ON THE MARCINKIEWICZ-FEJÉR MEANS OF DOUBLE FOURIER SERIES WITH RESPECT TO THE WALSH-KACZMARZ SYSTEM

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ABSTRACT. The main aim of this paper is to prove that the maximal operator of Marcinkiewicz-Fejér means of double Fourier series with respect to the Walsh-Kaczmarz system is bounded from the dyadic Hardy-Lorentz space H_{pq} into Lorentz space L_{pq} for every $p > 2/3$ and $0 < q \leq \infty$. As a consequence, we obtain the a.e. convergence of Marcinkiewicz-Fejér means of double Fourier series with respect to the Walsh-Kaczmarz system. That is, $\sigma_n(f, x^1, x^2) \to f(x^1, x^2)$ a.e. as $n \to \infty$.

1. Introduction

In 1939 for the two-dimensional trigonometric Fourier partial sums $S_{j,j}(f)$ Marcinkiewicz [12] has proved for $f \in L \log L([0, 2\pi]^2)$ that the means

$$\sigma_n f = \frac{1}{n} \sum_{j=1}^{n} S_{j,j}(f)$$

converge a.e. to f as $n \to \infty$. Zhizhiashvili [22] improved this result for $f \in L([0, 2\pi]^2)$. Dyachenko [2] proved this result for dimensions greater than 2.

For the two-dimensional Walsh-Fourier series Weisz [19] proved that the maximal operator

$$\sigma^* f = \sup_{n \geq 1} \frac{1}{n} \left| \sum_{j=1}^{n} S_{j,j}(f) \right|$$

is bounded from the two-dimensional dyadic martingale Hardy-Lorentz space H_{pq} to the Lorentz space L_{pq} for $p > 2/3$ and $0 < q \leq \infty$ and is of weak type (1,1). Goginava [9] generalized the theorem of Weisz for d-dimensional Walsh-Fourier series. The a. e. convergence of the arithmetic means of quadratical partial sums of double Vilenkin-Fourier series was studied by Gát [4].

It is well-known that $\varepsilon_n |D_n(x)| \to 0$ as $n \to \infty$ [13] for every $x \in K$. Dirichlet kernels in the Kaczmarz ordering do not satisfy this property [17]. Namely, in 1948 Šneider [17] introduced the Walsh-Kaczmarz system and showed that the inequality

$$\limsup_{n \to \infty} \frac{D_n^{\kappa}(x)}{\log n} \geq C > 0$$

holds a.e. Consequently, it is harder to obtain pointwise convergence results for Walsh-Kaczmarz-Fourier series than for Walsh-Fourier series. In 1974 Schipp [14] and Young [18] proved that the Walsh-Kaczmarz system is a convergence system. Skvorcov in 1981 [16]

The author is supported by the Hungarian National Foundation for Scientific Research (OTKA), grant no. T 048780.

2000 Mathematics Subject Classification. 42C10.

Key words and phrases: Walsh-Kaczmarz system, Marcinkiewicz-Fejér means, Maximal operator.

showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to f for any continuous functions f. Gát [3] proved, for any integrable functions, that the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere to the function itself. Gát's Theorem was extended by Simon [15] to H_{pq} spaces, namely that the maximal operator of Fejér means of one-dimensional Fourier series is bounded from Hardy-Lorentz space into Lorentz space for $p > 1/2$ and $0 < q \leq \infty$. He also showed the (H_{pq}, L_{pq})-boundedness for every $0 < p \leq 1$, while the maximal operator of the Fejér means is considered only of order 2^n.

In [5] it was proved that the maximal operator of Marcinkiewicz-Fejér means of double Fourier series with respect to the Walsh-Kaczmarz system is bounded from the dyadic Hardy-Lorentz space H_{pq} into Lorentz L_{pq} space for every $p > 1/2$ and $0 < q \leq \infty$ provided that the supremum in the maximal operator is taken over special indices. As a consequence, we obtain a.e. convergence of Marcinkiewicz-Fejér means of double Fourier series for special indices with respect to the Walsh-Kaczmarz system. That is $\sigma_{2^n}(f, x^1, x^2) \to f(x^1, x^2)$ a.e. as $n \to \infty$.

In this present paper we generalize this result for maximal operator σ^* and prove that the maximal operator of Marcinkiewicz-Fejér means of double Fourier series with respect to the Walsh-Kaczmarz system is bounded from the dyadic Hardy-Lorentz space H_{pq} into Lorentz space L_{pq} for every $p > 2/3$ and $0 < q \leq \infty$ (the supremum in the maximal operator is taken over all indices.). As a consequence, we obtain the a.e. convergence of Marcinkiewicz-Fejér means of double Fourier series with respect to the Walsh-Kaczmarz system, that is $\sigma_n(f, x^1, x^2) \to f(x^1, x^2)$ a.e. as $n \to \infty$.

2. Definitions and Notation

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Denote Z_2 the discrete cyclic group of order 2, that is $Z_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is $1/2$. Let K be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of K are of the form $x = (x_0, x_1, ..., x_k, ...)$ with $x_k \in \{0, 1\}$ $(k \in \mathbf{N})$. The group operation on K is the coordinate-wise addition, the measure (denote by μ) and the topology are the product measure and topology. The compact Abelian group K is called the Walsh group. A base for the neighborhoods of K can be given in the following way:

$$I_0(x) := K, \ \ I_n(x) := I_n(x_0, ..., x_{n-1}) := \{y \in K : \ y = (x_0, ..., x_{n-1}, y_n, y_{n+1}, ...)\},$$

$$(x \in K, n \in \mathbf{N}).$$

These sets are called dyadic intervals. Let $0 = (0 : i \in \mathbf{N}) \in K$ denote the null element of K, $I_n := I_n(0)$ $(n \in \mathbf{N})$. Set $e_n := (0, ..., 0, 1, 0, ...) \in K$, the nth coordinate of which is 1 and the rest are zeros $(n \in \mathbf{N})$.

For $k \in \mathbf{N}$ and $x \in K$ denote

$$r_k(x) := (-1)^{x_k}$$

the kth Rademacher function. If $n \in \mathbf{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0, 1\}$ $(i \in \mathbf{N})$, i. e. n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum\limits_{k=0}^{|n|-1} n_k x_k} \qquad (x \in K, n \in \mathbf{P}).$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 := 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k}.$$

For $A \in \mathbf{N}$ define the transformation $\tau_A : K \to K$ by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, ..., x_0, x_A, x_{A+1}, ...).$$

By the definition of τ_A (see [16]), we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbf{N}, x \in K).$$

The σ-algebra generated by the dyadic 2-dimensional cube I_k^2 of length $2^{-k} \times 2^{-k}$ will be denoted by F_k $(k \in \mathbf{N})$. Let $L^p(K^2)$ denote the usual Lebesgue spaces on K^2 with the corresponding norm $\|.\|_p$.

The Lorentz space $L_{pq}(K^2)$, $0 < p, q \leq \infty$ with norms or quasi-norms $\|\cdot\|_{pq}$ is defined in the usual way (For details see e.g. Weisz [20]).

