ON THE MAXIMAL OPERATOR OF (C,α)-MEANS OF WALSH-KACZMARZ-FOURIER SERIES

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Abstract. Simon **[J. Approx. Theory, 127 (2004), 39-60]** proved that the maximal operator $\sigma^{\alpha,\kappa,*}$ of the (C,α)-means of the Walsh-Kaczmarz-Fourier series is bounded from the martingale Hardy space H_p to the space L_p for $p>1/(1+\alpha), 0<\alpha\leq 1$.

Recently, Gát and Goginava [2] proved that this boundedness result does not hold if $p\leq 1/(1+\alpha)$. However, in the endpoint case $p=1/(1+\alpha)$ the maximal operator $\sigma^{\alpha,\kappa,*}$ is bounded from the martingale Hardy space $H_{1/(1+\alpha)}$ to the space weak-$L_{1/(1+\alpha)}$.

The main aim of this paper is to prove a stronger result, that is for any $0<p\leq 1/(1+\alpha)$ there exists a martingale $f\in H_p$ such that the maximal operator $\sigma^{\alpha,\kappa,*}$ does not belong to the space L_p.

1. Introduction

In 1948 Šneider [8] introduced the Walsh-Kaczmarz system and showed that the inequality

$$\limsup_{n\to\infty}\frac{D_n^{\kappa}(x)}{\log n}\geq C>0$$

holds a.e. In 1974 Schipp [4] and Young [9] proved that the Walsh-Kaczmarz system is a convergence system. Skvortsov in 1981 [7] showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to f for any continuous functions f. Gát [1] proved, for any integrable functions, that the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere to the function and Gát proved that $\|\sigma^{\kappa *}\|_1\leq C\||f|\|_{H_1}$. Gát's result was extended to the Hardy space by Simon [6], who proved that $\sigma^{\kappa *}$ is of type (H_p,L_p) for $p>1/2$. Weisz [11] showed that in endpoint case $p=1/2$ the maximal operator is of weak type $\left(H_{1/2},L_{1/2}\right)$.

In paper [5] Simon proved the (H_p,L_p)-boundedness of the maximal operator of (C,α)-means of Walsh-Kaczmarz-Fourier series, where $0<\alpha\leq 1$ and $1/(1+\alpha)<p\leq 1$.

In the paper [2] Gát and Goginava proved that in theorem of Simon the assumption $p>1/(1+\alpha)$ is essential, namely, this boundedness result does not hold if $p\leq 1/(1+\alpha)$. However, in the endpoint case $p=1/(1+\alpha)$ the maximal operator $\sigma^{\alpha,\kappa,*}$ is bounded from the martingale Hardy space $H_{1/(1+\alpha)}$ to the space weak-$L_{1/(1+\alpha)}$.

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The main aim of this paper is to prove a stronger result, for any $0 < p \leq 1/(1+\alpha)$ there exists a martingale $f \in H_p$ such that

$$\|\sigma^{\alpha,\kappa,*}f\|_p = +\infty.$$

2. Dyadic Hardy space and (C,α)-means

Now, we give a brief introduction to the theory of dyadic analysis [3]. Let denote by $\mathbf{Z}_2$ the discrete cyclic group of order 2, the group operation is the modulo 2 addition and every subset is open. The normalized Haar measure on $\mathbf{Z}_2$ is given in the way that the measure of a singleton is 1/2. Let $G := \underset{k=0}{\overset{\infty}{\times}} \mathbf{Z}_2$, G be called the Walsh group. The elements of G are sequences $x = (x_0, x_1, ..., x_k, ...)$ with $x_k \in \{0,1\}$ $(k \in \mathbf{N})$.

The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure (denoted by μ) and the topology are the product measure and topology. Dyadic intervalls are defined by

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y = (x_0, ..., x_{n-1}, y_n, y_{n+1} ...)\}$$

for $x \in G, n \in \mathbf{P}$. They form a base for the neighborhoods of G. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \mathbf{N}$.

