

Negative results concerning Fourier series on the complete product of \mathfrak{S}_3

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Abstract

The aim of this paper is to continue the studies about convergence in L^p -norm of the Fourier series based on representative product systems on the complete product of finite groups. We restrict our attention to bounded groups with unbounded sequence Ψ . The most simple example of this groups is the complete product of \mathfrak{S}_3 . In this case we proved the existence of an $1 < p < 2$ number for which exists an $f \in L^p$ such that its n -th partial sum of Fourier series S_n do not converge to the function f in L^p -norm (see [2]). In this paper we extend this "negative" result for all $1 < p < \infty$ and $p \neq 2$ numbers.

In Section 1 we introduce basic concepts in the study of representative product systems and Fourier analysis. We also introduce the system with which we work on the complete product of \mathfrak{S}_3 , i.e. the symmetric group on 3 elements (see [2]). Section 2 extends the definition of the sequence Ψ for all $p \geq 1$. Finally, we use the results of Section 2 to study the convergence in L^p -norm ($p \geq 1$) of the Fourier series on bounded groups with unbounded sequence Ψ , supposing all the same finite groups appearing in the product of G have the same system φ at all of their occurrences. These results appear in Section 3 and they complete the statement proved by G. Gát and the author of this paper in [2] for the complete product of \mathfrak{S}_3 .

Throughout this work denote by \mathbf{N} , \mathbf{P} , \mathbf{C} the set of nonnegative, positive integers and complex numbers, respectively. The notation which we used in this paper is similar to the one appeared in [3].

1 Representative product systems

Let $m := (m_k, k \in \mathbf{N})$ be a sequence of positive integers such that $m_k \geq 2$ and G_k a finite group with order m_k , ($k \in \mathbf{N}$). Suppose that each group has discrete topology and normalized Haar measure μ_k . Let G be the compact group formed by the complete direct product of G_k with the product of the topologies, operations and measures (μ). Thus each $x \in G$ consist of sequences $x := (x_0, x_1, \dots)$, where $x_k \in G_k$, ($k \in \mathbf{N}$). We call this sequence the *expansion* of x . The compact totally disconnected group G is called a *bounded group* if the sequence m is bounded.

If $M_0 := 1$ and $M_{k+1} := m_k M_k$, $k \in \mathbf{N}$, then every $n \in \mathbf{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \leq n_k < m_k$, $n_k \in \mathbf{N}$. This allows us to say that the sequence (n_0, n_1, \dots) is the expansion of n with respect to m .

Denote by Σ_k the dual object of G_k . Let $\{\varphi_k^s : 0 \leq s < m_k\}$ be the set of all *normalized coordinate functions* of the group G_k and suppose that $\varphi_k^0 \equiv 1$. Thus for every $0 \leq s < m_k$ there exists a $\sigma \in \Sigma_k$, $i, j \in \{1, \dots, d_\sigma\}$ such that

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \quad (x \in G_k).$$

Let ψ be the product system of φ_k^s , namely

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \quad (x \in G),$$

where n is of the form $n = \sum_{k=0}^{\infty} n_k M_k$ and $x = (x_0, x_1, \dots)$. Thus we say that ψ is the *representative product system* of φ . The Weyl-Peter's theorem (see [3]) secure that the system ψ is orthonormal and complete on $L^2(G)$.

The functions ψ_n ($n \in \mathbf{N}$) are not necessary uniformly bounded, so define

$$\Psi_k := \max_{n < M_k} \|\psi_n\|_1 \|\psi_n\|_\infty \quad (k \in \mathbf{N}).$$

Ψ_k is the multiplication of the greatest product of the square root of the dimension and L^1 -norm of the functions φ appeared in all group G_{m_j} for $0 \leq j < k$. It seems that the boundedness of the sequence Ψ plays an important role in the norm convergence of Fourier series.

