CONVERGENCE OF SEQUENCES OF TWO-DIMENSIONAL FEJÉR MEANS OF TRIGONOMETRIC FOURIER SERIES OF INTEGRABLE FUNCTIONS

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ABSTRACT. The aim of this paper is to prove the a.e. convergence of sequences of the Fejér means of the trigonometric Fourier series of two variable integrable functions. That is, let $a = (a_1, a_2) : \mathbb{N} \to \mathbb{N}^2$ such that $a_j(n+1) \geq \alpha \sup_{k \leq n} a_j(k)$ $(j = 1, 2,\, n \in \mathbb{N})$ for some $\alpha > 0$ and $a_1(+\infty) = a_2(+\infty) = +\infty$. Then for each integrable function $f \in L^1(\mathbb{T}^2)$ we have the a.e. relation $\lim_{n\to\infty} \sigma_{a(n)} f = f$. It will be a straightforward and easy consequence of this result the historical cone restricted a.e. convergence result with respect to the two-dimensional Fejér means of integrable functions due to Marcinkiewicz and Zygmund [7].

First, we give a brief introduction to the theory of the Fourier series. Let $\mathbb{N}$ denote the set of natural number, that is, $\mathbb{N} = \{0, 1, \dots\}$ and $\mathbb{P} = \mathbb{N} \setminus \{0\}$.

The system of functions

$$e^{\imath n x} \quad (n = 0, \pm 1, \pm 2, \dots)$$

$(x \in \mathbb{R}, \imath = \sqrt{-1})$ is called the trigonometric system. It is orthogonal over any interval of length 2π, specially over $\mathbb{T} := [-\pi, \pi)$. Let $f \in L^1(\mathbb{T})$, that is integrable on $\mathbb{T}$. The kth Fourier coefficient of f is

$$\hat{f}(k) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-\imath k t} dt,$$

where k is any integer number. The nth $(n \in \mathbb{N})$ partial sum of the Fourier series of f is

$$S_n f(y) := \sum_{k=-n}^{n} \hat{f}(k) e^{\imath k y}.$$

The nth $(n \in \mathbb{N})$ Fejér or $(C, 1)$ mean of function f is defined in the following way:

$$\sigma_n f(y) := \frac{1}{n+1} \sum_{k=0}^{n} S_k f(y).$$

It is known that

$$\sigma_n f(y) = \frac{1}{\pi} \int_T f(x) K_n(y - x) dx,$$

where the function K_n is known as the nth Fejér kernel; we find an appropriate expression for it e.g. in the book of Bary [1].

$$K_n(u) = \frac{1}{2(n+1)} \left(\frac{\sin(\frac{u}{2}(n+1))}{\sin(\frac{u}{2})} \right)^2.$$

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From this expression one immediately derive the following properties of the kernel. They will play an essential role later.

$$K_n(u) \geq 0.$$

$$K_n(u) \leq \frac{\pi^2}{2(n+1)u^2} \quad (0 < |u| \leq \pi).$$

Let f be an integrable function that is let $f \in L^1(\mathbb{T}^2)$. The $k = (k_1, k_2)$th Fourier coefficient of f is

$$\hat{f}(k) = \hat{f}(k_1, k_2) := \frac{1}{4\pi^2} \int_{\mathbb{T}\times\mathbb{T}} f(x_1, x_2) e^{-\imath(k_1 t_1 + k_2 t_2)} d(t_1, t_2),$$

where k_1, k_2 are integers. The nth ($n \in \mathbb{N}^2$) rectangual partial sum of the Fourier series of f is

$$S_n f(y) = S_{(n_1,n_2)} f(y_1, y_2) := \sum_{k_1=-n_1}^{n_1} \sum_{k_2=-n_2}^{n_2} \hat{f}(k_1, k_2) e^{\imath(k_1 y_1 + k_2 y_2)}.$$

