MAXIMAL OPERATORS OF FEJÉR MEANS OF WALSH-KACZMARZ-FOURIER SERIES

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Abstract. The main aim of this paper is to prove that there exists a martingale $f \in H_{1/2}$ such that the maximal Fejér operator with respect to Walsh-Kaczmarz system does not belong to the space $L_{1/2}$. For the two-dimensional case, we prove that there exists a martingale $f \in H_{1/2}^{\square}(f \in H_{1/2})$ such that the restricted (unrestricted) maximal operator of Fejér means of two-dimensional Walsh-Kaczmarz-Fourier series does not belong to the space weak-$L_{1/2}$.

1. Introduction

The first result with respect to the a.e. convergence of the Walsh-Fejér means $\sigma_n f$ is due to Fine [1]. Later, Schipp [9] showed that the maximal operator $\sigma^* f := \sup_n |\sigma_n f|$ is of weak type $(1,1)$, from which the a. e. convergence follows by standard argument. Schipp's result implies by interpolation also the boundedness of $\sigma^* : L_p \to L_p$ $(1 < p \leq \infty)$. This fails to hold for $p = 1$, but Fujii [2] proved that σ^* is bounded from the dyadic Hardy space H_1 to the space L_1. Fujii's theorem was extended by Weisz [19]. Namely, he proved that the maximal operator of the Fejér means of the one-dimensional Walsh-Fourier series is bounded from the martingale Hardy space $H_p(G)$ to the space $L_p(G)$ for $p > 1/2$. Simon [13] gave a counterexample, which shows that this boundedness does not hold for $0 < p < 1/2$. In the endpoint case $p = 1/2$ Weisz [21] proved that σ^* is bounded from the Hardy space $H_{1/2}(G)$ to the space weak-$L_{1/2}(G)$ (see also [14]). In [5] the first author proved that the maximal operator σ^* is not bounded from the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$.

In 1948 Šneider [16] introduced the Walsh-Kaczmarz system and showed that the inequality

$$\limsup_{n\to\infty} \frac{D_n^{\kappa}(x)}{\log n} \geq C > 0$$

holds a.e. In 1974 Schipp [10] and Young [17] proved that the Walsh-Kaczmarz system is a convergence system. Skvortsov in 1981 [15] showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to f for any continuous functions f. Gát [3] proved, for any integrable functions, that the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere to the function and Gát proved that $\|\sigma^{\kappa *}\|_1 \leq C\||f|\|_{H_1}$. The result of Gát was extended to the Hardy space by Simon [11], who proved that $\sigma^{\kappa *}$ is of type (H_p, L_p) for $p > 1/2$. Weisz [21] showed that in endpoint case $p = 1/2$ the maximal operator is of weak type $\left(H_{1/2}, L_{1/2}\right)$.

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In this paper we will prove a stronger result than the unboundedness of the maximal operator from the Hardy space $H_{1/2}$ to the space $L_{1/2}$, in particular, we prove that there exists a martingale $f \in H_{1/2}$ such that

$$\|\sigma^{\kappa,*}\|_{1/2} = +\infty.$$

For the two-dimensional Walsh-Kaczmarz-Fourier series Simon proved [12] that the restricted maximal operator $\sigma_{\lambda}^{\kappa *}$ is bounded from the Hardy space H_p to the space L_p for all $p > 1/2$.

In the paper [7] it was proved that the assumption $p > 1/2$ is essential. Namely, the maximal operator $\sigma^{\kappa *} := \sup_n |\sigma_{n,n}^{\kappa}|$ of the Fejér means of double Fourier series with respect to the Walsh-Kaczmarz system is not bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$. In this paper we will prove a stronger result than in the paper [7], in particular, we prove that there exists a martingale $f \in H_{1/2}^{\Box}(f \in H_{1/2})$ such that

$$\|\sigma^{\kappa,\Box}\|_{weak-L_{1/2}} = +\infty \quad (\|\sigma^{\kappa,*}\|_{weak-L_{1/2}} = +\infty).$$

Thus, as regards boundedness of $\sigma^{\kappa,*}$ the case of two-dimensional Walsh-Kaczmarz series differs from the case of one-dimensional Walsh-Kaczmarz series.

Let denote by $\mathbf{Z}_2$ the discrete cyclic group of order 2, the group operation is the modulo 2 addition and every subset is open. The normalized Haar measure on $\mathbf{Z}_2$ is given in the way that the measure of a singleton is 1/2. Let $G := \mathop{\times}\limits_{k=0}^{\infty} \mathbf{Z}_2$, G be called the Walsh group. The elements of G are sequences $x = (x_0, x_1, ..., x_k, ...)$ with $x_k \in \{0, 1\}$ $(k \in \mathbf{N})$.

The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure (denoted by μ) and the topology are the product measure and topology. Dyadic intervalls are defined by

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y = (x_0, ..., x_{n-1}, y_n, y_{n+1} ...)\}$$

for $x \in G, n \in \mathbf{P}$. They form a base for the neighborhoods of G. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \mathbf{N}$.

Let L_p denote the usual Lebesgue spaces on G (with the corresponding norm or quasinorm $\|.\|_p$). The space weak-L_p consists of all measurable functions f for which

$$\|f\|_{\text{weak}-L_p} := \sup_{\lambda > 0} \lambda \mu \left(|f| > \lambda\right)^{1/p} < +\infty.$$

The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbf{N}).$$

Let the Walsh-Paley functions be the product functions of the Rademacher functions. Namely, each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0, 1\} \ (i \in \mathbf{N}),$$

where only a finite number of n_i's different from zero. Let the order of $n > 0$ be denoted by $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$. Walsh-Paley functions are $w_0 = 1$ and for $n \geq 1$

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k}.$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 = 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} \left(r_{|n|-1-k}(x)\right)^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The set of Walsh-Kaczmarz functions and the set of Walsh-Paley functions is the same in dyadic blocks. Namely,

$$\left\{\kappa_n : 2^k \leq n < 2^{k+1}\right\} = \left\{w_n : 2^k \leq n < 2^{k+1}\right\}$$

for all $k \in \mathbf{P}$ and $\kappa_0 = w_0$.