For an integrable complex function f defined in G we define the Fourier coefficients and partial sums by

$$\widehat{f}_k := \int_{G_m} f \overline{\psi}_k d\mu \quad (k \in \mathbf{N}), \quad S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \quad (n \in \mathbf{P}).$$

According to the theorem of Banach-Steinhaus, $S_n f \rightarrow f$ as $n \rightarrow \infty$ in L^p norm for $f \in L^p(G)$ if and only if there exists a $C_p > 0$ such that

$$\|S_n f\|_p \leq C_p \|f\|_p \quad (f \in L^p(G)).$$

Thus, we say that the operator S_n is of *type* (p, p) . Since system ψ forms an orthonormal base in the Hilbert space $L^2(G)$, it is obvious that S_n is of type $(2, 2)$.

The representative product systems are the generalization of the well known Walsh-Paley and Vilenkin systems. Indeed, we obtain the Walsh-Paley system if $m_k = 2$ and $G_k := \mathcal{Z}_2$, the cyclic group of order 2 for all $k \in \mathbf{N}$. Moreover, we obtain the Vilenkin systems if the sequence m is an arbitrary sequence of integers greater than 1 and $G_k := \mathcal{Z}_{m_k}$, the cyclic group of order m_k for all $k \in \mathbf{N}$.

Let $m_k = 6$ for all $k \in \mathbf{N}$ and \mathcal{S}_3 be the *symmetric group* on 3 elements. Let $G_k := \mathcal{S}_3$ for all $k \in \mathbf{N}$. \mathcal{S}_3 has two characters and a 2-dimensional representation. Using a calculation of the matrices corresponding to the 2-dimensional

representation we construct the functions φ_k^s . In the notation the index k is omitted because all of the groups G_k are the same.

	e	(12)	(13)	(23)	(123)	(132)	$\ \varphi^s\ _1$	$\ \varphi^s\ _\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

Notice that the functions φ_k^s can take the value 0, and the product system of φ is not uniformly bounded. This facts encumber the study of this systems. On the other hand, $\max_{0 \leq s < 6} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3}$, thus $\Psi_k = \left(\frac{4}{3}\right)^k \rightarrow \infty$ if $k \rightarrow \infty$. More examples of representative product systems are appeared in [2] and [5].

2 The sequence of functions $\Psi_k(p)$

We extend the definition of the sequence Ψ for all $p \geq 1$ as follows:

$$\Psi_k(p) := \max_{n < M_k} \|\psi_n\|_p \|\psi_n\|_q \quad (p \geq 1, \frac{1}{p} + \frac{1}{q} = 1, k \in \mathbf{N})$$

(if $p = 1$ then $q = \infty$). Notice that $\Psi_k = \Psi_k(1)$ for all $k \in \mathbf{N}$. Clearly, the functions $\Psi_k(p)$ can be written by the form

$$\Psi_k(p) = \prod_{i=0}^{k-1} \max_{s < m_i} \|\varphi_i^s\|_p \|\varphi_i^s\|_q =: \prod_{i=0}^{k-1} \Upsilon_i(p) \quad (p \geq 1, \frac{1}{p} + \frac{1}{q} = 1, k \in \mathbf{N}).$$

Therefore, we study the product $\|f\|_p \|f\|_q$ for normalized functions on finite groups. On this regard we use the Hölder's inequality in the following form.

Theorem 1 (Hölder's inequality). (see [3], p 137)

Let $f \in L^p(X)$ and $g \in L^q(X)$, ($\frac{1}{p} + \frac{1}{q} = 1, p \neq \infty$). Then $fg \in L^1(X)$; in fact

$$(i) \quad \left| \int_X fg \, d\mu \right| \leq \int_X |fg| \, d\mu$$

and

$$(ii) \quad \int_X |fg| \, d\mu \leq \|f\|_p \|g\|_q$$

Equality holds in (i) if and only if $\text{sgn}(f(x)g(x))$ ($\text{sgn} : \mathbb{C} \rightarrow \mathbb{C}, \text{sgn}(z) := \frac{z}{|z|}$) is constant almost everywhere on the set where $fg \neq 0$. Equality holds in (ii) if and only if there are nonnegative numbers A and B not both 0 such that

$$(*) \quad A|f(x)|^p = B|g(x)|^q$$

almost everywhere.

First, we prove the following lemma.