Denote by $f = (f_n, n \in \mathbf{N})$ a one-parameter martingale with respect to $(F_n, n \in \mathbf{N})$. The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbf{N}} |f_n|.$$

For $0 < p, q \leq \infty$ the Hardy-Lorentz martingale space $H_{p,q}(K^2)$ consists all martingales for which

$$\|f\|_{H_{p,q}} = \|f^*\|_{p,q} < \infty.$$

A bounded measurable function a is a p-atom, if there exists a dyadic 2-dimensional cube I^2, such that

a) $\int\limits_{I^2} a d\mu = 0$;

b) $\|a\|_\infty \leq \mu(I^2)^{-1/p}$;

c) supp $a \subset I^2$.

An operator T which maps the set of martingale into the collection of measurable functions will be called p-quasi-local, if there exists a constant $C_p > 0$ such that for every p-atom a

$$\int\limits_{K^2 \backslash I^2} |Ta|^p \leq C_p < \infty,$$

where I^2 is the support of the atom a.

The Dirichlet kernels are defined by

$$D_n^{\alpha}(x) := \sum_{k=0}^{n-1} \alpha_k(x),$$

where $\alpha_k = w_k$ or κ_k. Recall that

$$D_{2^n}(x) := D_{2^n}^{w}(x) = D_{2^n}^{\kappa}(x) = \begin{cases} 2^n, & \text{if } x \in I_n(0), \\ 0, & \text{if } x \notin I_n(0). \end{cases} \tag{1}$$

The Fejér kernels are defined as follows

$$K_n^{\alpha}(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k^{\alpha}(x).$$

The Kroneker product $(\alpha_{m,n} : n, m \in \mathbf{N})$ of two Walsh(Kaczmarz) system is said to be the two-dimensional Walsh(Kaczmarz) system. Thus,

$$\alpha_{m,n}(x, y) = \alpha_n(x)\, \alpha_m(y).$$

If $f \in L(K^2)$, then the number $\hat{f}^{\alpha}(n, m) := \int\limits_{K^2} f \alpha_{m,n}$ $(n, m \in \mathbf{N})$ is said to be the (n, m)th Walsh-Fourier coefficient of f. We can extend this definition to martingales in the usual way (see Weisz [20, 21]). Denote by $S_{n,m}^{\alpha}$ the (n, m)th partial sum of the Walsh-Fourier series of a martingale f, namely,

$$S_{n,m}^{\alpha}(f, x) := \sum_{k=0}^{n-1} \sum_{i=0}^{m-1} \hat{f}^{\alpha}(k, i) \alpha_{k,i}(x).$$

The Marcinkiewicz-Fejér means of a martingale f are defined by

$$\sigma_n^{\alpha} f\left(x^1, x^2\right) := \frac{1}{n} \sum_{k=0}^{n-1} S_{k,k}^{\alpha}(f, x^1, x^2).$$

The 2-dimensional Dirichlet kernels and Marcinkiewicz-Fejér kernels are defined by

$$D_{k,l}^{\alpha}(x^1, x^2) := D_k^{\alpha}(x^1) D_l^{\alpha}(x^2), \quad K_n^{\alpha}(x^1, x^2) := \frac{1}{n} \sum_{k=0}^{n-1} D_{k,k}^{\alpha}(x^1, x^2).$$

Let

$$K_{a,b}^{\alpha} := \sum_{j=a}^{a+b-1} D_{j,j}^{\alpha},$$

and $n^{(s)} := \sum\limits_{i=s}^{\infty} n_i 2^i$ $(n, s \in \mathbf{N})$. By simple calculations we get

$$n K_n^{\alpha} = \sum_{s=0}^{|n|} n_s K_{n^{(s+1)}, 2^s}^{\alpha}.$$

For the martingale f we consider the maximal operators

$$\sigma^* f(x^1,x^2) = \sup_n |\sigma_n^\kappa(f,x^1,x^2)|$$

and

$$\sigma^\# f(x^1,x^2) = \sup_A |\sigma_{2^A}^\kappa(f,x^1,x^2)|.$$

3. Formulation of Main Results

Theorem 1. *Let* $f \in H_{p,q}(K^2)$, $p > \frac{2}{3}$, $0 < q \le \infty$. *Then*

$$\|\sigma^* f\|_{p,q} \le C\,(p,q)\,\|f\|_{H_{p,q}}.$$

Corollary 1. *Let* $f \in L_1(K^2)$. *Then*

$$\|\sigma^* f\|_{weak\_L_1} \le C\,\|f\|_{L_1}.$$

Corollary 2 (Nagy [11])**.** *Let* $f \in L_1(K^2)$. *Then*

$$\sigma_n(f,x^1,x^2) \to f(x^1,x^2) \quad a.e. \quad as \quad n \to \infty.$$

4. Auxiliary Propositions

We shall need the following lemmas (see [3, 4, 10, 5, 19]).

Lemma 1 (Weisz)**.** *Suppose that the operator* T *is sublinear and* p*-quasi-local for each* $0 < p_0 < p \le 1$. *If* T *is bounded from* $L_\infty(K^2)$ *to* $L_\infty(K^2)$, *then*

$$\|Tf\|_{pq} \le C\,(p,q)\,\|f\|_{pq} \qquad \left(f \in H_{pq}\left(K^2\right)\right)$$

for every $0 < p_0 < p < \infty$ *and* $0 < q \le \infty$. *In particular for* $f \in L_1(K^2)$, *holds*

$$\|Tf\|_{1,\infty} = \|Tf\|_{weak\_L_1(K^2)} \le C\,\|f\|_1.$$

Lemma 2 (Gát)**.** *Suppose that* $s,a,n \in \mathbf{N}$ *and* $x \in I_a \backslash I_{a+1}$. *If* $s \le a \le |n|$, *then* $|K^w_{n^{(s+1)},2^s}(x)| \le c2^{s+a}$, *while if* $a < s \le |n|$, *then*

$$K^w_{n^{(s+1)},2^s}(x) = \begin{cases} 0 & \text{if } x - x_a e_a \notin I_s, \\ w_{n^{(s+1)}}(x) 2^{s+a-1} & \text{if } x - x_a e_a \in I_s. \end{cases}$$

Lemma 3 (Nagy)**.** *Suppose that* $s,a,n \in \mathbf{N}$, $(x^1,x^2) \in I_{|n|+1} \times (I_a \backslash I_{a+1})$. *If* $s \le a \le |n|$, *then*

$$\left| K^w_{n^{(s+1)},2^s}(x^1,x^2) \right| \le c2^{s+a}(n^{(s+1)} + 2^s).$$

If $a < s \le |n|$, *then we have*

$$K^w_{n^{(s+1)},2^s}(x^1,x^2) =$$

$$\begin{cases} 0 & \text{if } \exists l, a < a+l < s, x^2 - x_a^2 e_a - e_{a+l} \notin I_s, x_{a+l}^2 \ne 0, \\ w_{n^{(s+1)}}(x^2) 2^{2a+s+l-2} & \text{if } \exists l, a < a+l < s, x^2 - x_a^2 e_a - e_{a+l} \in I_s, x_{a+l}^2 \ne 0, \\ w_{n^{(s+1)}}(x^2) 2^{a-2} n(s,a) & \text{if } x^2 - x_a^2 e_a \in I_s, \end{cases}$$

where $n(s,a) = [n^{(s+1)}2^{s+1} - 2^a(2^s - 2^{a-1} + \frac{1}{2}) - 2^s(2^s - 2)]$.