V. A. Skvortsov (see [15]) gave a relation between the Walsh-Kaczmarz functions and the Walsh-Paley functions by the help of the transformation $\tau_A : G \to G$ defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, ..., x_1, x_0, x_A, x_{A+1}, ...)$$

for $A \in \mathbf{N}$. By the definition of τ_A, we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbf{N}, x \in G).$$

The Dirichlet kernels are defined by

$$D_n^{\alpha} := \sum_{k=0}^{n-1} \alpha_k,$$

where $\alpha_n = w_n$ or κ_n $(n \in \mathbf{P})$, $D_0^{\alpha} := 0$. The 2^nth Dirichlet kernels have a closed form (see e.g. [8])

$$D_{2^n}^{w}(x) = D_{2^n}^{\kappa}(x) = D_{2^n}(x) = \begin{cases} 0, & \text{if } x \notin I_n \\ 2^n, & \text{if } x \in I_n. \end{cases}$$

The σ-algebra generated by the dyadic intervals of measure 2^{-k} will be denoted by F_k $(k \in \mathbf{N})$.

Denote by $f = \left(f^{(n)}, n \in \mathbf{N}\right)$ a martingale with respect to $(F_n, n \in \mathbf{N})$ (for details see, e. g. [20]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbf{N}} \left|f^{(n)}\right|.$$

In case $f \in L_1(G)$, the maximal function can also be given by

$$f^*(x) = \sup_{n \in \mathbf{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right|, \quad x \in G.$$

For $0 < p < \infty$ the Hardy martingale space $H_p(G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(G)$, then it is easy to show that the sequence $(S_{2^n} f : n \in \mathbf{N})$ is a martingale. If f is a martingale, that is $f = (f^{(0)}, f^{(1)}, ...)$ then the Walsh-(Kaczmarz)-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i) = \lim_{k \to \infty} \int_G f^{(k)}(x)\, \alpha_i(x)\, d\mu(x) \quad (\alpha = w \text{ or } \kappa).$$

The Walsh-(Kaczmarz)-Fourier coefficients of $f \in L_1(G)$ are the same as the ones of the martingale $(S_{2^n} f : n \in \mathbf{N})$ obtained from f.

The two-dimensional dyadic cubes are of the form

$$I_{n,n}(x,y) := I_n(x) \times I_n(y).$$

The σ-algebra generated by the dyadic rectangles $\{I_{n,n}(x,y) : (x,y) \in G \times G\}$is denoted by $F_{n,n}$.

Denote by $f = \left(f^{(n,n)}, n \in N\right)$ a martingale with respect to $(F_{n,n}, n \in \mathbf{N})$ (for details see, e. g. [20]). The maximal function of a martingale f is defined by

$$f^{\square} = \sup_{n \in N} \left| f^{(n,n)} \right|.$$

In case $f \in L_1(G \times G)$, the maximal function can also be given by

$$f^{\square}(x,y) = \sup_{n \in \mathbf{N}} \frac{1}{\mu(I_{n,n}(x,y))} \left| \int\limits_{I_{n,n}(x,y)} f(u,v) \, d\mu(u,v) \right|,$$

$$(x,y) \in G \times G,$$

For $0 < p < \infty$ the Hardy martingale space $H_p^{\square}(G \times G)$ consists of all martingales for which

$$\|f\|_{H_p} := \left\| f^{\square} \right\|_p < \infty.$$

Let

$$I_{n,m}(x,y) := I_n(x) \times I_m(y).$$

The σ-algebra generated by the dyadic rectangles $\{I_{n,m}(x,y) : (x,y) \in G \times G\}$ will be denoted by $F_{n,m}(n,m \in \mathbf{N})$.

Denote by $f = \left(f^{(n,m)}, n,m \in \mathbf{N}\right)$ a martingale with respect to $(F_{n,m}, n,m \in \mathbf{N})$ (for details see, e. g. [20]).

The maximal function of a martingale f is defined by

$$f^{*} = \sup_{n,m \in \mathbf{N}} \left| f^{(n,m)} \right|.$$

For $0 < p < \infty$ the Hardy martingale space $H_p(G \times G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^{*}\|_p < \infty$$

.

In case $f \in L_1(G \times G)$, maximal functions can also be given by

$$f^{*}(x,y) = \sup_{n,m \in \mathbf{N}} \frac{1}{\mu(I_{n,m}(x,y))} \left| \int\limits_{I_{n,m}(x,y)} f(u,v) \, d\mu(u,v) \right|$$

2. THE ONE-DIMENSIONAL MAXIMAL OPERATOR

For $n = 1, 2, ...$ and a martingale f the Fejér means of the Walsh-(Kaczmarz)-Fourier series of the function f is given by

$$\sigma_n^\alpha f(x) = \frac{1}{n} \sum_{j=0}^{n-1} S_j^\alpha(f; x) \quad (\alpha = w \text{ or } \kappa).$$

For a martingale f we consider the maximal operator

$$\sigma^{\alpha,*} f = \sup_{n\in\mathbf{P}} \left|\sigma_n^\alpha f(x)\right| \quad (\alpha = w \text{ or } \kappa).$$

The nth Fejér kernel of the Walsh-(Kaczmarz)-Fourier series defined by

$$K_n^\alpha(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k^\alpha(x) \quad (\alpha = w \text{ or } \kappa).$$

A bounded measurable function a is a p-atom, if there exists a dyadic interval I, such that

a) $\int_I a d\mu = 0$;
b) $\|a\|_\infty \le \mu(I)^{-1/p}$;
c) $\operatorname{supp} a \subset I$.

