_n \in [a, b]$ be distinct points (nodes) and let $w \in C^2(a, b)$ be a given function (weight function).

Find a polynomial R_n of minimal degree such that

$$R_n(x_i) = y_i, \quad R_n'(x_i) = y_i', \quad (w^2 R_n)'''(x_i) = y_i''',$$

for $i = 1, \ldots, n$, where y_i, y_i', y_i''' are arbitrary given real numbers.

• With additional Balázs-type condition

⇒ K. K. Mathur, R. B. Saxena (1993) on the zeros of Hermite polynomials

⇒ A. Krebsz (2004) on the zeros of classical orthogonal polynomials

• With additional interpolatory condition

⇒ P. Bajpai (1994) on the zeros of extended Chebyshev polynomials

⇒ S. Datta, P. Mathur (2001) on the zeros of Hermite polynomials

• Without any additional condition

⇒ A. Krebsz, M. L. (2004)

2. Weighted (0,...,r-2,r)-interpolation

For $n \in \mathbb{N}$ let $x_1, \ldots, x_n \in [a, b]$ be distinct points (nodes) and let $w \in C^r(a, b)$ be a given function (weight function) and $r \geq 2$ fixed integer.

Find a minimal degree polynomial S_n such that

$$S_n^{(j)}(x_i) = y_i^{(j)}, \qquad \left(w^{r-1} S_n\right)^{(r)}(x_i) = y_i^{(r)}$$

for $i = 1, \ldots, n, \quad j = 0, \ldots, r-2$, where $y_i^{(j)}$ are arbitrary given real numbers.

- $r = 2$ weighted $(0, 2)$-interpolation
- $r = 3$ weighted $(0, 1, 3)$-interpolation

In general the problem is not regular $\quad \Rightarrow \quad$ additional conditions are needed.

In what follows, let

$$p_n(x) = c(x - x_1)(x - x_2)\cdots(x - x_n)$$

be defined on the nodal points, $n \geq 2$, and

$$\ell_k(x) = \frac{p_n(x)}{p_n'(x_k)(x - x_k)} \qquad (k = 1, \ldots, n)$$

be the fundamental polynomials of Lagrange interpolation.

Let $w \in C^r(a, b)$ be a weight function,
and let $v(x)$ be a polynomial associated with the additional conditions,

$$w(x_i) \neq 0, \quad v(x_i) \neq 0 \qquad (i = 1, \ldots, n).$$

Lemma. *If* $q = q_1 q_2$, $w = w_1 w_2$, *and*

$$(q_1 w_1 p_n)''(x_i) = 0 \quad \& \quad (q_2 w_2)'(x_i) = 0 \quad (i = 1, \ldots, n)$$

then

$$\left((qwp_n)^{r-1} Q\right)^{(r)}(x_i) = r!(qwp_n')^{r-1}(x_i) Q'(x_i) \quad (i = 1, \ldots, n)$$

for any $Q \in C^r(a, b)$.

The classical orthogonal polynomials satisfy the homogeneous differential equation

$$(wy)'' + f \cdot (wy) = 0$$

with an appropriate weight function w and function f.

The fundamental polynomials

$\Rightarrow A_{j,k}(x) \quad (k = 1, \dots, n,\ j = 0, \dots, r-2)$

$$
\begin{aligned}
A_{j,k}^{(l)}(x_i) &= \delta_{jl}\delta_{ik}, \qquad (i, k = 1, \dots, n;\ \ j, l = 0, \dots, r-2) \\
\left(w^{r-1}A_{j,k}\right)^{(r)}(x_i) &= 0, \qquad (i = 1, \dots, n).
\end{aligned}
$$

$\Rightarrow A_{r,k}(x) \quad (k = 1, \dots, n)$

$$
\begin{aligned}
A_{r,k}^{(l)}(x_i) &= 0, \qquad (i, k = 1, \dots, n;\ \ l = 0, \dots, r-2) \\
\left(w^{r-1}A_{r,k}\right)^{(r)}(x_i) &= \delta_{ik}, \qquad (i = 1, \dots, n).
\end{aligned}
$$

$$A_{j,k}(x) = \frac{1}{j!v(x_k)}\left\{v(x)(x-x_k)^j\ \ell_k^r(x) + (qp_n)^{r-1}(x)Q_{j,k}(x) + \sum_{l=j+1}^{r-2} d_{j,k}^{[l]} A_{l,k}(x)\right\}$$