Lemma 4 (Nagy)**.** *Let* $A,s,l\in\mathbf{N}, s\le l<A, (x^1,x^2)\in(I_a\backslash I_{a+1})\times(I_b\backslash I_b)$. *Suppose that* $s\le a\le b\le A$, *then*

$$K^w_{n^{(s+1)},2^s}(x^1,x^2)\le c2^{a+b+s}.$$

If $a\le b<s\le A$ *then*

$$K^w_{n^{(s+1)},2^s}(x^1,x^2)=$$

$$\begin{cases} 0 \text{ if} \exists i\in B_1, x^1_i\ne x^2_i, \\ 0 \text{ if} \forall i\in B_1, x^1_i=x^2_i, \exists m\in B_2, x^1-e_a-e_m\notin I_{b+1}, x^1_m=1, \\ -w_{n^{(s+1)}}(x^1+x^2)2^{s+a+m-2} \text{ if} \forall i\in B_1, x^1_i=x^2_i, \exists m\in B_2, x^1-e_a-e_m\in I_{b+1}, x^1_m=1, \\ w_{n^{(s+1)}}(x^1+x^2)2^{s+2a-1} \text{ if } x^1-e_a\in I_{b+1}(\forall i\in B_1, x^1_i=x^2_i), \end{cases}$$

where $B_1=\{b+1,...,s-1\}, B_2=\{a+1,...,b\}$.

Lemma 5 (Nagy)**.** *For* $k\in\mathbf{P}$ *and* $(x^1,x^2)\in K^2$ *holds*

$$\begin{aligned} k\quad & K^\kappa_k(x^1,x^2)=2^{|k|}K^\kappa_{2^{|k|}}(x^1,x^2)\\ +\quad & (k-2^{|k|})(D_{2^{|k|},2^{|k|}}(x^1,x^2)+D_{2^{|k|}}(x^1)r_{|k|}(x^2)K^w_{k-2^{|k|}}(\tau_{|k|}(x^2))\\ +\quad & D_{2^{|k|}}(x^2)r_{|k|}(x^1)K^w_{k-2^{|k|}}(\tau_{|k|}(x^1))\\ +\quad & r_{|k|}(x^1+x^2)K^w_{k-2^{|k|}}(\tau_{|k|}(x^1),\tau_{|k|}(x^2))). \end{aligned}$$

Corollary 3. *We have*

$$\sup_k\int\limits_{K^2}\left|K^\kappa_k(x^1,x^2)\right|dx^1dx^2<\infty.$$

Proof. Since [7, 8] $\sup\limits_k\int\limits_{K^2}|K^w_k(x^1,x^2)|\,dx^1dx^2<\infty$ and $\sup\limits_k\int\limits_K|K^w_k(x)|\,dx<\infty$ we obtain the proof of Corollary from Lemma 5.

We need the following lemma proved in [5].

Lemma 6 (Gát, Goginava, Nagy)**.** *The operator* $\sigma^\#$ *is* p*-quasi-local for each* $1/2<p\le 1$.

Lemma 7. *Let* $(x^1,x^2)\in(I_a\backslash I_{a+1})\times(I_b\backslash I_{b+1})$, *where* $a<s\le b$. *Then*

$$\left|K^w_{n^{(s+1)},2^s}\left(x^1,x^2\right)\right|\le c2^{a+b+s}\sum_{m=a}^{s-1}\mathbf{1}_{I_s(0,\dots,0,x_a=1,0,\dots,0,x_m=1,0,\dots,0)}\left(x^1\right).$$

Proof. We can write

$$K^w_{n^{(s+1)},2^s}\left(x^1,x^2\right)=\sum_{j=0}^{2^s-1}D^w_{j+n^{(s+1)}}\left(x^1\right)D^w_{j+n^{(s+1)}}\left(x^2\right).$$

It is evident that

$$D^w_{j+n^{(s+1)}}\left(x^1\right)=D^w_{n^{(s+1)}}\left(x^1\right)+w_{n^{(s+1)}}\left(x^1\right)D^w_j\left(x^1\right)=w_{n^{(s+1)}}\left(x^1\right)D^w_j\left(x^1\right),$$

$$D^w_{j+n^{(s+1)}}\left(x^2\right)=D^w_{n^{(s+1)}}\left(x^2\right)+w_{n^{(s+1)}}\left(x^2\right)D^w_j\left(x^2\right).$$

Then we obtain

$$K^w_{n^{(s+1)},2^s}\left(x^1,x^2\right)=w_{n^{(s+1)}}\left(x^1\right)D^w_{n^{(s+1)}}\left(x^2\right)2^sK^w_{2^s}\left(x^1\right)$$

$$+w_{n^{(s+1)}}\left(x^1+x^2\right)\sum_{j=0}^{2^s-1}D^w_j\left(x^1\right)D^w_j\left(x^2\right)=I+II. \tag{2}$$

Using Lemma 2, we have the following estimation

$$|I| \leq c2^{a+b+s}\mathbf{1}_{I_s(0,\ldots,0,x_a=1,0,\ldots,0)}\left(x^1\right). \tag{3}$$

To estimate II we write

$$|II| \leq \left|\sum_{j=0}^{2^s-1} jD_j\left(x^1\right)\right|. \tag{4}$$

Since

$$D_j^w\left(x^1\right) = w_j\left(x^1\right)\left(\sum_{k=0}^{a-1} j_k 2^k - j_a 2^a\right), (x^1 \in I_a \backslash I_{a+1}),$$

we have

$$\sum_{j=0}^{2^s-1} jD_j^w\left(x^1\right) = \sum_{j=0}^{2^s-1} jw_j\left(x^1\right)\left(\sum_{k=0}^{a-1} j_k 2^k - j_a 2^a\right).$$

If there exist coordinates $x_{a+l} \neq 0$ and $x_{a+q} \neq 0$ $(a < a+l < a+q < s)$, then we have

$$\sum_{j=0}^{2^s-1} jD_j^w\left(x^1\right) = \sum_{j_i=0, i\neq a+l, a+q}^{1} \sum_{j_{a+l}=0}^{1} \sum_{j_{a+q}=0}^{1} \left(j_{a+l}2^{a+l} + j_{a+q}2^{a+q} + \Phi_1\right) \times (-1)^{j_{a+l}+j_{a+q}}\Phi_2,$$

where the functions Φ_1 and Φ_2 does not depend on j_{a+l} and j_{a+q}. Consequently, we can write

$$\sum_{j_{a+l}=0}^{1} \left(j_{a+q}2^{a+q} + \Phi_1\right) \times (-1)^{j_{a+l}+j_{a+q}}\Phi_2 = 0$$

and

$$\sum_{j_{a+q}=0}^{1} j_{a+l}2^{a+l} \times (-1)^{j_{a+l}+j_{a+q}}\Phi_2 = 0.$$

These give that

$$\sum_{j=0}^{2^s-1} jD_j^w\left(x^1\right) = 0 \text{ for } x - e_{a+l}x_{a+l} \notin I_s \text{ for } l = 1, ..., s-a-1. \tag{5}$$

Combining (2)-(5), we complete the proof of Lemma 7.