The basic result of atomic decomposition is the following one.

Theorem A. *(Weisz [20]). A martingale* $f = \left(f^{(n)} : n \in \mathbf{N}\right)$ *is in* $H_p\,(0 < p \le 1)$ *if and only if there exists a sequence* $(a_k, k \in \mathbf{N})$ *of* p*-atoms and a sequence* $(\mu_k, k \in \mathbf{N})$ *of real numbers such that for every* $n \in \mathbf{N}$,

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = f^{(n)}, \tag{1}$$

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p\right)^{1/p},$$

where the infimum is taken over all decompositions of f *of the form (1).*

We will use the following lemma of Goginava:

Lemma 1. *(Goginava [6]) Let* $2 < A \in \mathbf{P}$ *and* $q_A := 2^{2A} + 2^{2A-2} + \cdots + 2^2 + 2^0$. *Then*

$$q_{A-1} \left|K_{q_{A-1}}^w(x)\right| \ge 2^{2m+2s-3}$$

for $x \in I_{2A}(0, ..., 0, x_{2m} = 1, 0, ..., 0, x_{2s} = 1, x_{2s+1}, ..., x_{2A-1})$, $\quad m = 0, 1, ..., A-3, \quad s = m+2, m+3, ..., A-1.$

We will prove the following theorem.

Theorem 1. *There exists a martingale* $f \in H_{1/2}(G)$ *such that*

$$\left\|\sigma^{\kappa,*}f\right\|_{1/2} = +\infty.$$

Proof. Since $\frac{2^{m_k}}{m_k} \uparrow \infty$ as $k \to \infty$ it is easy to show that there exists an increasing sequence of positive integers $(m_k : k \in \mathbf{N})$ such that

$$\sum_{k=0}^{\infty} \frac{1}{m_k^{1/2}} < \infty, \tag{2}$$

$$\sum_{l=0}^{k-1} \frac{2^{4m_l}}{m_l} < \frac{2^{4m_k}}{m_k}, \tag{3}$$

$$\frac{k 2^{4m_{k-1}}}{m_{k-1}} \leq \frac{2^{2m_k}}{m_k}. \tag{4}$$

Let

$$f^{(A)}(x) := \sum_{k, 2m_k < A} \lambda_k a_k, \text{ where } \lambda_k := \frac{1}{m_k}$$

and

$$a_k(x) := 2^{2m_k}\left(D_{2^{2m_k+1}}(x) - D_{2^{2m_k}}(x)\right).$$

The martingale $f := \left(f^{(0)}, f^{(1)}, ..., f^{(A)}, ...\right)$ is in $H_{1/2}(G)$. Indeed, since

$$S_{2^A} a_k(x) = \begin{cases} 0, & \text{if } A \leq 2m_k, \\ a_k(x), & \text{if } A > 2m_k, \end{cases}$$

and

$$f^{(A)}(x) = \sum_{k: 2m_k < A} \lambda_k a_k(x) = \sum_{k=0}^{\infty} \lambda_k S_{2^A} a_k(x)$$

by (2) and Theorem A we conclude that $f \in H_{1/2}(G)$.

Now, we investigate the Fourier coefficients.

Let $j \in \{2^{2m_k}, ..., 2^{2m_k+1} - 1\}$ for some $k = 0, 1, 2, ...$ Then it is evident that

$$\widehat{f}^{\kappa}(j) := \lim_{A\to\infty} \widehat{f^{(A)}}^{\kappa}(j) = \frac{2^{2m_k}}{m_k}$$

and $\widehat{f}^{\kappa}(j) = 0$, if $j \notin \{2^{2m_k}, ..., 2^{2m_k+1} - 1\}$, $k = 0, 1, 2, ...$

Now, we decompose the q_{m_k}th Walsh-Kaczmarz-Fejér means as follows. (For the definition of q_{m_k} see Lemma 1 of Goginava.)

$$\sigma^{\kappa}_{q_{m_k}} f(x) = \frac{1}{q_{m_k}} \sum_{j=0}^{2^{2m_k}-1} S^{\kappa}_j f(x) + \frac{1}{q_{m_k}} \sum_{j=2^{2m_k}}^{q_{m_k}-1} S^{\kappa}_j f(x) = I + II. \tag{5}$$

Let $j < 2^{2m_k}$. Then (3) gives that

$$\left|S_j^\kappa f(x)\right| \leq \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} \left|\widehat{f}^\kappa(v)\right| \leq \sum_{l=0}^{k-1} \frac{2^{4m_l}}{m_l} < 2\frac{2^{4m_{k-1}}}{m_{k-1}}$$

and

$$I \leq c\frac{1}{q_{m_k}} \sum_{j=0}^{2^{2m_k}-1} \left|S_j^\kappa f(x)\right| \leq c\frac{2^{4m_{k-1}}}{m_{k-1}}. \tag{6}$$

Now, we discuss II.