The nth ($n \in \mathbb{N}^2$) two-dimensional Fejér or $(C, 1)$ mean of function f is defined in the following way:

$$\sigma_n f(y) = \sigma_{(n_1,n_2)} f(y) := \frac{1}{(n_1+1)(n_2+1)} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} S_k f(y),$$

where $y \in \mathbb{T}^2$. In 1939 Marcinkiewicz and Zygmund [7] proved their celebrated theorem on the convergence of the two-dimensional restricted $(C, 1)$ means of trigonometric Fourier series. They proved for any integrable function $f \in L^1(\mathbb{T}^2)$ the a.e. convergence

$$\sigma_{(n_1,n_2)} f \longrightarrow f$$

provided $n_1/\beta \leq n_2 \leq \beta n_1$, where $\beta > 1$ is a fixed constant. So, the set of indices (n_1, n_2) remains in some positive cone around the identical function. Actually, their proof is not a simple one. Among others, the main theorem of this paper, that is, Theorem 1 provide an easy proof for this celebrated result of Marcinkiewicz and Zygmund.

We also mention that Jessen, Marcinkiewicz and Zygmund [8] also proved the a.e. convergence $\sigma_n f \to f$ without any restriction on the indices (other than $\min\{n_1, n_2\} \to \infty$), but for functions in $L \log^+ L$. For a joint generalization of these results of Marcinkiewicz – Zygmund and Jessen – Marcinkiewicz – Zygmund see the paper of the author [4]. For another proof of the "cone restricted" convergence of the two dimensional Fejér means see the paper of Weisz [10] and the result of Marcinkiewicz and Zygmund with respect to the multi-dimensional case was also proved with a different proof by Weisz [11]. With respect to this issue one can find some interesting and important read in [6] and [5].

For another modern treatise on the theory of Fourier series see for instance the book of Edwards [3].

We study the a.e. convergence of subsequences of the two-dimensional $(C, 1)$ means $\sigma_{a(n)} f$ of integrable functions that is, $f \in L^1(\mathbb{T}^2)$, where $a : \mathbb{N} \to \mathbb{N}^2$.

Theorem 1. *Let $a = (a_1, a_2) : \mathbb{N} \to \mathbb{N}^2$ be a sequence with property $a_j(+\infty) = +\infty$ $(j = 1, 2)$. Suppose that there exists an $\alpha > 0$ such that $a_j(n+1) \geq \alpha \sup_{k \leq n} a_j(k)$ $(j = 1, 2,\ n \in \mathbb{N})$. Then for each integrable function $f \in L^1(\mathbb{T}^2)$ we have the a.e. relation*

$$\lim_{n\to\infty} \sigma_{a(n)} f = f.$$

This Theorem, which is the main result of this paper is a consequence of the following lemma.

Lemma 2. *Let $a = (a_1, a_2) : \mathbb{N} \to \mathbb{N}^2$ be a sequence with property $a_j(+\infty) = +\infty$ $(j = 1, 2)$. Suppose that $\lfloor \log_2 a_j \rfloor$ ($\lfloor x \rfloor$ denotes the lower integer part of x) is monotone increasing $(j = 1, 2)$. Then for each integrable function $f \in L^1(\mathbb{T}^2)$ we have the a.e. relation*

$$\lim_{n\to\infty} \sigma_{a(n)} f = f.$$

A straightforward and easy consequence of Lemma 2 is the celebrated result of Marcinkiewicz and Zygmund [7] with respect to the „cone restricted" almost everywhere convergence of two-dimensional Fejér means of integrable function.

Corollary 3. *Let $\beta > 1$ and $f \in L^1(I^2)$. Then we have the a.e. relation*

$$\lim_{\substack{n_1,n_2\to\infty \\ 1/\beta \le n_1/n_2 \le \beta}} \sigma_{(n_1,n_2)} f = f.$$