Let L_p denote the usual Lebesgue spaces on G (with the corresponding norm or quasinorm $\|.\|_p$).

The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbf{N}).$$

Let the Walsh-Paley functions be the product functions of the Rademacher functions. Namely, each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0,1\} \ (i \in \mathbf{N}),$$

where only a finite number of n_i's different from zero. Let the order of $n > 0$ be denoted by $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$. Walsh-Paley functions are $w_0 = 1$ and for $n \geq 1$

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k}.$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 = 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The set of Walsh-Kaczmarz functions and the set of Walsh-Paley functions is the same in dyadic blocks. Namely,

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{w_n : 2^k \leq n < 2^{k+1}\}$$

for all $k \in \mathbf{P}$ and $\kappa_0 = w_0$.

Skvortsov (see [7]) gave a relation between the Walsh-Kaczmarz functions and the Walsh-Paley functions by the help of the transformation $\tau_A : G \to G$ defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, ..., x_1, x_0, x_A, x_{A+1}, ...)$$

for $A \in \mathbf{N}$. By the definition of τ_A, we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbf{N}, x \in G).$$

The Dirichlet kernels are defined by

$$D_n^{\psi} := \sum_{k=0}^{n-1} \psi_k,$$

where $\psi_n = w_n$ or κ_n $(n \in \mathbf{P})$, $D_0^{\alpha} := 0$. The 2^nth Dirichlet kernels have a closed form (see e.g. [3])

$$D_{2^n}^{w}(x) = D_{2^n}^{\kappa}(x) = D_{2^n}(x) = \begin{cases} 0, & \text{if } x \notin I_n \\ 2^n, & \text{if } x \in I_n. \end{cases}$$

If $f \in L_1(G)$, then the number

$$\widehat{f}^{\alpha}(n) = \int_G f \alpha_n$$

is said to the nth Walsh-(Kaczmarz)-Fourier coefficient.

Denote by S_n^{α} the nth partial sums of the Walsh-(Kaczmarz)-Fourier series of a function f, namely

$$S_n^{\alpha}(f; x) = \sum_{k=0}^{n-1} \widehat{f}^{\alpha}(k) \alpha_k.$$

The σ-algebra generated by the dyadic intervals of measure 2^{-k} will be denoted by F_k $(k \in \mathbf{N})$.

Denote by $f = \left(f^{(n)}, n \in \mathbf{N}\right)$ a martingale with respect to $(F_n, n \in \mathbf{N})$ (for details see, e. g. [10]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbf{N}} \left| f^{(n)} \right|.$$

In case $f \in L_1(G)$, the maximal function can also be given by

$$f^*(x) = \sup_{n \in \mathbf{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right|, \quad x \in G.$$

For $0 < p < \infty$ the Hardy martingale space $H_p(G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(G)$, then it is easy to show that the sequence $(S_{2^n} f : n \in \mathbf{N})$ is a martingale. If f is a martingale, that is $f = (f^{(0)}, f^{(1)}, ...)$ then the Walsh-(Kaczmarz)-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i) = \lim_{k \to \infty} \int_G f^{(k)}(x)\, \psi_i(x)\, d\mu(x) \quad (\psi = w \text{ or } \kappa).$$

The Walsh-(Kaczmarz)-Fourier coefficients of $f \in L_1(G)$ are the same as the ones of the martingale $(S_{2^n} f : n \in \mathbf{N})$ obtained from f.