Lemma 8. *Let $n < 2^{A+1}, A > N$ and $x \in I_N\left(x_0, ..., x_m = 1, 0, ..., 0, x_l = 1, 0, ..., 0\right)$, $l = 0, ..., N-1$, $m = -1, 0, ..., l$. Then*

$$\int_{I_N} n\left|K_n^w\left(\tau_A\left(x+t\right)\right)\right| dt \leq c\frac{2^A}{2^{m+l}},$$

where

$$\begin{aligned} & I_N\left(x_0, ..., x_m = 1, 0, ..., 0, x_l = 1, 0, ..., 0\right) \\ := \ & I_N\left(0, ..., 0, x_l = 1, 0, ..., 0\right), \ \textit{for } m = -1. \end{aligned}$$

Proof. From Lemma 2, we obtain that $K^w_{n^{(s+1)},2^s}(\tau_A(x+t))=0 \quad \text{for } \ s\geq A-m$. Hence, we can suppose that $s<A-m$.

Using Lemma 2, $K^w_{n^{(s+1)},2^s}(\tau_A(x+t))\neq 0$ implies that

1)
$$t\in I_N(0,...,0,x_N,...,x_{A-1}) \quad \text{if} \ \ 0\leq s<A-m;$$

2)
$$t\in I_A(0,...,0,x_N,...,x_{q-1},1-x_q,x_{q+1},...,x_{A-1}) \quad \text{if} \ \ A-N<s<A-l;$$

3)
$$t\in I_A(0,...,0,t_N,...,t_{A-s-1},x_{A-s},...,x_{q-1},1-x_q,x_{q+1},...,x_{A-1}) \quad \text{if} \ \ 1\leq s\leq A-N;$$

4)
$$t\in I_A(0,...,0,t_N,...,t_{q-1},1-x_q,x_{q+1},...,x_{A-s},...,x_{A-1}) \quad \text{if} \ \ 1\leq s<A-N.$$

Consequently, we can write

$$\int_{I_N} n\left|K^w_n(\tau_A(x+t))\right|dt\leq \sum_{s=0}^{A-m}\int_{I_N}\left|K^w_{n^{(s+1)},2^s}(\tau_A(x+t))\right|dt$$

$$\leq c\left\{\sum_{s=0}^{A-m}\frac{2^{s+A-l}}{2^A}+\sum_{s=A-N}^{A-l}\sum_{q=N}^{A}\frac{2^{s+A-q}}{2^A}+\sum_{s=0}^{A-N}\sum_{q=A-s}^{A}\frac{2^{s+A-q}2^{A-s-N}}{2^A}+\sum_{s=0}^{A-N}\sum_{q=N}^{A-s}\frac{2^{s+A-q}2^{q-N}}{2^A}\right\}$$

$$\leq c\left\{\frac{2^A}{2^{m+l}}+\frac{2^A}{2^{N+l}}+\frac{2^A}{2^{2N}}+\sum_{s=0}^{A-m}\frac{2^s(A-s-N+1)}{2^{2N}}\right\}\leq c\frac{2^A}{2^{m+l}}.$$

Lemma 8 is proved.

Lemma 9. *Let* $(x^1,x^2)\in I_N\left(x^1_0,...,x^1_{m^1}=1,0,...,0\right)\times I_N\left(x^2_0,...,x^2_{m^2}=1,0,...,0\right), m^1\leq m^2, A>N$ *and* $n<2^{A+1}$. *Then*

$$\int_{I_N\times I_N} n\left|K^w_n\left(\tau_A\left(x^1+t^1\right),\tau_A\left(x^2+t^2\right)\right)\right|dt^1dt^2$$

$$\leq c\left\{\sum_{r=0}^{m^1-1}\sum_{q^2=m^1}^{m^2}\frac{2^A}{2^{m^2+q^2+r}}1_{I_N\left(x^2_0,...,x^2_r,x^1_{r+1},...,x^1_{m^1-1},x^2_{m^1},0,...,0,x^2_{q^2}=1,0,...,0,x^2_{m^2}=1,0,...,0\right)}\left(x^2\right)\right.$$

$$\left.+\sum_{r=m^1}^{m^2-1}\frac{2^A}{2^{m^1+m^2+r}}\sum_{q^2=r}^{m^2}1_{I_N\left(x^2_0,...,x^2_r,0,...,0,x^2_{q^2}=1,0...,0,x^2_{m^2}=1,0,...,0\right)}\left(x^2\right)\right\}.$$

Proof. Let $0\leq A-s<m^1$. Then using Lemma 4, $K^w_{n^{(s+1)},2^s}\left(\tau_A\left(x^1+t^1\right),\tau_A\left(x^2+t^2\right)\right)\neq 0$ implies that

1)
$$\left(t^1,t^2\right)\in I_A\left(0,...,0,x^1_N,...,x^1_{A-1}\right)\times I_A\left(0,...,0,x^2_N,...,x^2_{A-1}\right),$$
$$x^1_{A-s+1}=x^2_{A-s+1},...,x^1_{m^1-1}=x^2_{m^1-1},x^2_{m^1}=\cdots=x^2_{q^2-1}=x^2_{q^2+1}=\cdots=x^2_{m^2-1}=0,x^2_{q^2}=1,$$
where $m^1<q^2<m^2$;

2)

$$\left(t^1,t^2\right)\in I_A\left(0,...,0,x^1_N,...,x^1_{A-1}\right)\times I_A\left(0,...,0,x^2_N,...,x^2_{q^2-1},1-x^2_{q^2},x^2_{q^2+1},...,x^2_{A-1}\right),$$
$$x^1_{A-s+1}=x^2_{A-s+1},...,x^1_{m^1-1}=x^2_{m^1-1},x^2_{m^1}=\cdots=x^2_{m^2-1}=0;$$

3)

$$\left(t^1,t^2\right)\in I_A\left(0,...,0,x^1_N,...,x^1_{q^1-1},1-x^1_{q^1},x^1_{q^1+1},...,x^1_{l^1-1},1-x^1_{l^1},x^1_{l^1+1},...,x^1_{A-1}\right)$$
$$\times I_A\left(0,...,0,x^2_N,...,x^2_{A-1}\right),$$
$$x^1_{A-s+1}=x^2_{A-s+1},...,x^1_{m^1-1}=x^2_{m^1-1},x^2_{m^1}=1,x^2_{m^1+1}=\cdots=x^2_{m^2-1}=0.$$