Proof. The proof of this corollary comes directly from Lemma 2. So, let $\gamma := \lceil \log_2 \beta \rceil$. For $k, l \in \mathbb{N}$ set $N_{\gamma,l,k} := \left\{(n_1, n_2) \in \mathbb{N}^2 : 2^k \le n_1 < 2^{k+1}, 2^{k-\gamma+l} \le n_2 < 2^{k-\gamma+l+1}\right\}$. Let $N_{\gamma,l}$ be the union of the disjoint sets $N_{\gamma,l,k}$. It is easy to give a sequence $a : \mathbb{N} \to \mathbb{N}^2$ such that $\lfloor \log_2 a_1 \rfloor, \lfloor \log_2 a_2 \rfloor$ are monotone increasing (for $n \in N_{\gamma,l,k}$ we have $\lfloor \log_2 n_1 \rfloor = k$, $\lfloor \log_2 n_2 \rfloor = k - \gamma + l$) and $a(\mathbb{N}) = N_{\gamma,l}$. This by Lemma 2 gives that for each integrable function f

$$\sigma_{(n_1,n_2)} f \to f$$

a.e. provided by $n \in N_{\gamma,l}$ and $n_1, n_2 \to \infty$. Hence, we also have this a.e. relation for $n \in \bigcup_{l=0}^{2\gamma} N_{\gamma,l} =: N_\gamma$ and $n_1, n_2 \to \infty$. After then, let $n \in \mathbb{P}^2$ be such that $1/\beta \le n_1/n_2 \le \beta$. Denote by k the natural number for which $2^k \le n_1 < 2^{k+1}$. Then, $2^{k-\gamma} \le 2^k/\beta \le n_2 < 2^{k+1}\beta \le 2^{k+\gamma+1}$. Consequently, $n \in N_\gamma$. This completes the proof of this corollary. $\square$

Extend the map $a : \mathbb{N} \to \mathbb{N}^2$ to $[1, +\infty)$ linearly in a way that for $n \le x \le n + 1$ set

$$\begin{aligned}
\tilde{a}(x) &= (\tilde{a}_1(x), \tilde{a}_2(x)) \\
&= ((x - n)a_1(n + 1) + (n + 1 - x)a_1(n), (x - n)a_2(n + 1) + (n + 1 - x)a_2(n)) \\
&= (x - n)a(n + 1) + (n + 1 - x)a(n).
\end{aligned}$$

Without the loss of generality we can suppose from now that $a_j(n) \ge 1$ for $n \in \mathbb{N}$ and $j = 1, 2$. Also set $\beta(x) = \lfloor \log_2(\tilde{a}(x)) \rfloor : [1, +\infty) \to \mathbb{N}^2$.

The following Calderon- Zygmund type decomposition lemma on $\mathbb{T}^2$ will play a fundamental role in the proof of Lemma 2.

The dyadic subintervals of $\mathbb{T}$ are defined in the following way.

$$\mathcal{I}_0 := \{\mathbb{T}\}, \quad \mathcal{I}_1 := \{[-\pi, 0), [0, \pi)\},$$
$$\mathcal{I}_2 := \{[-\pi, -\pi/2), [-\pi/2, 0), [0, \pi/2), [\pi/2, \pi)\}, \ldots$$

$$\mathcal{I} := \bigcup_{n=0}^{\infty} \mathcal{I}_n.$$

The elements of $\mathcal{I}$ are said to be dyadic intervals. If $F \in \mathcal{I}$, then there exists a unique $n \in \mathbb{N}$ such that $F \in \mathcal{I}_n$, and consequently $\mathrm{mes}(F) = \frac{2\pi}{2^n}$ (the Lebesgue measure). Each $\mathcal{I}_n$

has 2^n disjoint elements ($n \in \mathbb{N}$). $\mathfrak{I} \times \mathfrak{I}$ is the set of dyadic rectangles. For $x \in \mathbb{T}$ denote by $I_n(x)$ the element of $\mathfrak{I}_n$ for which $x \in I_n(x)$.