Set $A_n^\alpha := \frac{(1+\alpha)\ldots(n+\alpha)}{n!}$ for any $n \in \mathbf{N}, \alpha \in \mathbf{R}$ $(\alpha \neq -1, -2, \ldots)$. It is known that $A_n^\alpha \sim n^\alpha$. For $n = 1, 2, \ldots$ and a martingale f the (C, α)-means of the Walsh-(Kaczmarz)-Fourier series of the function f is given by

$$\sigma_n^{\alpha,\psi} f(x) = \frac{1}{A_{n-1}^\alpha} \sum_{j=1}^{n} A_{n-j}^{\alpha-1} S_j^\psi (f; x) \quad (\psi = w \text{ or } \kappa).$$

For a martingale f we consider the maximal operator

$$\sigma^{\alpha,\psi,*} f = \sup_{n \in \mathbf{P}} \left| \sigma_n^{\alpha,\psi} f(x) \right| \quad (\psi = w \text{ or } \kappa).$$

The nth (C, α)-kernel of the Walsh-(Kaczmarz)-Fourier series defined by

$$K_n^{\alpha.\psi}(x) := \frac{1}{A_{n-1}^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} D_k^\psi (x) \quad (\psi = w \text{ or } \kappa).$$

A bounded measurable function a is a p-atom, if there exists a dyadic interval I, such that

a) $\int_I a d\mu = 0$;
b) $\|a\|_\infty \leq \mu(I)^{-1/p}$;
c) $\operatorname{supp} a \subset I$.

The basic result of atomic decomposition is the following one.

Theorem A. *(Weisz [10]). A martingale* $f = \left(f^{(n)} : n \in \mathbf{N}\right)$ *is in* H_p $(0 < p \leq 1)$ *if and only if there exists a sequence* $(a_k, k \in \mathbf{N})$ *of* p*-atoms and a sequence* $(\mu_k, k \in \mathbf{N})$ *of real numbers such that for every* $n \in \mathbf{N}$*,*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = f^{(n)}, \tag{1}$$

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of f *of the form (1).*

In the paper [5] Simon proved the following theorem

Theorem B. *Let* $0 < \alpha \leq 1$ *and* $1/(1+\alpha) < p \leq 1$*. Then there exists a constant* C *such that*

$$\|\sigma^{\alpha,\kappa,*} f\|_p \leq C \|f\|_{H_p}$$

for all $f \in H_p(G)$*.*

In this paper we prove that in theorem of Simon the assumption $p > 1/(1+\alpha)$ is essential. Moreover, we prove that the following is true.

Theorem 1. *Let* $0 < \alpha \leq 1$ *and* $0 < p \leq 1/(1+\alpha)$. *Then there exists a martingale* $f \in H_p(G)$ *such that*

$$\left\|\sigma^{\alpha,\kappa,*} f\right\|_p = +\infty.$$

3. PROOF OF MAIN RESULT

Proof. Let $(m_k : k \in \mathbf{N})$ be an increasing sequence of positive integers such that

$$\sum_{k=0}^{\infty} \frac{1}{m_k^p} < \infty, \tag{2}$$

$$\sum_{l=0}^{k-1} \frac{2^{2m_l/p}}{m_l} < \frac{2^{2m_k/p}}{m_k}, \tag{3}$$

$$\frac{2^{2m_{k-1}/p}}{m_{k-1}} \leq \frac{2^{m_k}}{m_k}. \tag{4}$$

Let

$$f^{(A)}(x) := \sum_{k, 2m_k < A} \lambda_k a_k, \text{ where } \lambda_k := \frac{2}{m_k}$$

and

$$a_k(x) := 2^{2(1/p-1)m_k - 1}\left(D_{2^{2m_k+1}}(x) - D_{2^{2m_k}}(x)\right).$$

The martingale $f := \left(f^{(0)}, f^{(1)}, ..., f^{(A)}, ...\right)$ is in $H_p(G)$. Indeed, since

$$||a_k||_\infty = 2^{2m_k(1/p-1)-1} 2^{2m_k+1} = (\operatorname{supp} a_k)^{-1/p},$$

$$S_{2^A} a_k(x) = \begin{cases} 0, & \text{if } A \leq 2m_k, \\ a_k(x), & \text{if } A > 2m_k, \end{cases}$$

and

$$f^{(A)}(x) = \sum_{k: 2m_k < A} \lambda_k a_k(x) = \sum_{k=0}^{\infty} \lambda_k S_{2^A} a_k(x)$$

by (2) and Theorem A we conclude that $f \in H_p(G)$.