For $2^{2m_k} \leq j < q_{m_k}$ we have the following:

$$\begin{aligned} S_j^\kappa f(x) &= \sum_{v=0}^{2^{2m_{k-1}+1}-1} \widehat{f}^\kappa(v)\kappa_v(x) + \sum_{v=2^{2m_k}}^{j-1} \widehat{f}^\kappa(v)\kappa_v(x) \\ &= \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} \widehat{f}^\kappa(v)\kappa_v(x) + \sum_{v=2^{2m_k}}^{j-1} \widehat{f}^\kappa(v)\kappa_v(x) \\ &= \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} \frac{2^{2m_l}}{m_l}\kappa_v(x) + \frac{2^{2m_k}}{m_k} \sum_{v=2^{2m_k}}^{j-1} \kappa_v(x) \\ &= \sum_{l=0}^{k-1} \frac{2^{2m_l}}{m_l}\left(D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)\right) + \frac{2^{2m_k}}{m_k}\left(D_j^\kappa(x) - D_{2^{2m_k}}(x)\right). \end{aligned} \tag{7}$$

This gives that

$$\begin{aligned} II &= \frac{\left(q_{m_k} - 2^{2m_k}\right)}{q_{m_k}} \sum_{l=0}^{k-1} \frac{2^{2m_l}}{m_l}\left(D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)\right) \\ &\quad + \frac{2^{2m_k}}{q_{m_k} m_k} \sum_{j=2^{2m_k}}^{q_{m_k}-1} \left(D_j^\kappa(x) - D_{2^{2m_k}}(x)\right) \\ &=: II_1 + II_2. \end{aligned}$$

To discuss II_1, we use (3) and $|D_{2^n}(x)| \leq 2^n$. Thus, we can write

$$|II_1| \leq c\sum_{l=0}^{k-1} \frac{2^{4m_l}}{m_l} < c\frac{2^{4m_{k-1}}}{m_{k-1}}. \tag{8}$$

From $\sigma_{q_{m_k}}^\kappa f(x) = I + II_1 + II_2$. and (6), (8) we have

$$|\sigma_{q_{m_k}}^\kappa f(x)| \geq |II_2| - |I| - |II_1| \geq |II_2| - c\frac{2^{4m_{k-1}}}{m_{k-1}}. \tag{9}$$

Now, we discuss II_2.

We can write the nth Dirichlet kernel with respect to the Walsh-Kaczmarz system in the following form:

$$\begin{aligned} D_n^{\kappa}(x) &= D_{2^{|n|}}(x)+\sum_{k=2^{|n|}}^{n-1} r_{|k|}(x) w_{k-2^{|n|}}(\tau_{|k|}(x)) \\ &= D_{2^{|n|}}(x)+r_{|n|}(x) D_{n-2^{|n|}}^{w}(\tau_{|n|}(x)). \end{aligned} \tag{10}$$

By the help of this, we immediately get

$$\begin{aligned} |II_2| &= \frac{2^{2m_k}}{q_{m_k} m_k}\left|\sum_{j=0}^{q_{m_k-1}-1}\left(D_{j+2^{2m_k}}^{\kappa}(x)-D_{2^{2m_k}}(x)\right)\right| \\ &= \frac{2^{2m_k}}{q_{m_k} m_k}\left|r_{2m_k}(x)\sum_{j=0}^{q_{m_k-1}-1} D_j^{w}\left(\tau_{2m_k}(x)\right)\right| \\ &= \frac{2^{2m_k}}{m_k}\frac{q_{m_k-1}}{q_{m_k}}\left|K_{q_{m_k-1}}^{w}\left(\tau_{2m_k}(x)\right)\right| \\ &\geq c\frac{q_{m_k-1}}{m_k}\left|K_{q_{m_k-1}}^{w}\left(\tau_{2m_k}(x)\right)\right|. \end{aligned}$$

Thus, from (9) we have

$$|\sigma_{q_{m_k}}^{\kappa} f(x)| \geq c\frac{q_{m_k-1}}{m_k}\left|K_{q_{m_k-1}}^{w}\left(\tau_{2m_k}(x)\right)\right|-c\frac{2^{4m_{k-1}}}{m_{k-1}}.$$

Define the set $J_{2A}^{l,s}(x)$ for $l<s<A$ by

$$J_{2A}^{l,s}(x):=I_{2A}(x_0,x_1,\ldots,x_{2A-2s-2},x_{2A-2s-1}=1,0,\ldots,0,x_{2A-2l-1}=1,0,\ldots,0).$$

Let $x\in J_{2m_k}^{l,s}(x)$, for some $l=[m_k/2],[m_k/2]+1,\ldots,m_k-3$, and $s=l+2,l+3,\ldots,m_k-1$, then from Lemma 1 and (4) we have

$$\left|\sigma_{q_{m_k}}^{\kappa} f(x)\right| \geq \frac{c2^{2l+2s}}{m_k}-\frac{c2^{4m_{k-1}}}{m_{k-1}} \geq \frac{c2^{2l+2s-1}}{m_k}.$$

Hence,

$$\begin{aligned} \int_G |\sigma^{\kappa,*} f(x)|^{1/2} d\mu(x) &\geq \int_G |\sigma_{q_{m_k}}^{\kappa} f(x)|^{1/2} d\mu(x) \\ &\geq \sum_{l=[m_k/2]}^{m_k-3}\sum_{s=l+2}^{m_k-1}\sum_{\substack{x_i=0 \\ i\in\{0,1,\ldots,2m_k-2s-2\}}}^{1}\int_{J_{2m_k}^{l,s}(x)} |\sigma_{q_{m_k}}^{\kappa} f(x)|^{1/2} d\mu(x) \\ &\geq \frac{c}{m_k^{1/2}}\sum_{l=[m_k/2]}^{m_k-3}\sum_{s=l+2}^{m_k-1} 2^{2m_k-2s}\frac{1}{2^{2m_k}}2^{l+s} \\ &\geq \frac{c}{m_k^{1/2}}\sum_{l=[m_k/2]}^{m_k-3}\sum_{s=l+2}^{m_k-1}\frac{2^l}{2^s} \geq c m_k^{1/2} \to \infty \text{ as } k\to\infty. \end{aligned}$$