Consequently, from Lemma 4 we can write

$$\sum_{s=A-m^1+1}^{A}\int_{I_N\times I_N}\left|K^w_{n^{(s+1)},2^s}\left(\tau_A\left(x^1+t^1\right),\tau_A\left(x^2+t^2\right)\right)\right|dt^1dt^2$$

$$\leq c\sum_{s=A-m^1+1}^{A}\left\{\sum_{q^2=m^1}^{m^2}\frac{2^{s+A-m^2+A-q^2}}{2^{2A}}1_{I_N\left(x^2_0,...,x^2_{A-s},x^1_{A-s+1},...,x^1_{m^1-1},0,...,0,x^2_{q^2}=1,0,...,0,x^2_{m^2}=1,0,...,0\right)}\left(x^2\right)\right.$$
$$+\sum_{q^2=N}^{A}\frac{2^{s+A-m^2+A-q^2}}{2^{2A}}1_{I_N\left(x^2_0,...,x^2_{A-s},x^1_{A-s+1},...,x^1_{m^1-1},0,...,0,x^2_{m^2}=1,0,...,0\right)}\left(x^2\right)$$
$$\left.+\sum_{q^1=N}^{A}\sum_{l^1=q^1}^{A}\frac{2^{s+A-l^1+A-q^1}}{2^{2A}}1_{I_N\left(x^2_0,...,x^2_{A-s},x^1_{A-s+1},...,x^1_{m^1-1},0,...,0,x^2_{m^2}=1,0,...,0\right)}\left(x^2\right)\right\}$$

$$\leq c\left\{\sum_{r=0}^{m^1-1}\sum_{q^2=m^1}^{m^2}\frac{2^A}{2^{m^2+q^2+r}}1_{I_N\left(x^2_0,...,x^2_r,x^1_{r+1},...,x^1_{m^1-1},0,...,0,x^2_{q^2}=1,0,...,0,x^2_{m^2}=1,0,...,0\right)}\left(x^2\right)\right\}. \tag{6}$$

Let $m^1\leq A-s<m^2$. Then using Lemmas 3 and 7, $K^w_{n^{(s+1)},2^s}\left(\tau_A\left(x^1+t^1\right),\tau_A\left(x^2+t^2\right)\right)\neq 0$ implies that

4)

$$\left(t^1,t^2\right)\in I_A\left(0,...,0,x^1_N,...,x^1_{q^1-1},1-x^1_{q^1},x^1_{q^1+1},...,x^1_{l^1-1},1-x^1_{l^1},x^1_{l^1+1},...,x^1_{A-1}\right)$$
$$\times I_A\left(0,...,0,x^2_N,...,x^2_{A-1}\right),\quad x^2_{A-s+1}=\cdots=x^2_{m^2-1}=0;$$

5)

$$\left(t^1,t^2\right)\in I_A\left(0,...,0,x^1_N,...,x^1_{A-1}\right)\times I_A\left(0,...,0,x^2_N,...,x^2_{A-1}\right),$$
$$x^2_{A-s}=\cdots=x^2_{q^2-1}=x^2_{q^2+1}=\cdots=x^2_{m^2-1}=0,\quad x^2_{q^2}=1;$$

6)

$$\left(t^1,t^2\right)\in I_A\left(0,...,0,x^1_N,...,x^1_{A-1}\right)\times I_A\left(0,...,0,x^2_N,...,x^2_{q^2-1},1-x^2_{q^2},x^2_{q^2+1},...,x^2_{A-1}\right),$$
$$x^2_{A-s}=\cdots=x^2_{m^2-1}=0.$$

Consequently, we can write

$$\sum_{s=A-m^2+1}^{A-m^1}\int_{I_N\times I_N}\left|K^w_{n^{(s+1)},2^s}\left(\tau_A\left(x^1+t^1\right),\tau_A\left(x^2+t^2\right)\right)\right|dt^1dt^2$$

$$\leq c\sum_{s=A-m^2+1}^{A-m^1}\left\{\sum_{q^1=N}^{A-1}\sum_{l^1=q^1}^{A-1}\frac{2^{s+A-l^1+A-q^1}}{2^{2A}}+\frac{2^{s+A-m^2+A-m^1}}{2^{2A}}\sum_{q^2=A-s}^{m^2}\right.$$

$$1_{I_N\left(x_0^2,\dots,x_{A-s-1}^2,0,\dots,0,x_{q^2}^2=1,0\dots,0,x_{m^2}^2=1,0,\dots,0\right)}\left(x^2\right)$$

$$\left.+\sum_{q^2=N}^{A-1}\frac{2^{s+A-q^2+A-m^1}}{2^{2A}}1_{I_N\left(x_0^2,\dots,x_{A-s}^2,0,\dots,0,x_{m^2}^2=1,0,\dots,0\right)}\left(x^2\right)\right\}$$

$$\leq c\left\{\frac{2^A}{2^{2N+m^1}}+\sum_{r=m^1}^{m^2-1}\frac{2^A}{2^{m^1+m^2+r}}\sum_{q^2=r}^{m^2}1_{I_N\left(x_0^2,\dots,x_r^2,0,\dots,0,x_{q^2}^2=1,0\dots,0,x_{m^2}^2=1,0,\dots,0\right)}\left(x^2\right)\right.$$

$$\left.+\sum_{r=m^1}^{m^2-1}\frac{2^A}{2^{N+m^1+r}}1_{I_N\left(x_0^2,\dots,x_r^2,0,\dots,0,x_{m^2}^2=1,0,\dots,0\right)}\left(x^2\right)\right\}$$

$$\leq c\left\{\sum_{r=m^1}^{m^2-1}\frac{2^A}{2^{m^1+m^2+r}}\sum_{q^2=r}^{m^2}1_{I_N\left(x_0^2,\dots,x_r^2,0,\dots,0,x_{q^2}=1,0\dots,0,x_{m^2}^2=1,0,\dots,0\right)}\left(x^2\right)\right\}. \tag{7}$$

Let $m^2\leq A-s<N$. Then using Lemmas 3 and 7, $K^w_{n^{(s+1)},2^s}\left(\tau_A\left(x^1+t^1\right),\tau_A\left(x^2+t^2\right)\right)\neq 0$ implies that

7)

$$\left(t^1,t^2\right)\in I_A\left(0,\dots,0,x_N^1,\dots,x_{A-1}^1\right)\times I_A\left(0,\dots,0,x_N^2,\dots,x_{A-1}^2\right),$$

8)

$$\left(t^1,t^2\right)\in I_A\left(0,\dots,0,t_N^1,\dots,t_{q^2-1}^1,x_{q^2}^1,\dots,x_{q^1-1}^1,1-x_{q^1}^1,x_{q^1+1}^1,\dots,x_{l^1-1}^1,1-x_{l^1}^1,x_{l^1+1}^1,\dots,x_{A-1}^1\right)$$

$$\times I_A\left(0,\dots,0,t_N^1+x_N^2+x_N^1,\dots,t_{q^2-1}^1+x_{q^2-1}^2+x_{q^2-1}^1,1-x_{q^2}^2,x_{q^2+1}^2,\dots,x_{A-1}^2\right),$$

9)

$$\left(t^1,t^2\right)\in I_A\left(0,\dots,0,x_N^1,\dots,x_{A-1}^1\right)$$

$$\times I_A\left(0,\dots,0,x_N^2,\dots,x_{q^2-1}^2,1-x_{q^2}^2,x_{q^2+1}^2,\dots,x_{l^2-1}^2,1-x_{l^2}^2,x_{l^2+1}^2,\dots,x_{A-1}^2\right).$$