Now, we investigate the Fourier coefficients.

Let $j \in \{2^{2m_k}, ..., 2^{2m_k+1} - 1\}$ for some $k = 0, 1, 2, ...$ Then it is evident that

$$\widehat{f}^{\kappa}(j) := \lim_{A \to \infty} \widehat{f^{(A)}}^{\kappa}(j) = \frac{2^{2m_k(1/p-1)}}{m_k}$$

and $\widehat{f}^{\kappa}\ (j) = 0$, if $j \notin \{2^{2m_k}, ..., 2^{2m_k+1} - 1\}$, $k = 0, 1, 2, ...$

Set $q_{A.s} := 2^{2A} + 2^{2s}$ for any $A > s$. Now, we decompose the $q_{m_k,s}$th Walsh-Kaczmarz (C,α)-means as follows

$$\begin{aligned} \sigma^{\alpha,\kappa}_{q_{m_k,s}} f(x) &= \frac{1}{A_{q_{m_k,s}-1}} \sum_{j=1}^{2^{2m_k}-1} A^{\alpha-1}_{q_{m_k,s}-j} S^{\kappa}_{j} f(x) \\ &+ \frac{1}{A_{q_{m_k,s}-1}} \sum_{j=2^{2m_k}}^{q_{m_k,s}} A^{\alpha-1}_{q_{m_k,s}-j} S^{\kappa}_{j} f(x) \\ &= I + II. \end{aligned} \tag{5}$$

Let $j < 2^{2m_k}$. Then (3) gives that

$$\left|S^{\kappa}_{j} f(x)\right| \le \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} |\widehat{f}^{\kappa}(v)| \le \sum_{l=0}^{k-1} \frac{2^{2m_l(1/p-1)}}{m_l} 2^{2m_l} \le 2\frac{2^{2m_{k-1}/p}}{m_{k-1}}$$

and

$$|I| \le c\frac{1}{A_{q_{m_k,s}-1}} \sum_{j=1}^{2^{2m_k}-1} A^{\alpha-1}_{q_{m_k,s}-j} \left|S^{\kappa}_{j} f(x)\right| \le c(\alpha)\frac{2^{2m_{k-1}/p}}{m_{k-1}}. \tag{6}$$

Now, we discuss II.

For $2^{2m_k} \le j < q_{m_k,s}$ we have the following:

$$\begin{aligned} S^{\kappa}_{j} f(x) &= \sum_{v=0}^{2^{2m_{k-1}+1}-1} \widehat{f}^{\kappa}(v)\, \kappa_v(x) + \sum_{v=2^{2m_k}}^{j-1} \widehat{f}^{\kappa}(v)\, \kappa_v(x) \\ &= \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} \widehat{f}^{\kappa}(v)\, \kappa_v(x) + \sum_{v=2^{2m_k}}^{j-1} \widehat{f}^{\kappa}(v)\, \kappa_v(x) \\ &= \sum_{l=0}^{k-1} \sum_{v=2^{2m_l}}^{2^{2m_l+1}-1} \frac{2^{2m_l(1/p-1)}}{m_l} \kappa_v(x) + \frac{2^{2m_k(1/p-1)}}{m_k} \sum_{v=2^{2m_k}}^{j-1} \kappa_v(x) \\ &= \sum_{l=0}^{k-1} \frac{2^{2m_l(1/p-1)}}{m_l} \left(D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)\right) + \frac{2^{2m_k(1/p-1)}}{m_k} \left(D^{\kappa}_{j}(x) - D_{2^{2m_k}}(x)\right). \end{aligned} \tag{7}$$