That is $\|\sigma^{\kappa,*} f\|_{1/2}=+\infty$. The proof is complete. □

3. THE TWO-DIMENSIONAL RESTRICTED MAXIMAL OPERATOR

For $\alpha = w$ or κ the rectangular partial sums of the double Walsh-(Kaczmarz)-Fourier series are defined as follows:

$$S^{\alpha}_{M,N}f(x,y) := \sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\widehat{f}(i,j)\,\alpha_i(x)\,\alpha_j(y),$$

where the number

$$\widehat{f}(i,j) = \int\limits_{G\times G} f(x,y)\,\alpha_i(x)\,\alpha_j(y)\,d\mu(x,y).$$

is said to be the (i,j)th Walsh-(Kaczmarz)-Fourier coefficient of the function f.

If $f \in L_1(G\times G)$ then it is easy to show that the sequence $(S_{2^n,2^n}(f) : n \in \mathbf{N})$ is a martingale. If f is a martingale, that is $f = (f^{(n,n)} : n \in \mathbf{N})$ then the Walsh-(Kaczmarz)-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}^{\alpha}(i,j) = \lim_{k\to\infty}\int\limits_{G\times G} f^{(k)}(x,y)\,\alpha_i(x)\,\alpha_j(y)\,d\mu(x,y).$$

The Walsh-(Kaczmarz)-Fourier coefficients of $f \in L_1(G\times G)$ are the same as the ones of the martingale $(S_{2^n,2^n}(f) : n \in \mathbf{N})$ obtained from f.

For $n,m \in \mathbf{P}$ and a martingale f the (n,m)th Fejér mean of the double Walsh-(Kaczmarz)-Fourier series is given by

$$\sigma^{\alpha}_{n,m}f(x,y) = \frac{1}{nm}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}S^{\alpha}_{i,j}f(x,y).$$

For the martingale f the restricted maximal operator is defined by

$$\sigma^{\alpha,\square}_{\lambda}f(x,y) = \sup_{2^{-\lambda}\le n/m\le 2^{\lambda}}|\sigma^{\alpha}_{n,m}f(x,y)|.$$

A bounded measurable function a is a p-atom, if there exists a dyadic 2-dimensional cube $I\times I$, such that

a) $\int\limits_{I\times I} a\,d\mu = 0$;
b) $\|a\|_{\infty} \le \mu(I\times I)^{-1/p}$;
c) $\operatorname{supp} a \subset I\times I$.

The basic result of atomic decomposition is the following one.

Theorem B. *(Weisz [20]). A martingale $f = \left(f^{(n,n)} : n \in \mathbf{N}\right)$ is in $H^{\square}_p$ $(0 < p \le 1)$ if and only if there exists a sequence $(a_k, k \in \mathbf{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbf{N})$ of real numbers such that for every $n \in \mathbf{N}$,*

$$\sum_{k=0}^{\infty}\mu_k S_{2^n,2^n}a_k = f^{(n,n)}, \tag{11}$$

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p^{\square}} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

We will prove the following theorem.

Theorem 2. *There exists a martingale* $f \in H_{1/2}^{\square}(G \times G)$ *such that*

$$\left\|\sigma^{\kappa,\square} f\right\|_{weak-L_{1/2}} = +\infty.$$

Proof. To prove Theorem 2 we modify the sequence $\{m_k : k \in \mathbf{P}\}$ and atoms a_k given in the previous section in the following way.

Let $\{m_k : k \in \mathbf{N}\}$ be an increasing sequence of positive integers such that

$$\sum_{k=0}^{\infty} \frac{1}{m_k^{1/2}} < \infty, \tag{12}$$

$$\sum_{l=0}^{k-1} \frac{2^{8m_l}}{m_l} < \frac{2^{8m_k}}{m_k}, \tag{13}$$

$$\frac{2^{8m_{k-1}}}{m_{k-1}} < \frac{2^{m_k}}{km_k}. \tag{14}$$

Let

$$f^{(A,A)}(x,y) := \sum_{k, 2m_k < A} \lambda_k a_k(x,y), \text{ where } \lambda_k := \frac{1}{m_k}$$

and

$$a_k(x,y) := 2^{4m_k} \left(D_{2^{2m_k+1}}(x) - D_{2^{2m_k}}(x)\right) \left(D_{2^{2m_k+1}}(y) - D_{2^{2m_k}}(y)\right).$$

The martingale $f := \left(f^{(0,0)}, f^{(1,1)}, ..., f^{(A,A)}, ...\right) \in H_{1/2}^{\square}(G \times G)$. Indeed,

$$S_{2^A,2^A} a_k(x,y) = \begin{cases} 0, & \text{if } A \le 2m_k, \\ a_k(x,y), & \text{if } A > 2m_k, \end{cases}$$

$$f^{(A,A)}(x) = \sum_{k, 2m_k < A} \lambda_k a_k(x,y) = \sum_{k=0}^{\infty} \lambda_k S_{2^A,2^A} a_k(x,y)$$

from (12) and Theorem B we conclude that $f \in H_{1/2}^{\square}(G \times G)$.