Lemma 1. *Let G be a finite group with discrete topology and normalized Haar measure μ , and let f be a normalized complex valued function on G ($\|f\|_2 = 1$). Thus*

(a) *if $\|f\|_1 \|f\|_\infty = 1$, then $\|f\|_p \|f\|_q = 1$ for all $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

(b) *if $\|f\|_1 \|f\|_\infty > 1$, then $\|f\|_p \|f\|_q > 1$ for all $p \geq 1$, $p \neq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof.

(a) The conditions imply the equality

$$\int_G |f| d\mu \cdot \|f\|_\infty = 1 = \int_G |f|^2 d\mu.$$

Let $f_0 := \frac{f}{\|f\|_\infty}$. Then

$$(1) \quad |f_0(x)| \leq 1 \quad (x \in G)$$

and

$$(2) \quad \int_G |f_0| d\mu = \int_G |f_0|^2 d\mu.$$

Thus by (1) we obtain $|f_0(x)| - |f_0(x)|^2 \geq 0$ ($x \in G$) and by (2) we obtain

$$\int_G |f_0| - |f_0|^2 d\mu = 0.$$

Hence $|f_0(x)| = |f_0(x)|^2$ for all $x \in G$. Thus, we have $|f_0| = 1$ or $|f_0| = 0$ for all $x \in G$, therefore $|f| = \|f\|_\infty$ or $|f| = 0$ for all $x \in G$. For this reason the condition (*) in the Hölder's inequality is true for all ($1 < p < \infty$) and the equality $1 = \int_G |f|^2 d\mu = \|f\|_p \|f\|_q$ holds.

(b) Suppose there is a $1 < p < 2$ such that

$$\|f\|_p \|f\|_q = 1 = \int_G |f|^2 d\mu.$$

Then the equality (ii) in the Hölder's inequality holds. For this reason there are nonnegative numbers A and B not both 0 such that

$$A|f(x)|^p = B|f(x)|^q \quad (x \in G).$$

Thus, there is a $c > 0$ such that $|f| = c$ or $|f| = 0$ for all $x \in G$ ($c = \|f\|_\infty$). Then $|f| \cdot \|f\|_\infty = |f|^2$. Integrating both part of the last equation we have $\|f\|_1 \|f\|_\infty = 1$. We obtain a contradiction. \square

However, the following lemma states much more.

Lemma 2. *Let G be a finite group with discrete topology and normalized Haar measure μ , and let f be a complex valued function on G . Thus, the function $\Psi(p) := \|f\|_p \|f\|_q$ ($\frac{1}{p} + \frac{1}{q} = 1$) is a monotone decreasing function on the interval $[1, 2]$.*

Proof. Let $f_0 := \frac{f}{\|f\|_\infty}$. Then $\Psi(p) = \|f\|_\infty^2 \|f_0\|_p \|f_0\|_q$. Let m be the order of the group G . We take the elements of G in some order, $G = \{g_1, g_2, \dots, g_m\}$, to obtain the numbers

$$a_i := |f_0(g_i)| \leq 1 \quad (i = 1, \dots, m),$$

with which we write

$$\Psi(p) = \frac{\|f\|_\infty^2}{m} \left(\sum_{i=1}^m a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^m a_i^q \right)^{\frac{1}{q}}.$$

Since $q = \frac{p}{p-1}$ we have

$$\frac{\partial q}{\partial p} = -\frac{1}{(p-1)^2} = -\frac{q^2}{p^2}.$$

Therefore,

$$\begin{aligned} \frac{\partial \Psi}{\partial p} = & \Psi(p) \left[-\frac{1}{p^2} \log \left(\sum_{i=1}^m a_i^p \right) + \frac{1}{p} \frac{\sum_{i=1}^m a_i^p \log a_i}{\sum_{i=1}^m a_i^p} \right] \\ & + \Psi(p) \left[-\frac{1}{q^2} \log \left(\sum_{i=1}^m a_i^q \right) + \frac{1}{q} \frac{\sum_{i=1}^m a_i^q \log a_i}{\sum_{i=1}^m a_i^q} \right] \left(-\frac{q^2}{p^2} \right). \end{aligned}$$