Consequently, we can write

$$\sum_{s=A-N}^{A-m^2-1}\int_{I_N\times I_N}\left|K^w_{n^{(s+1)},2^s}\left(\tau_A\left(x^1+t^1\right),\tau_A\left(x^2+t^2\right)\right)\right|dt^1dt^2$$

$$\leq c\sum_{s=A-N}^{A-m^2-1}\left\{\frac{2^{s+A-m^1+A-m^2}}{2^{2A}}+\sum_{q^2=N}^{A-1}\sum_{l^1=q^2}^{A}\sum_{q^1=q^2}^{l^1}\frac{2^{s+A-l^1+A-q^1}2^{q^2-N}}{2^{2A}}+\sum_{q^2=N}^{A-1}\sum_{l^2=q^2}^{A-1}\frac{2^{s+A-l^2+A-m^1}}{2^{2A}}\right\}$$

$$\leq c\left\{\frac{2^A}{2^{m^1+2m^2}}+\frac{2^A}{2^{m^2+2N}}+\frac{2^A}{2^{N+m^1+m^2}}\right\}$$

$$\leq c\left\{\sum_{r=m^1}^{m^2-1}\frac{2^A}{2^{m^1+m^2+r}}\sum_{q^2=r}^{m^2}1_{I_N\left(x_0^2,\dots,x_r^2,0,\dots,0,x_{q^2}=1,0\dots,0,x_{m^2}^2=1,0,\dots,0\right)}\left(x^2\right)\right\}. \tag{8}$$

Let $0 \leq s \leq A-N$. Then using Lemmas 3 and 7, $K^{w}_{n^{(s+1)},2^{s}}\left(\tau_A\left(x^1+t^1\right),\tau_A\left(x^2+t^2\right)\right)\neq 0$ implies that

10)
$$\left(t^1,t^2\right)\in I_A\left(0,...,0,x^1_N,...,x^1_{A-1}\right)\times I_A\left(0,...,0,x^2_N,...,x^2_{A-1}\right),$$

11)
$$\left(t^1,t^2\right)\in I_A\left(0,...,0,x^1_N,...,x^1_{A-1}\right)\times I_A\left(0,...,0,t^2_N,...,t^2_{q^2-1},1-x^2_{q^2},x^2_{q^2+1},...,x^2_{A-1}\right),$$

12)
$$\left(t^1,t^2\right)\in I_A\left(0,...,0,t^1_N,...,t^1_{q^1-1},1-x^1_{q^1},x^1_{q^1+1},...,x^1_{A-1}\right)$$
$$\times I_A\left(0,...,0,t^2_N,...,t^2_{q^2-1},1-x^2_{q^2},x^2_{q^2+1},...,x^2_{A-1}\right),$$

13)
$$\left(t^1,t^2\right)\in I_A\left(0,...,0,x^1_N,...,x^1_{A-1}\right)$$
$$\times I_A\left(0,...,0,t^2_N,...,t^2_{A-s-1},x^2_{A-s},...,x^2_{q^2-1},1-x^2_{q^2},x^2_{q^2+1},...,x^2_{l^2-1},1-x^2_{l^2},x^2_{l^2+1},...,x^2_{A-1}\right),$$

14)
$$\left(t^1,t^2\right)\in I_A\left(0,...,0,t^1_N,...,t^1_{q^1-1},1-x^1_{q^1},...,x^1_{A-s},...,x^1_{q^1+1}...,x^1_{A-1}\right)$$
$$\times I_A\left(0,...,0,t^2_N,...,t^2_{A-s-1},x^2_{A-s},...,x^2_{q^2-1},1-x^2_{q^2},x^2_{q^2+1},...,x^2_{l^2-1},1-x^2_{l^2},x^2_{l^2+1},...,x^2_{A-1}\right),$$

15)
$$\left(t^1,t^2\right)\in I_A\left(0,...,0,t^1_N,...,t^1_{A-s},...,t^1_{q^1-1},1-x^1_{q^1},x^1_{q^1+1}...,x^1_{A-1}\right)$$
$$\times I_A\left(0,...,0,t^2_N,...,t^2_{A-s},x^2_{A-s+1}+x^1_{A-s+1}+t^1_{A-s+1},...,x^2_{q^1-1}+x^1_{q^1-1}+t^1_{q^1-1},\right.$$
$$\left.x^2_{q^1},...,x^2_{q^2-1},1-x^2_{q^2},x^2_{q^2+1},...,x^2_{l^2-1},1-x^2_{l^2},x^2_{l^2+1},...,x^2_{A-1}\right).$$

Hence, we can write

$$\sum_{s=0}^{A-N}\int_{I_N\times I_N}\left|K^{w}_{n^{(s+1)},2^{s}}\left(\tau_A\left(x^1+t^1\right),\tau_A\left(x^2+t^2\right)\right)\right|dt^1dt^2$$
$$\leq c\sum_{s=0}^{A-N}\left\{\frac{2^{s+A-m^1+A-m^2}}{2^{2A}}+\sum_{q^2=N}^{A-s}\frac{2^{s+A-m^1+A-q^2}2^{q^2-N}}{2^{2A}}\right.$$
$$+\sum_{q^2=A-s}^{A-1}\sum_{l^2=q^2}^{A-1}\frac{2^{s+A-l^2+A-m^1}2^{A-s-N}}{2^{2A}}+\sum_{q^1=N}^{A-s}\sum_{q^2=A-s}^{A-1}\sum_{l^2=q^2}^{A-1}\frac{2^{s+A-l^2+A-q^1}2^{q^1-N}2^{A-s-N}}{2^{2A}}$$
$$\left.+\sum_{q^1=A-s}^{A}\sum_{l^2=q^1}^{A-1}\sum_{q^2=q^1}^{l^2}\frac{2^{s+A-l^2+A-q^2}2^{q^1-N}2^{A-s-N}}{2^{2A}}\right\}\leq c\left\{\frac{2^A}{2^{m^1+m^2+N}}+\sum_{s=0}^{A-N}\frac{2^s\left(A-s-N\right)}{2^{m^1+N}}\right.$$
$$\left.+\sum_{s=0}^{A-N}\sum_{q^2=A-s}^{A-1}\sum_{l^2=q^2}^{A-1}\frac{2^A}{2^{m^1+l^2+N}}+\sum_{s=0}^{A-N}\sum_{q^1=N}^{A-s}\sum_{q^2=A-s}^{A-1}\sum_{l^2=q^2}^{A-1}\frac{2^A}{2^{l^2+N}}+\sum_{s=0}^{A-N}\sum_{q^1=A-s}^{A}\sum_{l^2=q^1}^{A-1}\sum_{q^2=q^1}^{l^2}\frac{2^{A+q^1}}{2^{q^2+l^2+2N}}\right\}$$
$$\leq c\left\{\frac{2^A}{2^{m^1+m^2+N}}+\frac{2^A}{2^{m^1+2N}}+\frac{2^A}{2^{m^1+2N}}+\frac{2^A}{2^{3N}}+\frac{2^A}{2^{3N}}\right\}$$

(9)
$$\leq c\left\{\sum_{r=m^1}^{m^2-1}\frac{2^A}{2^{m^1+m^2+r}}\sum_{q^2=r}^{m^2}1_{I_N\left(x_0^1,\dots,x_{m^1}^1=1,0,\dots,0\right)}\left(x^1\right)1_{I_N\left(x_0^2,\dots,x_r^2,0,\dots,0,x_{q^2}=1,0\dots,0,x_{m^2}^2=1,0,\dots,0\right)}\left(x^2\right)\right\}.$$

By (6)-(9), we complete the proof of Lemma 9.