Now, we investigate the Fourier coefficients. Since

$$\int_{G \times G} f^{(A)}(x,y) \kappa_i(x) \kappa_j(y) \, d\mu(x,y)$$

$$= \begin{cases} 0, & (i,j) \notin \bigcup_{k=0}^{\infty} \{2^{2m_k}, ..., 2^{2m_k+1}-1\} \times \{2^{2m_k}, ..., 2^{2m_k+1}-1\}, \\ 0, & (i,j) \in \{2^{2m_k}, ..., 2^{2m_k+1}-1\} \times \{2^{2m_k}, ..., 2^{2m_k+1}-1\}, A = 0,1,...,2m_k, \\ \frac{2^{4m_k}}{m_k}, & (i,j) \in \{2^{2m_k}, ..., 2^{2m_k+1}-1\} \times \{2^{2m_k}, ..., 2^{2m_k+1}-1\}, A > 2m_k, \end{cases}$$

we can write

$$\widehat{f}^{\kappa}(i,j)=\begin{cases}\frac{2^{4m_k}}{m_k}, & (i,j)\in\{2^{2m_k},...,2^{2m_k+1}-1\}\times\{2^{2m_k},...,2^{2m_k+1}-1\},\\ 0, & (i,j)\notin\bigcup_{k=1}^{\infty}\{2^{2m_k},...,2^{2m_k+1}-1\}\times\{2^{2m_k},...,2^{2m_k+1}-1\}.\end{cases}\tag{15}$$

We decompose the (q_{m_k},q_{m_k})th Fejér means as follows

$$\begin{aligned}\sigma^{\kappa}_{q_{m_k}q_{m_k}}f(x,y) &= \frac{1}{q^2_{m_k}}\sum_{i=0}^{q_{m_k}-1}\sum_{j=0}^{q_{m_k}-1}S^{\kappa}_{i,j}f(x,y)\\ &= \frac{1}{q^2_{m_k}}\sum_{i=0}^{2^{2m_k}-1}\sum_{j=0}^{2^{2m_k}-1}S^{\kappa}_{i,j}f(x,y)+\frac{1}{q^2_{m_k}}\sum_{i=2^{2m_k}}^{q_{m_k}-1}\sum_{j=0}^{2^{2m_k}-1}S^{\kappa}_{i,j}f(x,y)\\ &\quad+\frac{1}{q^2_{m_k}}\sum_{i=0}^{2^{2m_k}-1}\sum_{j=2^{2m_k}}^{q_{m_k}-1}S^{\kappa}_{i,j}f(x,y)+\frac{1}{q^2_{m_k}}\sum_{i=2^{2m_k}}^{q_{m_k}-1}\sum_{j=2^{2m_k}}^{q_{m_k}-1}S^{\kappa}_{i,j}f(x,y)\\ &= I+II+III+IV.\end{aligned}\tag{16}$$

Let

$$\begin{aligned}(i,j) &\in \left(\left\{2^{2m_k},...,q_{m_k}-1\right\}\times\left\{0,1,...,2^{2m_k}-1\right\}\right)\\ &\quad\cup\left(\left\{0,1,...,2^{2m_k}-1\right\}\times\left\{2^{2m_k},...,q_{m_k}-1\right\}\right)\\ &\quad\cup\left(\left\{0,1,...,2^{2m_k}-1\right\}\times\left\{0,1,...,2^{2m_k}-1\right\}\right).\end{aligned}$$

for some k. Then from (15) and (13) it is easy to show that

$$\left|S^{\kappa}_{i,j}f(x,y)\right|\leq\sum_{l=0}^{k-1}\sum_{\nu=2^{2m_l}}^{2^{2m_l+1}-1}\sum_{\mu=2^{2m_l}}^{2^{2m_l+1}-1}\left|\widehat{f}^{\kappa}(\nu,\mu)\right|\leq\sum_{l=0}^{k-1}\frac{2^{8m_l}}{m_l}\leq\frac{C2^{m_k}}{km_k}.$$

Consequently, we have

$$|I|\leq\frac{1}{q^2_{m_k}}\sum_{i=0}^{2^{2m_k}-1}\sum_{j=0}^{2^{2m_k}-1}\left|S^{\kappa}_{i,j}f(x,y)\right|\leq C\frac{2^{4m_k}}{q^2_{m_k}}\frac{2^{m_k}}{km_k}\leq\frac{C2^{m_k}}{km_k},\tag{17}$$

$$|II|\leq\frac{2^{2m_k}(q_{m_k}-2^{2m_k})}{q^2_{m_k}}\frac{2^{m_k}}{km_k}\leq C\frac{2^{m_k}}{km_k}\tag{18}$$

and

$$|III|\leq\frac{C2^{m_k}}{km_k}.\tag{19}$$

Combining (16)-(19) we obtain that

$$\left|\sigma^{\kappa}_{q_{m_k},q_{m_k}}f(x,y)\right|\geq|IV|-\frac{C2^{m_k}}{km_k}.\tag{20}$$

Now, we discuss IV.

Let $(i,j)\in\{2^{2m_k},...,q_{m_k}-1\}\times\{2^{2m_k},...,q_{m_k}-1\}$. Then from (15) we have

$$\begin{aligned}
S_{i,j}^{\kappa}f(x,y) &= \sum_{\nu=0}^{i-1}\sum_{\mu=0}^{j-1}\widehat{f}^{\kappa}(\nu,\mu)\,\kappa_{\nu}(x)\kappa_{\mu}(y)\\
&= \sum_{l=0}^{k-1}\sum_{\nu=2^{2m_l}}^{2^{2m_l+1}-1}\sum_{\mu=2^{2m_l}}^{2^{2m_l+1}-1}\widehat{f}^{\kappa}(\nu,\mu)\,\kappa_{\nu}(x)\kappa_{\mu}(y)\\
&\quad+\sum_{\nu=2^{2m_k}}^{i-1}\sum_{\mu=2^{2m_k}}^{j-1}\widehat{f}^{\kappa}(\nu,\mu)\,\kappa_{\nu}(x)\kappa_{\mu}(y)\\
&= \sum_{l=0}^{k-1}\frac{2^{4m_l}}{m_l}\left(D_{2^{2m_l+1}}(x)-D_{2^{2m_l}}(x)\right)\times\left(D_{2^{2m_l+1}}(y)-D_{2^{2m_l}}(y)\right)\\
&\quad+\frac{2^{4m_k}}{m_k}\left(D_i^{\kappa}(x)-D_{2^{2m_k}}(x)\right)\left(D_j^{\kappa}(y)-D_{2^{2m_k}}(y)\right)
\end{aligned}\tag{21}$$