Lemma 4. *Let $f \in L^1(\mathbb{T}^2)$, and $\lambda > \|f\|_1/(2\pi)^2$. Suppose that the functions $\beta_j(x) = \lfloor \log_2 \tilde{a}_j(x) \rfloor : [1, +\infty) \to \mathbb{N}$ are monotone increasing, where functions $\tilde{a}_j$ are continuous ($j = 1, 2$). Then there exists a sequence of integrable functions (f_i) and disjoint rectangles $I_{\beta_1(s_i)}(u_{i,1}) \times I_{\beta_2(s_i)}(u_{i,2}) \in \mathfrak{I}_{\beta_1(s_i)} \times \mathfrak{I}_{\beta_2(s_i)}$ such that*

$$f = \sum_{i=0}^{\infty} f_i,$$

$$\|f_0\|_\infty \le C_\beta \lambda, \quad \|f_0\|_1 \le 3\|f\|_1, \quad \textit{and}$$

$$\operatorname{supp} f_i \subset I_{\beta_1(s_i)}(u_{i,1}) \times I_{\beta_2(s_i)}(u_{i,2})$$

for some $s_i \ge 1, u_i \in \mathbb{T}^2$ $(i \in \mathbb{P})$. Moreover, $\int_{\mathbb{T}^2} f_i(x)dx = 0$ $(i \ge 1)$ and for

$$F := \bigcup_{i=1}^{\infty} \left(I_{\beta_1(s_i)}(u_{i,1}) \times I_{\beta_2(s_i)}(u_{i,2}) \right) \quad \textit{we have} \quad \operatorname{mes}(F) \le \|f\|_1/\lambda.$$

Proof. Let $s_1 := 1$ and

$$\Omega_1 := \left\{ J = J_1 \times J_2 \in \mathfrak{I}_{\beta_1(s_1)} \times \mathfrak{I}_{\beta_2(s_1)} : \operatorname{mes}(J)^{-1} \int_J |f(x)|dx > \lambda \right\}.$$

Since for each $J \in \Omega_1$, we have

$$\operatorname{mes}(J)^{-1} = \frac{2^{\beta_1(1)+\beta_2(1)}}{4\pi^2},$$

then we also have

$$\lambda < \operatorname{mes}(J)^{-1} \int_J |f(x)|dx \le 2^{\beta_1(1)+\beta_2(1)} \frac{1}{4\pi^2} \int_{\mathbb{T}^2} |f(x)|dx < 2^{\beta_1(1)+\beta_2(1)} \lambda \le C_\beta \lambda.$$

Let $s_2 := \inf\{s \in [s_1, +\infty) : \sum_{j=1}^{2} |\beta_j(s) - \beta_j(s_1))| \ge 1\}$. Recall that the function $\tilde{a}$ is continuous, and $\beta(x) = \lfloor \log_2 \tilde{a}(x) \rfloor$ is monotone increasing and continuous from the right with respect to its both variables. Then we have the following three cases:

Case 1. $\beta_1(s_2) = \beta_1(s_1) + 1$ and $\beta_2(s_2) = \beta_1(s_1)$,

Case 2. $\beta_1(s_2) = \beta_1(s_1)$ and $\beta_2(s_2) = \beta_1(s_1) + 1$,

Case 3. $\beta_1(s_2) = \beta_1(s_1) + 1$ and $\beta_2(s_2) = \beta_1(s_1) + 1$.

We decompose the dyadic rectangles contained in

$$\left[\mathfrak{I}_{\beta_1(s_1)} \times \mathfrak{I}_{\beta_2(s_1)} \right] \setminus \{J : J \in \Omega_1\}.$$

That is,

$$\Omega_2 := \left\{ J \in \mathfrak{I}_{\beta_1(s_2)} \times \mathfrak{I}_{\beta_2(s_2)} : \operatorname{mes}(J)^{-1} \int_J |f(x)|dx > \lambda \,\text{and}\, \nexists K \in \Omega_1 \,\text{such as}\, J \subset K \right\}.$$

Consequently, for all $J \in \Omega_2$ we get

$$\lambda < \operatorname{mes}(J)^{-1} \int_J |f(x)|dx \le 4\lambda.$$

(In case 1 and 2 we even have 2λ, but it makes no problem to take 4λ, instead.) Generally, for $\mathbb{N} \ni n \geq 3$

$$s_n := \inf\{s \in [s_{n-1}, +\infty) : \sum_{j=1}^{2} |\beta_j(s) - \beta_j(s_{n-1}))| \geq 1\}.$$