where $\quad d_{j,k}^{[l]} = -\binom{l}{j} j!(v\ell_k^r)^{(l-j)}(x_k),$

$$Q_{j,k}(x) = \frac{1}{{p'_n}^{r-1}(x_k)} \int_{x_0}^{x} \frac{v(t)}{q^{r-1}(t)} \left[\frac{q_{j,k}(t)}{(t-x_k)^{r-j-1}} + a_{j,k}\ell_k(t) + b_{j,k}p_n(t)\right] dt + c_{j,k}$$

$$q_{j,k}(x) = \ell_k(x)\left[\ell'_k(x_k) + \sum_{l=1}^{r-j-2} \gamma_{l,k}(x-x_k)^l\right] - \ell'_k(x),$$

$$\gamma_{s,k} = \frac{1}{s!}\left\{\ell_k^{(s+1)}(x_k) - \ell'_k(x_k)\ell_k^{(s)}(x_k) - \sum_{l=1}^{s-1} \gamma_{l,k}\binom{s}{l} l!\ell_k^{(s-l)}(x_k)\right\},$$

$$a_{j,k} = -\frac{\left(w^{r-1}v\ell_k^r\right)^{(r-j)}(x_k)}{(r-j)!w^{r-1}(x_k)v(x_k)} - \frac{q_{j,k}^{(r-j-1)}(x_k)}{(r-j-1)!},$$

furthermore

$$A_{r,k}(x) = \beta_{r,k}\, p_n^{r-1}(x) \int_{x_0}^{x} v(t)\big[\ell_k(t) + b_{r,k} p_n(t)\big]\, dt + c_{r,k},$$

where

$$\beta_{r,k} = \frac{1}{r! p_n'^{r-1}(x_k) w^{r-1}(x_k) v(x_k)}.$$

- Find $b_{j,k}$, $c_{j,k}$, such that the polynomials $A_{j,k}$ are of minimal degree and they fulfil additional interpolational conditions.
- Characterize the nodes and the weight function w, for which the problem is regular.

Examples

• The additional conditions: derivatives are prescribed at x_0 up to the order m.

The problem is regular.

$v(x) = (x - x_0)^{m-1}$,
$b_{j,k} = 0$ and $c_{j,k}$ are determined from the conditions $A_{j,k}^{(m)}(x_0) = 0$.
$\Rightarrow$ P. Mathur, S. Datta (2001) $\quad m = r - 1$

•• The additional conditions: derivatives are prescribed at x_0 up to the order $r - 2$ & the weighted r-th derivative.

$$\textit{The problem is regular} \Leftrightarrow \left((wqp_n)^{r-1}\right)^{(r)}(x_0) \neq 0.$$

$v(x) = (x - x_0)^{r-1}$,
$b_{j,k} = 0$ and $c_{j,k}$ are determined from the conditions $\left(w^{r-1}A_{j,k}\right)^{(r)}(x_0) = 0$.

••• Pál-type weighted $(0, \ldots, r-2)$-interpolation with Hermite-type interpolation

• $\bar{x}_1, \ldots, \bar{x}_n$ distinct zeros of $q(x) = p_n(x) \Leftarrow$ Hermite-type conditions
$x_1, \ldots, x_{n-1}$ the zeros of $p_{n-1}(x) = p_n'(x) \Leftarrow$ weighted interpolation

• On the zeros of $p_n(x) = H_n(x)$

$\Rightarrow$ R. Srivastava, K. K. Mathur (1996) with Balázs-type condition for $R_n'(0)$

$\Rightarrow$ P. Mathur, A. Datta (2001) with interpolatory condition for $R_n(0)$ and $R_n'(0)$

$\Rightarrow$ P. Mathur, A. K. Srivastava (1997) $r = 3$, with Balázs-type condition for $R_n''(0)$

• On the zeros of classical orthogonal polynomials
$\Rightarrow$ M. L. (2004) $r = 2$, with arbitrary interpolatory conditions

•• $x_1, \ldots, x_n$ distinct zeros of $p_n(x) \Leftarrow$ weighted interpolation
$\bar{x}_1, \ldots, \bar{x}_{n-1}$ the zeros of $q(x) = p_n'(x) \Leftarrow$ Hermite-type conditions

• On the zeros of $p_n(x) = H_n(x)$
$\Rightarrow$ P. Mathur, A. Datta (2001) $r = 2$, with interpolatory condition for $R_n(0)$ and $R_n'(0)$

• On the zeros of classical orthogonal polynomials
$\Rightarrow$ M. L. (2004) $r = 2$, with arbitrary interpolatory condition