and

$$\begin{aligned}
IV &= \frac{1}{q_{m_k}^2}\left(q_{m_k}-2^{2m_k}\right)^2\sum_{l=0}^{k-1}\frac{2^{4m_l}}{m_l}\left(D_{2^{2m_l+1}}(x)-D_{2^{2m_l}}(x)\right)\left(D_{2^{2m_l+1}}(y)-D_{2^{2m_l}}(y)\right)\\
&\quad+\frac{1}{q_{m_k}^2}\frac{2^{4m_k}}{m_k}\left(\sum_{i=2^{2m_k}}^{q_{m_k}-1}\sum_{j=2^{2m_k}}^{q_{m_k}-1}\left(D_i^{\kappa}(x)-D_{2^{2m_k}}(x)\right)\left(D_j^{\kappa}(y)-D_{2^{2m_k}}(y)\right)\right)\\
&= IV_1+IV_2.
\end{aligned}$$

By (13), (14) and $|D_{2^n}(x)|\leq 2^n$ we get that

$$|IV_1|\leq C\sum_{l=0}^{k-1}\frac{2^{8m_l}}{m_l}\leq C\frac{2^{m_k}}{km_k}$$

and

$$\left|\sigma_{q_{m_k},q_{m_k}}^{\kappa}f(x,y)\right|\geq|IV_2|-\frac{C2^{m_k}}{km_k}.$$

By the help of the equation (10) we immediatelly have for IV_2

$$\begin{aligned}
IV_2 &= \frac{1}{q_{m_k}^2}\frac{2^{4m_k}}{m_k}r_{2m_k}(x)r_{2m_k}(y)\sum_{i=0}^{q_{m_k-1}-1}D_i^w(\tau_{2m_k}(x))\sum_{j=0}^{q_{m_k-1}-1}D_j^w(\tau_{2m_k}(y))\\
&= \frac{1}{q_{m_k}^2}\frac{2^{4m_k}}{m_k}r_{2m_k}(x)r_{2m_k}(y)q_{m_k-1}^2K_{q_{m_k-1}}^w(\tau_{2m_k}(x))K_{q_{m_k-1}}^w(\tau_{2m_k}(y)).
\end{aligned}$$

Let $(x,y)\in J_{2m_k}^{l_1,l_1+2}(x)\times J_{2m_k}^{l_2,l_2+2}(y)$, where $(l_1,l_2)\in\{0,1,...,m_k-3\}\times\{0,1,...,m_k-3\}$. Then from Lemma 1 we can write

$$q_{m_k-1}\left|K_{q_{m_k-1}}^w(\tau_{2m_k}(x))\right|\geq C2^{4l_1}\text{ and }q_{m_k-1}\left|K_{q_{m_k-1}}^w(\tau_{2m_k}(y))\right|\geq C2^{4l_2}.$$

Consequently,

$$\left|\sigma^{\kappa}_{q_{m_k},q_{m_k}} f(x,y)\right| \geq \frac{C}{m_k} 2^{4l_1+4l_2} - C\frac{2^{m_k}}{km_k}. \tag{22}$$

Denote

$$A(m_k) := \left\{(l_1,l_2) : 0 \leq l_2 \leq m_k - 3, 0 \leq l_1 \leq \frac{m_k}{4}, l_1 + l_2 \geq \frac{m_k}{4}\right\}$$

and

$$\lambda'_k := \frac{c2^{m_k}}{m_k}.$$

For $(l_1,l_2) \in A\left(m_k\right)$, we have

$$\left|\sigma^{\kappa}_{q_{m_k},q_{m_k}} f\left(x,y\right)\right| \geq \frac{c2^{m_k}}{m_k}$$

and

$$\begin{aligned} &\mu\left\{(x,y) \in G \times G : \sigma^{\kappa,\Box} f(x,y) \geq C\lambda'_k\right\} \geq \\ \geq &\sum_{(l_1,l_2)\in A(m_k)} \mu\left\{(x,y) \in J^{l_1,l_1+2}_{2m_k}(x) \times J^{l_2,l_2+2}_{2m_k}(y) : \left|\sigma^{\kappa}_{q_{m_k},q_{m_k}} f(x,y)\right| \geq \lambda'_k\right\} \\ \geq &\; C\sum_{l_1=0}^{[m_k/4]} \sum_{l_2=[m_k/4]-l_1}^{m_k-3} \sum_{x_0=0}^{1} \cdots \sum_{x_{2m_k-2l_1-2}=0}^{1} \sum_{y_0=0}^{1} \cdots \sum_{y_{2m_k-2l_2-2}=0}^{1} \mu\left(J^{l_1,l_1+2}_{2m_k}(x) \times J^{l_2,l_2+2}_{2m_k}(y)\right) \\ \geq &\; C\sum_{l_1=0}^{[m_k/4]} \sum_{l_2=[m_k/4]-l_1}^{m_k-3} \frac{1}{2^{2l_1+2l_2}} \geq \frac{Cm_k}{2^{m_k/2}}. \end{aligned}$$

Consequently,

$$\lambda'_k \left(\mu\left\{(x,y) : \sigma^{\kappa,\Box} f(x,y) \geq C\lambda'_k\right\}\right)^2 \geq C\frac{2^{m_k}}{m_k}\frac{m_k^2}{2^{m_k}} = Cm_k \to \infty \text{ as } k \to \infty.$$