That is, $\beta_j(s_n) = \beta_j(s_{n-1}) + 1$ for at last one j $(j = 1, 2)$. If for a j this is not valid, then $\beta_j(s_n) = \beta_j(s_{n-1})$. Also take

$$\Omega_n := \left\{ J \in \mathcal{I}_{\beta_1(s_n)} \times \mathcal{I}_{\beta_2(s_n)} : \operatorname{mes}(J)^{-1} \int_J |f(x)|dx > \lambda \,\text{and}\, \nexists K \in \bigcup_{i=1}^{n-1} \Omega_i \,\text{such as}\, J \subset K \right\}.$$

Similarly, as in the case of Ω_2 we have that for each $J \in \Omega_n$ the inequalities

$$\lambda < \operatorname{mes}(J)^{-1} \int_J |f(x)|dx \leq 4\lambda$$

hold. Denote by $l_n \in \mathbb{N}$ the number of elements of Ω_n, and the elements of Ω_n by $J_{n,k}(k = 1, \ldots, l_n, n \in \mathbb{N})$. Since $\mathcal{I}_{\beta_1(s_n)} \times \mathcal{I}_{\beta_2(s_n)}$ has $2^{\beta_1(s_n)+\beta_2(s_n)}$ (disjoint) elements, then $l_n \leq 2^{\beta_1(s_n)+\beta_2(s_n)}$ $(n \in \mathbb{N})$. For an arbitrary set $B \subset \mathbb{T}^2$ the characteristic function of B is denoted by 1_B. Let

$$f_{n,k} := \left(f - \operatorname{mes}(J_{n,k})^{-1} \int_{J_{n,k}} f(x)dx \right) 1_{J_{n,k}},$$

$k = 1, \ldots, l_n$, $n \in \mathbb{N}$ and $F := \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{l_n} J_{n,k}$. Since the dyadic rectangles $J_{n,k}$ are disjoint, then we have the following decomposition of the function f:

$$\begin{aligned} f &= \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} f 1_{J_{n,k}} + f 1_{\mathbb{T}^2 \setminus F} \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} \left(f - \operatorname{mes}(J_{n,k})^{-1} \int_{J_{n,k}} f(x)dx \right) 1_{J_{n,k}} \\ &+ \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} \left[\operatorname{mes}(J_{n,k})^{-1} \int_{J_{n,k}} f(x)dx \right] 1_{J_{n,k}} + f 1_{\mathbb{T}^2 \setminus F} \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} f_{n,k} + f_0. \end{aligned}$$

This means that $f_0 = \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} \left[\operatorname{mes}(J_{n,k})^{-1} \int_{J_{n,k}} f(x)dx \right] 1_{J_{n,k}} + f 1_{\mathbb{T}^2 \setminus F}$ and the functions f_i $(i = 1, 2, \ldots)$ in the statement of Lemma 4 will be the functions $f_{n,k}$ $(k = 1, \ldots, l_n,\, n \in \mathbb{N})$. $\operatorname{supp} f_{n,k} \subset J_{n,k}$ are disjoint dyadic rectangles,

$$\operatorname{mes}(J_{n,k}) = \frac{4\pi^2}{2^{\beta_1(s_n)+\beta_2(s_n)}},$$

$$\int_{\mathbb{T}^2} f_{n,k}(x)dx = \int_{J_{n,k}} f(x)dx - \operatorname{mes}(J_{n,k})^{-1} \int_{J_{n,k}} f(x)dx \cdot \operatorname{mes}(J_{n,k}) = 0.$$

$$\|f_{n,k}\|_1 \leq \|f 1_{J_{n,k}}\|_1 + \operatorname{mes}(J_{n,k})^{-1} \int_{J_{n,k}} |f(x)|dx \|1_{J_{n,k}}\|_1 = 2\|f 1_{J_{n,k}}\|_1.$$

Consequently,

$$\|\sum_{n=1}^{\infty}\sum_{k=1}^{l_n} f_{n,k}\|_1 \le 2\sum_{n=1}^{\infty}\sum_{k=1}^{l_n} \|f1_{J_{n,k}}\|_1 = 2\int_F |f(x)|dx \le 2\|f\|_1.$$

