LFP vs LP: comparative analysis of two optimization approaches

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Consider the following linear programming (LP) and linear-fractional programming (LFP) problems:

$$\begin{array}{c} P(x) \rightarrow \text{max} \\ x \in S, \end{array} \qquad (1)$$

$$\begin{array}{c} Q(x) = P(x)/C(x) \rightarrow \text{max} \\ x \in S, \end{array} \qquad (2)$$

where

$$P(x) = \sum_{j=1}^{n} p_j\, x_j + p_0,$$

$$C(x) = \sum_{j=1}^{n} c_j\, x_j + c_0$$

$C(x) > 0$ for all $x \in S$

$$S = \{x \in R^n : Ax \leq b,\ x \geq 0\},$$

A is $m \times n$ matrix, i.e. $A = \|a_{ij}\|_{m\times n}$,
$x = (x_1, x_2, \cdots, x_n)$, $b = (b_1, b_2, \cdots, b_m)^T$;
$a_{ij},\ b_i,\ p_j,\ c_j$ are scalar constants,
T denotes the transpose of a vector.

LP: a wide range of real-world applications.
LFP: may be applied in the same applications as LP (at least!).
Interpretation: $P(x)$-profit, $C(x)$-cost, $Q(x) = P(x)/C(x)$-efficiency, a company has to decide which objective function to apply.

Difference: Two different objective functions (linear and linear-fractional) to be optimized on the same feasible set (i.e. subject to the same constraints).

Well-known fact: Two or more objective functions defined on the same feasible set in general case lead to different (non-coincident) optimal solutions.

Conclusion: The economic interests expressed by linear objective function and linear-fractional objective function on the same feasible set in general case result different optimal solutions (and, hence, different decisions) which may conflict with one another.

1. Non-coincident optimal solutions

Let us consider the case when problems (1) and (2) have different optimal solutions.

Let vector x^* be an optimal solution of LP problem (1) and x' denote an optimal solution of LFP problem (2). Obviously, in this case we have the following inequalities:

$$P(x^*) \geq P(x') \tag{3}$$

and

$$Q(x^*) \leq Q(x'). \tag{4}$$

Which optimal plan should prefer the company? Let us return to inequality (4) and rewrite it in the following form

$$\frac{P(x^*)}{C(x^*)} \leq \frac{P(x')}{C(x')}$$

or (here where suppose that $P(x') > 0$ and $C(x^*) > 0$)

$$\frac{P(x^*)}{P(x')} \leq \frac{C(x^*)}{C(x')} \; . \tag{5}$$

From (3) we have that $P(x^*)/P(x') \geq 1$. Using the latter in (5) we obtain that

$$C(x^*)/C(x') \geq 1$$

or (assuming that $C(x') > 0$)

$$C(x^*) \geq C(x') \; . \tag{6}$$

Inequality (6) shows that the cost of optimal plan x^* can not be less than the cost of optimal plan x'. In other words, in general case we can say that

optimal plan x' is cheaper than x^*.

Thus, the answer for the question *which optimal plan the company should choose* may be formulated as follows: it depends on the costs $C(x^*)$ and $C(x')$, and the amount of money the company can spend.

So there are following three possible scenarios:

1. If the company **can not** spend $C(x^*)$ units of money since it has only $C(x')$ units to invest, there is only one possibility - to accept optimal plan x', implement it and obtain $P(x')$ units of profit;

2. If the company **can** spend $C(x^*)$ units of money, then there are following two cases:

   (a) to accept optimal plan x^*, implement it and obtain $P(x^*)$ units of profit, or

   (b) to accept optimal plan x', implement it $k = C(x^*)/C(x') \geq 1$ times and obtain $kP(x')$ units of profit.

Observe, that

$$\begin{aligned} kP(x') &= \frac{C(x^*)}{C(x')} P(x') = C(x^*) \frac{P(x')}{C(x')} = \\ &= C(x^*) Q(x') \geq C(x^*) Q(x^*) = \\ &= P(x^*) \end{aligned} \tag{7}$$

i.e.

$$kP(x') \geq P(x^*)$$

So profit in case (2.b) can not be less then profit of the case (2.a).

2. Coincident optimal solutions.

Let be given LFP problem (2):

$$Q(x) = P(x)/C(x) \rightarrow \max \\ x \in S,$$

and the following LP problem:

$$C(x) \rightarrow \max \\ x \in S. \tag{8}$$

Problems (2) and (8) have different optimal solutions and we have to redirect objective function $Q(x)$ in such a way that all optimization problems considered lead to the same optimal solution.

Interpretation: $P(x)$-profit, $C(x)$-man-power requirement, $Q(x) = P(x)/C(x)$-efficiency calculated as profit gained per unit of man-power requirement.

Assumption: There is an unemployment in the society, company prefers to optimize the $Q(x)$ criteria, economic interest of the society - maximization of $C(x)$-man-power requirement.

Notation: Vector x' solves LFP problem (2), vector x'' solves LP problem (8).
Obviously,

$$C(x') \leq C(x'') \tag{9}$$

i.e. man-power requirement in point x' is **less** (generally speaking) than in the point x''.

Redirection: using data from the optimal simplex tableau for LP problem (8) we construct some vector $t = (t_0, t_1, t_2, \cdots, t_n)$ and replace profit vector $p = (p_0, p_1, p_2, \cdots, p_n)$ in LFP problem (2) with new vector $p' = (p'_0, p'_1, p'_2, \cdots, p'_n)$, where

$$p'_j = p_j + t_j, \quad j = 0, 1, 2, \ldots, n \, . \tag{10}$$

Thus, we have a new LFP problem

$$Q'(x) = \frac{P'(x)}{C(x)} \rightarrow \max \tag{11}$$
$$x \in S,$$

where

$$P'(x) = \sum_{j=1}^{n} p'_j\, x_j + p'_0\, .$$

Mathematically it is easy to show that there is infinite number of such vector t such that x'' solves modified LFP problem (11).

Interpretation: t_j - **tax** or **subsidy** (depending on the sign) for each unit of j-th product.

Moreover: There is at least one such vector t that

$$T(x'') \;=\; \sum_{j=1}^{n} t_j\, x''_j + t_0 \;=\; 0\,, \qquad (12)$$

i.e. there is (at least one) vector t of such taxes and subsidies that their total sum in point x'' is equal to zero.

Resume

If a real-world optimization problem can be reduced to a linear programming model, often it automatically means that the problem may be re-formulated as an LFP problem too. In this case may appear the following question:

what type of objective function we have to apply - linear or linear-fractional?

The investigation of such situations has led us to the results presented above.

The mathematical background (statements and proofs) of the results presented in the talk may be obtained from the author:

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Many thanks for attention ...