Analogously, we can prove that the following are true.

Lemma 10. *Let* $(x^1,x^2)\in I_N(0,...,0)\times I_N\left(x_0^2,...,x_{m^2}^2=1,0,...,0\right),A>N$ *and* $n<2^{A+1}$. *Then*
$$\int\limits_{I_N\times I_N} n\left|K_n^w\left(\tau_A\left(x^1+t^1\right),\tau_A\left(x^2+t^2\right)\right)\right|dt^1dt^2$$
$$\leq c\left\{\sum_{r=0}^{m^2}\frac{2^A}{2^{m^2+r}}\sum_{q^2=r}^{m^2}1_{I_N\left(x_0^2,\dots,x_r^2,0,\dots,0,x_{q^2}=1,0\dots,0,x_{m^2}^2=1,0,\dots,0\right)}\left(x^2\right)\right\}.$$

Lemma 11. *Let* $(x^1,x^2)\in I_N\left(x_0^1,...,x_{m^1}^1=1,0,...,0\right)\times I_N(0,...,0),A>N$ *and* $n<2^{A+1}$. *Then*
$$\int\limits_{I_N\times I_N} n\left|K_n^w\left(\tau_A\left(x^1+t^1\right),\tau_A\left(x^2+t^2\right)\right)\right|dt^1dt^2$$
$$\leq c\left\{\sum_{r=0}^{m^1}\frac{2^A}{2^{m^1+r}}\sum_{q^1=r}^{m^1}1_{I_N\left(x_0^1,\dots,x_r^1,0,\dots,0,x_{q^1}=1,0\dots,0,x_{m^1}^1=1,0,\dots,0\right)}\left(x^1\right)\right\}.$$

5. Proofs of the Main Results

Proof of Theorem 1.

By Lemma 1, the proof of Theorem 1 will be complete, if we show that the operator σ^* is p-quasi-local for each $2/3<p\leq 1$ and bounded from $L^\infty(K^2)$ to $L^\infty(K^2)$.

The boundedness follows from Corollary 3.

Let a be an arbitrary atom with support $R=I\times J$ and $\mu(I)=\mu(J)=2^{-N}$. We may assume that $I=J=I_N$. It is easy to see that $\sigma_n(a)=0$ if $n\leq 2^N$. Therefore, we can suppose that $n>2^N$.

Step 1. Integrating over $(K\backslash I_N)\times(K\backslash I_N)$. Let $n=2^A+m$, where $0\leq m<2^A$. Then from Lemma 5 and by (1) we have
$$\sigma_n a\left(x^1,x^2\right)=\frac{2^A}{n}\sigma_{2^A}a\left(x^1,x^2\right)$$
$$+\frac{1}{n}\int\limits_{I_N\times I_N} a\left(t^1,t^2\right)r_A\left(x^1+t^1+x^2+t^2\right)mK_m^w\left(\tau_A\left(x^1+t^1\right),\tau_A\left(x^2+t^2\right)\right)dt^1dt^2$$
$$=\sigma_n^{(1)}a\left(x^1,x^2\right)+\sigma_n^{(2)}a\left(x^1,x^2\right). \tag{10}$$

Since $|a| \leq c2^{2N/p}$, for $\sigma_n^{(2)} a\left(x^1, x^2\right)$ we get

$$\left|\sigma_n^{(2)} a\left(x^1, x^2\right)\right| \leq \frac{c2^{2N/p}}{n} \int\limits_{I_N \times I_N} m\left|K_m^w\left(\tau_A\left(x^1+t^1\right), \tau_A\left(x^2+t^2\right)\right)\right| dt^1 dt^2.$$

Let $m^1, m^2 = 0, \ldots, N-1$ such that

$$\left(x^1, x^2\right) \in I_N\left(x_0^1, \ldots, x_{m^1}^1 = 1, 0, \ldots, 0\right) \times I_N\left(x_0^2, \ldots, x_{m^2}^2 = 1, 0, \ldots, 0\right).$$

Then, applying Lemma 10, we get

$$\left|\sigma_n^{(2)} a\left(x^1, x^2\right)\right|$$

$$\leq \frac{c2^{2N/p}}{n}\left\{\sum_{r=0}^{m^1-1} \sum_{q^2=m^1}^{m^2} \frac{2^A}{2^{m^2+q^2+r}} 1_{I_N\left(x_0^2, \ldots, x_r^2, x_{r+1}^1, \ldots, x_{m^1-1}^1, x_{m^1}^2, 0, \ldots, 0, x_{q^2}^2=1, 0, \ldots, 0, x_{m^2}^2=1, 0, \ldots, 0\right)}\left(x^2\right)\right.$$

$$\left.+\sum_{r=m^1}^{m^2-1} \frac{2^A}{2^{m^1+m^2+r}} 1_{I_N\left(x_0^2, \ldots, x_r^2, 0, \ldots, 0, x_{q^2}^2=1, 0 \ldots, 0, x_{m^2}^2=1, 0, \ldots, 0\right)}\left(x^2\right)\right\}.$$

Using the inequality

$$\left(\sum_{k=0}^{\infty} a_k\right)^p \leq \sum_{k=0}^{\infty} a_k^p \qquad \left(a_k \geq 0, \quad 0<p \leq 1\right),$$

we obtain

$$\int\limits_{(K \backslash I_N) \times (K \backslash I_N)} \sup_{n \geq 2^N}\left|\sigma_n^{(2)} a\left(x^1, x^2\right)\right|^p dx^1 dx^2$$

$$\leq c_p 2^{2N} \sum_{m^1=0}^{N-1} \sum_{m^2=m^1}^{N-1}\left\{\frac{1}{2^{m^2 p}} \sum_{r=0}^{m^1-1} 2^{r(1-p)} \sum_{q^2=m^1}^{m^2} \frac{1}{2^{q^2 p}} \frac{2^{m^1}}{2^{2N}}\right.$$

$$\left.+\frac{1}{2^{(m^1+m^2) p}} \sum_{r=m^1}^{m^2-1} 2^{r(1-p)} \frac{2^{m^1}}{2^{2N}}\right\}$$

$$\leq c_p\left\{\sum_{m^1=0}^{N-1} 2^{m^1} \sum_{m^2=m^1}^{N-1} \frac{1}{2^{m^2 p}} \sum_{r=0}^{m^1-1} 2^{r(1-p)} \sum_{q^2=m^1}^{m^2} \frac{1}{2^{q^2 p}}\right.$$

$$\left.+\sum_{m^1=0}^{N-1} 2^{m^1(1-p)} \sum_{m^2=m^1}^{N-1} \frac{1}{2^{m^2 p}} \sum_{r=m^1}^{m^2-1} 2^{r(1-p)}\right\}. \tag{11}$$