This immediately gives

$$\|f_0\|_1 = \|f - \sum_{n=1}^{\infty}\sum_{k=1}^{l_n} f_{n,k}\|_1 \le 3\|f\|_1.$$

Since F is the disjoint union of the dyadic rectangles $J_{n,k}$, then for the two-dimensional Lebesgue measure of F we get

$$\begin{aligned}\operatorname{mes}(F) &= \sum_{n=1}^{\infty}\sum_{k=1}^{l_n} \operatorname{mes}(J_{n,k})\\ &< \sum_{n=1}^{\infty}\sum_{k=1}^{l_n} \frac{1}{\lambda}\int_{J_{n,k}} |f(x)|dx\\ &= \frac{1}{\lambda}\int_F |f(x)|dx \le \frac{1}{\lambda}\|f\|_1.\end{aligned}$$

It remains to prove $\|f_0\| \le C_\beta\lambda$. The construction of Ω_n gives the inequality

$$\operatorname{mes}(J_{n,k})^{-1}\int_{J_{n,k}} |f(x)|dx \le C_\beta\lambda$$

(in the case of $n = 1$ we have $2^{\beta_1(1)+\beta_2(1)}$, and in the case of $n \ge 2$ we have number 4 as constant C). That is,

$$\begin{aligned}\|f_0\|_\infty &\le C_\beta\lambda\|\sum_{n=1}^{\infty}\sum_{k=1}^{l_n} 1_{J_{n,k}}\|_\infty + \|f1_{\mathbb{T}^2\setminus F}\|_\infty\\ &= C_\beta\lambda\|1_F\|_\infty + \|f1_{\mathbb{T}^2\setminus F}\|_\infty\\ &\le C_\beta\lambda + \|f1_{\mathbb{T}^2\setminus F}\|_\infty.\end{aligned}$$

Let $\mathcal{A}_n$ be the σ-algebra generated by the elements of $\mathcal{I}_{\beta_1(s_n)} \times \mathcal{I}_{\beta_2(s_n)}$ $(n \in \mathbb{N})$. Then we have an increasing sequence of σ algebras

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots.$$

The conditional expectation operator of the function f with respect to $\mathcal{A}_n$ at a given point $x \in \mathbb{T}^2$ is

$$\operatorname{mes}(J)^{-1}\int_J f(t)dt,$$

where J is the unique element of $\mathcal{I}_{\beta_1(s_n)} \times \mathcal{I}_{\beta_2(s_n)}$ such that $x \in J$. Since $\lim_{+\infty}\beta_1 = \lim_{+\infty}\beta_2 = +\infty$, then the martingale convergence theorem (see e.g. the book of Neveau [9]) gives that this integral mean value converges to $f(x)$ for almost all x in $\mathbb{T}^2$.

Now let $x \in \mathbb{T}^2 \setminus F$. Then the construction of the set Ω_n gives for each $J \in \mathcal{I}_{\beta_1(s_n)} \times \mathcal{I}_{\beta_2(s_n)}$ that $\operatorname{mes}(J)^{-1}\int_J |f(t)|dt \le \lambda$ (for all $n \in \mathbb{N}$). From the lines above there follows

$$|f(x)| \le \lambda$$

for almost all $x \in \mathbb{T}^2 \setminus F$, so

$$\|f1_{\mathbb{T}^2\setminus F}\|_\infty \le \lambda, \quad \|f_0\|_\infty \le C_\beta\lambda.$$

With this the proof of Lemma 4 is complete. □

For $A \in \mathbb{N}, x \in \mathbb{T}$ denote by $I_A^1(x) = I_A(x) + \frac{2\pi}{2^A} := \left\{y + \frac{2\pi}{2^A} : y \in I_A(x)\right\} \in \mathcal{I}_A$, $I_A^{-1}(x) = I_A(x) - \frac{2\pi}{2^A} := \left\{y - \frac{2\pi}{2^A} : y \in I_A(x)\right\} \in \mathcal{I}_A$ the two adjacent intervals of the one dimensional interval $I_A(x)$. Remark that we mean the addition $y + \frac{2\pi}{2^A}$ and the subtraction $y - \frac{2\pi}{2^A}$ by modulo 2π. That is, for $y \in \mathbb{T}$ we also have $y + \frac{2\pi}{2^A}, y - \frac{2\pi}{2^A} \in \mathbb{T}$. Also use the notation $I_A^0(x) = I_A(x)$.