This gives that

$$\begin{aligned} II &= \frac{1}{A^{\alpha}_{q_{m_k,s}-1}} \sum_{j=2^{2m_k}}^{q_{m_k,s}} A^{\alpha-1}_{q_{m_k,s}-j} \sum_{l=0}^{k-1} \frac{2^{2m_l(1/p-1)}}{m_l} \left(D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)\right) \\ &\quad + \frac{2^{2m_k(1/p-1)}}{A^{\alpha}_{q_{m_k,s}-1} m_k} \sum_{j=2^{2m_k}}^{q_{m_k,s}} A^{\alpha-1}_{q_{m_k,s}-j} \left(D^{\kappa}_{j}(x) - D_{2^{2m_k}}(x)\right) \\ &=: II_1 + II_2. \end{aligned}$$

To discuss II_1, we use (3) and $|D_{2^n}(x)| \le 2^n$. Thus, we can write

$$|II_1| \le c(\alpha) \sum_{l=0}^{k-1} \frac{2^{2m_l(1/p-1)}}{m_l} 2^{2m_l+1} \le c(\alpha) \sum_{l=0}^{k-1} \frac{2^{2m_l/p}}{m_l} < c(\alpha)\frac{2^{2m_{k-1}/p}}{m_{k-1}}. \tag{8}$$

From $\sigma^{\alpha,\kappa}_{q_{m_k,s}} f(x) = I + II_1 + II_2$ and (6), (8) we have

$$|\sigma^{\alpha,\kappa}_{q_{m_k,s}} f(x)| \geq |II_2| - |I| - |II_1| \geq |II_2| - c\frac{2^{2m_{k-1}/p}}{m_{k-1}}. \tag{9}$$

Now, we discuss II_2. We can write the nth Dirichlet kernel with respect to the Walsh-Kaczmarz system in the following form:

$$\begin{aligned} D^{\kappa}_n(x) &= D_{2^{|n|}}(x) + \sum_{k=2^{|n|}}^{n-1} r_{|k|}(x) w_{k-2^{|n|}}(\tau_{|k|}(x)) \\ &= D_{2^{|n|}}(x) + r_{|n|}(x) D^w_{n-2^{|n|}}(\tau_{|n|}(x)). \qquad (10) \end{aligned}$$

By the help of this, we immediately get

$$\begin{aligned} |II_2| &= \frac{2^{2m_k(1/p-1)}}{A^{\alpha}_{q_{m_k,s}-1} m_k} \left| \sum_{j=1}^{2^{2s}} A^{\alpha-1}_{q_{m_k,s}-j-2^{2m_k}} \left(D^{\kappa}_{j+2^{2m_k}}(x) - D_{2^{2m_k}}(x) \right) \right| \\ &= \frac{2^{2m_k(1/p-1)}}{A^{\alpha}_{q_{m_k,s}-1} m_k} \left| r_{2m_k}(x) \sum_{j=1}^{2^{2s}} A^{\alpha-1}_{2^{2s}-j} D^w_j \left(\tau_{2m_k}(x)\right) \right| \\ &= \frac{2^{2m_k(1/p-1)}}{m_k} \frac{A^{\alpha}_{2^{2s}-1}}{A^{\alpha}_{q_{m_k,s}-1}} \left| K^{\alpha,w}_{2^{2s}} \left(\tau_{2m_k}(x)\right) \right| \\ &\geq c(\alpha) \frac{2^{2m_k(1/p-1)-2m_k\alpha} A^{\alpha}_{2^{2s}-1}}{m_k} \left| K^{\alpha,w}_{2^{2s}} \left(\tau_{2m_k}(x)\right) \right|. \end{aligned}$$

Thus, from (9) and (4) we have

$$|\sigma^{\alpha,\kappa}_{q_{m_k,s}} f(x)| \geq c \frac{2^{2m_k(1/p-1)-2m_k\alpha} A^{\alpha}_{2^{2s}-1}}{m_k} \left| K^{\alpha,w}_{2^{2s}} \left(\tau_{2m_k}(x)\right) \right| - c\frac{2^{m_k}}{m_k}.$$