The condition $1 < p < 2$ secures the fact

$$-\frac{1}{q} \cdot \frac{q^2}{p^2} = -\frac{1}{p(p-1)} < -\frac{1}{p},$$

from which we have

$$\begin{aligned} \frac{1}{\Psi(p)} \frac{\partial \Psi}{\partial p} \leq & \frac{1}{p^2} \left[\log \left(\sum_{i=1}^m a_i^q \right) - \log \left(\sum_{i=1}^m a_i^p \right) \right] \\ & + \frac{1}{p} \left[\frac{\sum_{i=1}^m a_i^p \log a_i}{\sum_{i=1}^m a_i^p} - \frac{\sum_{i=1}^m a_i^q \log a_i}{\sum_{i=1}^m a_i^q} \right]. \end{aligned}$$

Both addends in the sum above are not positive. Indeed, the facts $a_i \leq 1$ for all $1 \leq i \leq m$ and $p < q$ imply $a_i^q \leq a_i^p$ for all $1 \leq i \leq m$, from which it is clear

$$(1) \quad \log \left(\sum_{i=1}^m a_i^q \right) - \log \left(\sum_{i=1}^m a_i^p \right) \leq 0.$$

Secondly,

$$h(x) := \frac{\sum_{i=1}^m a_i^x \log a_i}{\sum_{i=1}^m a_i^x}$$

is a monotone increasing function. Indeed,

$$\begin{aligned} h'(x) &= \frac{(\sum_{i=1}^m a_i^x \log^2 a_i) \sum_{i=1}^m a_i^x - (\sum_{i=1}^m a_i^x \log a_i)^2}{(\sum_{i=1}^m a_i^x)^2} \\ &= \frac{\sum_{i,j=1}^m a_i^x a_j^x (\log a_i - \log a_j)^2}{(\sum_{i=1}^m a_i^x)^2} \geq 0. \end{aligned}$$

Consequently, we have

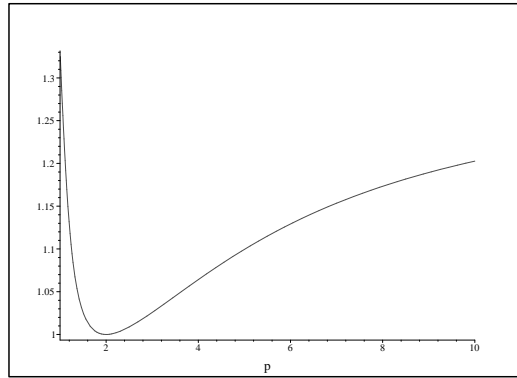
$$(2) \quad \frac{\sum_{i=1}^m a_i^p \log a_i}{\sum_{i=1}^m a_i^p} - \frac{\sum_{i=1}^m a_i^q \log a_i}{\sum_{i=1}^m a_i^q} \leq 0.$$

By (1) and (2) we obtain $\frac{\partial \Psi}{\partial p} \leq 0$ for all $1 < p < 2$, which completes the proof of the lemma. \square

We can apply Lemma 1 and Lemma 2 to obtain similar properties for $\Upsilon_k(p)$ and $\Psi_k(p)$ because these functions are the maximum value and the product of finite functions satisfying the conditions of the two lemmas. Consequently, we obtain:

Theorem 2. *Let G_k be a coordinate group of G such that $\|\varphi_k^s\|_1 = 1$ for all $s < m_k$. Then $\Upsilon_k(p) \equiv 1$. Otherwise, the function $\Upsilon_k(p)$ is a strictly monotone decreasing function on the interval $[1, 2]$. The function $\Psi_k(p) \equiv 1$ if $\|\varphi_i^s\|_1 = 1$ for all $s < m_i$ and $i \leq k$. Otherwise, the function $\Psi_k(p)$ is a strictly monotone decreasing function on the interval $[1, 2]$.*

It is important to remark the functions $\Upsilon_k(p)$ and $\Psi_k(p)$ are monotone increasing function if $p > 2$. It follows from the property $\Upsilon_k(p) = \Upsilon_k(\frac{p}{p-1})$. In order to illustrate these properties we plot the values of $\Upsilon(p)$ for the group S_3 .