Also define the integral mean values of the function $f \in L^1(\mathbb{T})$ at $x \in \mathbb{T}$

$$E_{A,\delta}f(x) := \frac{2^A}{2\pi}\int_{I_A^\delta(x)} f(t)dt,$$

where $\delta \in \{-1, 0, 1\}$. For $A = (A_1, A_2) \in \mathbb{N}^2$, $f \in L^1(\mathbb{T}^2)$, $\delta = (\delta_1, \delta_2) \in \{-1, 0, 1\} \times \{-1, 0, 1\}$, $I_A^\delta(x) = I_{A_1}^{\delta_1}(x_1) \times I_{A_2}^{\delta_2}(x_2)$, $x \in \mathbb{T}^2$ set the two dimensional integral mean values

$$E_{A,\delta}f(x) := \frac{2^{A_1+A_2}}{4\pi^2}\int_{I_A^\delta(x)} f(t)dt.$$

In the sequel we suppose that the functions $\beta_j(x) = \lfloor \log_2 \tilde{a}_j(x) \rfloor$ are monotone increasing and the functions $\tilde{a}_j(x)$ are continuous ($j = 1, 2$). Let δ be as above and f an integrable two variable function. Define the maximal operator $E^*_{\beta,\delta}f := \sup_{t\in[1,+\infty)} |E_{\beta(t),\delta}f|$. With the application of some lemmas below we prove that the maximal operator $\sigma^* f = \sup_n |\sigma_{a(n)}f|$ is of weak type (L^1, L^1). In order to have this the first lemma is:

Lemma 5. *$E^*_{\beta,\delta}$ is of weak type (L^1, L^1). That is,* $\operatorname{mes}\left\{x \in \mathbb{T}^2 : E^*_{\beta,\delta}f(x) > \lambda\right\} \le C_\beta \|f\|_1/\lambda$ *for every positive λ.*

Proof. Apply Lemma 4. Recall its notation. $\operatorname{supp} f_i \subset I_{\beta(s_i)}(u_i) = I_{\beta_1(s_i)}(u_{i,1}) \times I_{\beta_2(s_i)}(u_{i,2})$ $(i \in \mathbb{P})$ and $F = \bigcup_{i=1}^\infty I_{\beta(s_i)}(u_i)$. Set

$$9F = \bigcup_{\epsilon_1,\epsilon_2\in\{-1,0,1\}} \bigcup_{i=1}^\infty I^\epsilon_{\beta(s_i)}(u_i).$$

Obviously, $\operatorname{mes}(9F) \le 9\operatorname{mes}(F)$. Fix a $\delta \in \{-1, 0, 1\}^2$ and let $y \in \mathbb{T}^2 \setminus 9F = \overline{9F}$ and $i \in \mathbb{P}$. Then we prove $E^*_{\beta,\delta}f_i(y) = 0$.

Suppose that $1 \le t \le s_i$.

If there exists a $j \in \{1, 2\}$ such that the one dimensional intervals $I^{\delta_j}_{\beta_j(t)}(y_j)$ and $I_{\beta_j(s_i)}(u_{i,j})$ are disjoint, then we have $I^\delta_{\beta(t)}(y) \cap I_{\beta(s_i)}(u_i) = \emptyset$ and consequently

$$E_{\beta(t),\delta}f_i(y) = \frac{2^{\beta_1(t)+\beta_2(t)}}{4\pi^2}\int_{I^\delta_{\beta(t)}(y)} f_i(x)dx = \frac{2^{\beta_1(t)+\beta_2(t)}}{4\pi^2}\int_{I^\delta_{\beta(t)}(y)\cap I_{\beta(s_i)}(u_i)} f_i(x)dx = 0.$$