Let $2/3<p<1$. Then, we have

$$\int\limits_{(K \backslash I_N) \times (K \backslash I_N)} \sup_{n \geq 2^N}\left|\sigma_n^{(2)} a\left(x^1, x^2\right)\right|^p dx^1 dx^2 \leq c_p \sum_{m^1=0}^{N-1} 2^{m^1(2-3p)} \leq c_p<\infty, \tag{12}$$

while $p=1$, we get

(13)
$$\int\limits_{(K\backslash I_N)\times(K\backslash I_N)}\sup_{n\geq 2^N}\left|\sigma_n^{(2)}a\left(x^1,x^2\right)\right|dx^1dx^2\leq c\left\{\sum_{m^1=0}^{N-1}\frac{m^1}{2^{m^1}}+\sum_{m^1=0}^{N-1}\sum_{m^2=m^1}^{N-1}\frac{m^2-m^1}{2^{m^2}}\right\}\leq c<\infty.$$

Combining (11)-(13), we conclude that

$$\int\limits_{(K\backslash I_N)\times(K\backslash I_N)}\sup_{n\geq 2^N}\left|\sigma_n^{(2)}a\left(x^1,x^2\right)\right|^p dx^1dx^2\leq c_p<\infty,\text{ for }2/3<p\leq 1. \tag{14}$$

By (10), (14) and Lemma 6, we have

$$\int\limits_{(K\backslash I_N)\times(K\backslash I_N)}\sup_{n\geq 2^N}\left|\sigma_n a\left(x^1,x^2\right)\right|^p dx^1dx^2\leq c_p<\infty,\text{ for }2/3\leq p\leq 1. \tag{15}$$

Step 2. Integrating over $I_N\times(K\backslash I_N)$. By Lemma 5 and (1) we have

$$\sigma_n a\left(x^1,x^2\right)=\frac{2^A}{n}\sigma_{2^A}a\left(x^1,x^2\right)$$

$$+\frac{1}{n}\int\limits_{I_N\times I_N}a\left(t^1,t^2\right)D_{2^A}\left(x^1+t^1\right)r_A\left(x^2+t^2\right)mK_m^w\left(\tau_A\left(x^2+t^2\right)\right)dt^1dt^2$$

$$+\frac{1}{n}\int\limits_{I_N\times I_N}a\left(t^1,t^2\right)r_A\left(x^1+t^1+x^2+t^2\right)mK_m^w\left(\tau_A\left(x^1+t^1\right),\tau_A\left(x^2+t^2\right)\right)dt^1dt^2$$

$$=\sigma_n^{(1)}a\left(x^1,x^2\right)+\sigma_n^{(3)}a\left(x^1,x^2\right)+\sigma_n^{(4)}a\left(x^1,x^2\right). \tag{16}$$

Since $|a|\leq c2^{2N/p}$, for $\sigma_n^{(3)}a\left(x^1,x^2\right)$ we get

$$\left|\sigma_n^{(3)}a\left(x^1,x^2\right)\right|\leq\frac{c2^{2N/p}}{n}\int\limits_{I_N}m\left|K_m^w\left(\tau_A\left(x^2+t^2\right)\right)\right|dt^2.$$

Let $m^2=0,...,l^2$ and $l^2=0,...,N-1$ such that $x^2\in I_N\left(x_0^2,...,x_{m^2}^2=1,0,...,0,x_{l^2}^2=1,0,...,0\right)$. Lemma 2 gives

$$\left|\sigma_n^{(3)}a\left(x^1,x^2\right)\right|\leq\frac{c2^{2N/p}}{2^A}\frac{2^A}{2^{m^2+l^2}}\leq c\frac{2^{2N/p}}{2^{m^2+l^2}}.$$

Consequently, for $1/2<p\leq 1$ we have

$$\int\limits_{I_N\times(K\backslash I_N)}\sup_{n\geq 2^N}\left|\sigma_n^{(3)}a\left(x^1,x^2\right)\right|^p dx^1dx^2$$

$$\leq c_p\frac{2^{2N}}{2^N}\sum_{l^2=0}^{N-1}\frac{1}{2^{l^2}}\sum_{m^2=0}^{l^2}\frac{1}{2^{m^2p}}\frac{2^{m^2}}{2^N}\leq c_p\sum_{l^2=0}^{N-1}\frac{1}{2^{l^2p}}\sum_{m^2=0}^{l^2}2^{m^2(1-p)}\leq c_p<\infty. \tag{17}$$

Let $m^1, m^2 = 0, ..., N-1$ such that

$$\left(x^1, x^2\right) \in I_N\left(x_0^1, ..., x_{m^1}^1 = 1, 0, ..., 0\right) \times I_N\left(x_0^2, ..., x_{m^2}^2 = 1, 0, ..., 0\right).$$

Applying Lemma 10, we get

$$\left|\sigma_n^{(4)} a\left(x^1, x^2\right)\right| \leq \frac{c2^{2N/p}}{n} \int_{I_N \times I_N} m\left|K_m^w\left(\tau_A\left(x^1+t^1\right), \tau_A\left(x^2+t^2\right)\right)\right| dt^1 dt^2$$

$$\leq \frac{c2^{2N/p}}{n}\left\{\sum_{r=0}^{m^2} \frac{2^A}{2^{m^2+r}} \sum_{q^2=r}^{m^2} 1_{I_N\left(x_0^2,...,x_r^2,0,...,0,x_{q^2}^2=1,0,...,0,x_{m^2}^2=1,0,...,0\right)}\left(x^2\right)\right\}$$

$$\leq c2^{2N/p}\left\{\sum_{r=0}^{m^2} \frac{1}{2^{m^2+r}} \sum_{q^2=r}^{m^2} 1_{I_N\left(x_0^2,...,x_r^2,0,...,0,x_{q^2}^2=1,0,...,0,x_{m^2}^2=1,0,...,0\right)}\left(x^2\right)\right\}. \tag{18}$$

Hence, for $1/2 < p \leq 1$ we can write

$$\int_{I_N \times (K \backslash I_N)} \sup_{n \geq 2^N}\left|\sigma_n^{(4)} a\left(x^1, x^2\right)\right|^p dx^1 dx^2 \leq \frac{c_p 2^{2N}}{2^N} \sum_{m^2=0}^{N-1} \frac{1}{2^{m^2 p}} \sum_{r=0}^{m^2} \frac{1}{2^{pr}} \sum_{q^2=r}^{m^2} \frac{2^r}{2^N}$$

$$\leq c_p \sum_{m^2=0}^{N-1} \frac{1}{2^{m^2 p}} \sum_{r=0}^{m^2} 2^{r(1-p)}\left(m^2-r\right) \leq c_p < \infty. \tag{19}$$

Combining (18) and (19), we conclude that

$$\int_{I_N \times (K \backslash I_N)} \sup_{n \geq 2^N}\left|\sigma_n^{(4)} a\left(x^1, x^2\right)\right|^p dx^1 dx^2 \leq c_p < \infty, \text{ for } 1/2 < p \leq 1. \tag{20}$$

From (16), (17), (20) and Lemma 6, we get

$$\int_{I_N \times (K \backslash I_N)} \sup_{n \geq 2^N}\left|\sigma_n a\left(x^1, x^2\right)\right|^p dx^1 dx^2 \leq c_p < \infty, \text{ for } 1/2 < p \leq 1.$$

Step 3. Integrating over $(K \backslash I_N) \times I_N$.

This case is analogous to step 2.□

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