3 Negative results

Theorem 3. *Let p be a fix number in the interval $(1, 2)$ and $\frac{1}{p} + \frac{1}{q} = 1$. If G is a group with unbounded sequence $\Psi_k(p)$, then the operator S_n is not of type (p, p) or (q, q) .*

Proof. To prove this theorem choose $i_k < m_k$ the index for which the normalized coordinate function $\varphi_k^{i_k}$ of the finite group G_k satisfies

$$\|\varphi_k^{i_k}\|_p \|\varphi_k^{i_k}\|_q = \max_{s < m_k} \|\varphi_k^s\|_p \|\varphi_k^s\|_q.$$

Define

$$f_k(x) := \varphi_k^{i_k}(x) |\varphi_k^{i_k}(x)|^{q-2} \quad (x \in G_k).$$

Thus, $|f_k(x)|^p = |\varphi_k^{i_k}(x)|^q$ and also $f_k(x) \overline{\varphi_k^{i_k}(x)} = |\varphi_k^{i_k}(x)|^q \in \mathbb{R}^+$ if $\varphi_k^{i_k}(x) \neq 0$. Hence the both equalities hold in the Hölder's inequality. For this reason

$$(3) \quad \left| \int_{G_k} f_k \overline{\varphi_k^{i_k}} d\mu_k \right| \|\varphi_k^{i_k}\|_p = \|f_k\|_p \|\varphi_k^{i_k}\|_q \|\varphi_k^{i_k}\|_p.$$

If k is an arbitrary positive integer and $n := \sum_{j=0}^{k-1} i_j M_j$, then define $F_k \in L^p(G)$

by

$$F_k(x) := \prod_{j=0}^{k-1} f_j(x_j) \quad (x = (x_0, x_1, \dots) \in G).$$

Since $\|F_k\|_p = \prod_{j=0}^{k-1} \|f_j\|_p$ it follows from (3) that

$$(4) \quad \begin{aligned} \|S_{n+1}F_k - S_nF_k\|_p &= \left\| \int_G F_k \overline{\psi_n} d\mu \right\| \|\psi_n\|_p = \\ &= \prod_{j=0}^{k-1} \left\| \int_G f_j \overline{\varphi_j^{i_j}} d\mu_j \right\| \|\varphi_j^{i_j}\|_p \geq \Psi_k(p) \|F_k\|_p. \end{aligned}$$

On the other hand, if S_n is of type (p, p) , then there exists a $C_p > 0$ such that

$$\|S_{n+1}F_k - S_nF_k\|_p \leq \|S_{n+1}F_k\|_p + \|S_nF_k\|_p \leq 2C_p \|F_k\|_p$$

for each $k > 0$, which contradicts (4) because the sequence $\Psi_k(p)$ is not bounded. For this reason operators S_n are not uniformly of type (p, p) . By duality argument (see [4]) the operators S_n can not be uniformly of type (q, q) . This completes the proof of the theorem. \square

By Theorem 3 we obtain

Theorem 4. *Let G be a bounded group and suppose that all the same finite groups appearing in the product of G have the same system φ at all of their occurrences. If the sequence Ψ is unbounded, then the operator S_n is not of type (p, p) for all $p \neq 2$.*

Proof. If the sequence $\Psi_k = \Psi_k(1)$ is not bounded, there exists a finite group F with system $\{\varphi^s : 0 \leq s < |F|\}$ ($|F|$ is the order of the group F) which appears infinite many times in the product of G and

$$\Upsilon(1) := \max_{s < |F|} \|\varphi^s\|_1 \|\varphi^s\|_\infty > 1.$$

Hence by Theorem 2 we have

$$\Upsilon(p) := \max_{s < |F|} \|\varphi^s\|_p \|\varphi^s\|_q > 1$$

for all $p \neq 2$. Denote by $l(k)$ the number of times the group F appears in the first k coordinates of G . Thus $l(k) \rightarrow \infty$ if $k \rightarrow \infty$ and

$$\Psi_k(p) \geq \prod_{i=1}^{l(k)} \Upsilon(p) \rightarrow \infty \quad \text{if } k \rightarrow \infty,$$

for all $p \neq 2$. Consequently, the group G satisfies the conditions of Theorem 3 for all $1 < p < 2$. This completes the proof of the theorem. \square

Corollary 1. *If G is the complete product of \mathcal{S}_3 with the system φ appearing in Section 2, then the operator S_n is not of type (p, p) for all $p \neq 2$.*

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