On the set I_{2s}

$$A^{\alpha}_{2^{2s}-1} K^{\alpha,w}_{2^{2s}} = \sum_{l=0}^{2^{2s}-1} A^{\alpha-1}_{2^{2s}-l}\, l \geq C 2^{2s(1+\alpha)}$$

and

$$|\sigma^{\alpha,\kappa}_{q_{m_k,s}} f(x)| \geq C \frac{2^{2m_k(1/p-(1+\alpha))} 2^{2s(1+\alpha)}}{m_k} - c\frac{2^{m_k}}{m_k}.$$

We decompose the set G as the following disjoint union

$$G = I_A \cup \bigcup_{t=0}^{A-1} J^A_t,$$

where $A > t \geq 1$ and $J^A_t := \{x \in G : x_{A-1} = ... = x_{A-t} = 0, x_{A-t-1} = 1\}$, $J^A_0 := \{x \in G : x_{A-1} = 1\}$. Notice that, by the definition of τ_A we have $\tau_A(J^A_t) = I_t \backslash I_{t+1}$.

Therefore, we can write

$$
\begin{aligned}
\int_G |\sigma^{\alpha,\kappa,*}|^p d\mu &\geq \sum_{t=1}^{2m_k-1} \int_{J_t^{2m_k}} |\sigma^{\alpha,\kappa,*}|^p d\mu \\
&\geq \sum_{s=[\frac{m_k}{2}]+1}^{m_k-1} \int_{J_{2s}^{2m_k}} |\sigma^{\alpha,\kappa,*}|^p d\mu \\
&\geq \sum_{s=[\frac{m_k}{2}]+1}^{m_k-1} \int_{J_{2s}^{2m_k}} |\sigma^{\alpha,\kappa}_{q_{m_k,s}}|^p d\mu \\
&\geq \sum_{s=[\frac{m_k}{2}]+1}^{m_k-1} \int_{J_{2s}^{2m_k}} \left(c\frac{2^{2m_k(1/p-(1+\alpha))} A^{\alpha}_{2^{2s}-1}}{m_k} |K^{\alpha,w}_{2^{2s}} \circ \tau_{2m_k}| - \frac{c2^{m_k}}{m_k} \right)^p d\mu \\
&\geq \sum_{s=[\frac{m_k}{2}]+1}^{m_k-1} \int_{(I_{2s} \setminus I_{2s+1})} \left(c\frac{2^{2m_k(1/p-(1+\alpha))} A^{\alpha}_{2^{2s}-1}}{m_k} |K^{\alpha,w}_{2^{2s}}| - \frac{c2^{m_k}}{m_k} \right)^{p)} d\mu \\
&\geq \sum_{s=[\frac{m_k}{2}]+1}^{m_k-1} \int_{(I_{2s} \setminus I_{2s+1})} \left(C\frac{2^{2m_k(1/p-(1+\alpha))} 2^{2s(1+\alpha)}}{m_k} - c\frac{2^{m_k}}{m_k} \right)^p d\mu
\end{aligned}
$$

and

$$
\begin{aligned}
\int_G |\sigma^{\alpha,\kappa,*}|^p d\mu &\geq c \sum_{s=[\frac{m_k}{2}]+1}^{m_k-1} \int_{(I_{2s} \setminus I_{2s+1})} \left| \frac{2^{2m_k(1/p-(1+\alpha))} 2^{2s(1+\alpha)}}{m_k} \right|^p d\mu \\
&\geq c \sum_{s=[\frac{m_k}{2}]+1}^{m_k-1} \frac{2^{2s((1+\alpha)p-1)} 2^{2m_k(1-p(1+\alpha))}}{m_k^p} \\
&\geq \begin{cases} cm_k^{1-p}, & p = 1/\left(1+\alpha\right), \\ c\frac{2^{m_k(1-p(1+\alpha))}}{m_k^p}, & 0 < p < 1/(1+\alpha). \end{cases}
\end{aligned}
$$

That is $\|\sigma^{\alpha,\kappa,*} f\|_p = +\infty$ for $0 < p \leq 1/(1+\alpha)$. The proof is complete. □

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