This completes the proof of this theorem. □

4. The two-dimensional unrestricted maximal operator

If $f \in L_1\left(G \times G\right)$ then it is easy to show that the sequence $\left(S_{2^n,2^m}\left(f\right) : n,m \in \mathbf{N}\right)$ is a martingale. If f is a martingale, that is $f = \left(f^{(n,m)} : n,m \in \mathbf{N}\right)$ then the Walsh-(Kaczmarz)-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}\left(i,j\right) = \lim_{\min(k,l)\to\infty} \int_{G\times G} f^{(k,l)}\left(x,y\right) \alpha_i\left(x\right) \alpha_j\left(y\right) d\mu\left(x,y\right).$$

The Walsh-(Kaczmarz)-Fourier coefficients of $f \in L_1\left(G \times G\right)$ are the same as the ones of the martingale $\left(S_{2^n,2^m}\left(f\right) : n,m \in \mathbf{N}\right)$ obtained from f.

For the martingale f the unrestricted maximal operator of the Fejér mean is defined by

$$\sigma^{\alpha,*} f\left(x,y\right) = \sup_{n,m\in\mathbf{N}} |\sigma^{\alpha}_{n,m}(f;x,y)|.$$

A function $a \in L_2$ is caled a rectangle p-atom if there exists a dyadic rectangle R such that

(a) $\operatorname{supp} a \subset R$,

(b) $\|a\|_2 \leq |R|^{1/2-1/p}$,
(c) $\int\limits_G a(x,y)\,d\mu(x) = \int\limits_G a(x,y)\,d\mu(y) = 0$ for all $x, y \in G$.

The basic result of atomic decomposition is the following one.

Theorem C. *(Weisz [20]). A martingale $f = \left(f^{(n,m)} : n, m \in \mathbf{N}\right)$ is in $H_p\,(0 < p \leq 1)$ if there exists a sequence $(a_k, k \in \mathbf{N})$ of rectangle p-atoms and a sequence $(\mu_k, k \in \mathbf{N})$ of real numbers such that for every $n, m \in \mathbf{N}$,*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n,2^m} a_k = f^{(n,m)}, \tag{23}$$

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p\right)^{1/p},$$

We will prove the following theorem.

Theorem 3. *There exists a martingale $f \in H_{1/2}(G \times G)$ such that*

$$\|\sigma^{\kappa,*} f\|_{weak-L_{1/2}} = +\infty.$$

Proof. Now, we use the sequence $\{\mu_k : k \in \mathbf{N}\}$ and the atoms a_k defined in the previous proof. Let

$$f^{(A,B)}(x,y) := \sum_{l:2m_l < \min(A,B)} \lambda_l a_l(x,y).$$

First, we prove that the martingale $f := \left(f^{(A,B)} : A, B \in \mathbf{N}\right)$ belongs to the Hardy space $H_{1/2}(G \times G)$. Indeed, since

$$\|a_l\|_2 \leq c2^{6m_l},$$

$$S_{2^A,2^B} a_k(x,y) = \begin{cases} 0, & \text{if } \min(A,B) \leq 2m_k, \\ a_k(x,y), & \text{if } \min(A,B) > 2m_k, \end{cases}$$

$$f^{(A,B)}(x,y) := \sum_{l:2m_l < \min(A,B)} \lambda_l a_l(x,y) = \sum_{k=0}^{\infty} \lambda_k S_{2^A,2^B} a_k(x,y)$$

from (12) and Theorem C we conclude that $f \in H_{1/2}(G \times G)$.

Now, we investigate the Fourier coefficients. Since

$$\int\limits_{G\times G} f^{(A,B)}(x,y)\,\kappa_i(x)\,\kappa_j(y)\,d\mu(x,y)$$

$$= \begin{cases} 0, & (i,j) \notin \bigcup\limits_{k=0}^{\infty} \{2^{2m_k}, ..., 2^{2m_k+1}-1\} \times \{2^{2m_k}, ..., 2^{2m_k+1}-1\}, \\ 0, & (i,j) \in \{2^{2m_k}, ..., 2^{2m_k+1}-1\} \times \{2^{2m_k}, ..., 2^{2m_k+1}-1\}, \min(A,B) \leq 2m_k, \\ \frac{2^{4m_k}}{m_k}, & (i,j) \in \{2^{2m_k}, ..., 2^{2m_k+1}-1\} \times \{2^{2m_k}, ..., 2^{2m_k+1}-1\}, \min(A,B) > 2m_k, \end{cases}$$

we can write

$$\widehat{f^{\kappa}}(i,j)=\begin{cases}\frac{2^{4m_k}}{m_k}, & (i,j)\in\{2^{2m_k},...,2^{2m_k+1}-1\}\times\{2^{2m_k},...,2^{2m_k+1}-1\},\\ 0, & (i,j)\notin\bigcup\limits_{k=1}^{\infty}\{2^{2m_k},...,2^{2m_k+1}-1\}\times\{2^{2m_k},...,2^{2m_k+1}-1\}.\end{cases}$$

The estimation of

$$\mu\{(x,y):\sigma^{\kappa,*}f(x,y)\geq C\lambda_k\}$$

is analogous to the estimation of

$$\mu\left\{(x,y):\sigma^{\kappa,\Box}f(x,y)\geq C\lambda_k\right\}$$

and we have that

$$\sup_{\lambda>0}\lambda\mu\{(x,y):\sigma^{\kappa,*}f(x,y)\geq\lambda\}^2=\infty.$$

Theorem 3 is proved. □

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