If for both $j \in \{1, 2\}$ $I^{\delta_j}_{\beta_j(t)}(y_j) \cap I_{\beta_j(s_i)}(u_{i,j}) \ne \emptyset$, then since two one dimensional intervals are disjoint or one of them contains the other, then $\beta_j(t) \le \beta_j(s_i)$ (recall that β_j is monotone increasing) gives $I^{\delta_j}_{\beta_j(t)}(y_j) \supset I_{\beta_j(s_i)}(u_{i,j})$ $(j = 1, 2)$. That is, $I^\delta_{\beta(t)}(y) \supset I_{\beta(s_i)}(u_i)$ and this immediately gives

$$E_{\beta(t),\delta}f_i(y) = \frac{2^{\beta_1(t)+\beta_2(t)}}{4\pi^2}\int_{I^\delta_{\beta(t)}(y)} f_i(x)dx = \frac{2^{\beta_1(t)+\beta_2(t)}}{4\pi^2}\int_{I_{\beta(s_i)}(u_i)} f_i(x)dx = 0.$$

Now, we turn our attention to the other case:

Suppose that $t > s_i$.

$E_{\beta(t),\delta} f_i(y)$ is the integral mean value on the two dimensional rectangle $I^{\delta}_{\beta(t)}(y)$ and consequently we integrate function f_i on the set $I^{\delta}_{\beta(t)}(y) \cap I_{\beta(s_i)}(u_i)$. Therefore, if for either $j = 1$ or $j = 2$ $I^{\delta_j}_{\beta_j(t)}(y_j) \cap I_{\beta_j(s_i)}(u_{i,j}) = \emptyset$, then $E_{\beta(t),\delta} f_i(y) = 0$. That is, we can suppose that the intersection is not the empty set. Since β_j is monotone increasing, then $\beta_j(t) \geq \beta_j(s_i)$ and this implies $I^{\delta_j}_{\beta_j(t)}(y_j) \subset I_{\beta_j(s_i)}(u_{i,j})$ $(j = 1, 2)$. Thus, $I^{\delta_j}_{\beta_j(t)}(y_j) = I_{\beta_j(t)}(y_j) + \frac{2\pi\delta_j}{2^{\beta_j(t)}}$ gives $y_j + \frac{2\pi\delta_j}{2^{\beta_j(t)}} \in I_{\beta_j(s_i)}(u_{i,j})$. Consequently, if we add $-\frac{2\pi\delta_j}{2^{\beta_j(t)}}$ to $y_j + \frac{2\pi\delta_j}{2^{\beta_j(t)}}$ (modulo 2π), then applying the inequality $\left|\frac{2\pi\delta_j}{2^{\beta_j(t)}}\right| \leq \frac{2\pi}{2^{\beta_j(s_i)}}$ the result y_j will be an element of the union of the interval $I_{\beta_j(s_i)}(u_{i,j})$ and its two adjacent intervals. That is, $y_j \in I_{\beta_j(s_i)}(u_{i,j}) \cup I^1_{\beta_j(s_i)}(u_{i,j}) \cup I^{-1}_{\beta_j(s_i)}(u_{i,j})$ for $j = 1, 2$. This gives

$$y \in \bigcup_{\epsilon_1,\epsilon_2 \in \{-1,0,1\}} I^{\epsilon}_{\beta(s_i)}(u_i)$$

therefore $y \in 9F$ and this is a contradiction. That is, for all t we have $E_{\beta(t),\delta} f_i(y) = 0$ which gives $E^*_{\beta,\delta} f_i(y) = 0$ on $y \in \overline{9F}$. Now, turn back to the notation of Lemma 4.

$$\begin{aligned}
&\operatorname{mes}\left\{y \in \mathbb{T}^2 : E^*_{\beta,\delta} f(y) > 2C_\beta\lambda\right\} \\
&\leq \operatorname{mes}\left\{y \in \mathbb{T}^2 : E^*_{\beta,\delta} f_0(y) > C_\beta\lambda\right\} + \operatorname{mes}\left\{y \in \mathbb{T}^2 : E^*_{\beta,\delta}(\sum_{i=1}^{\infty} f_i)(y) > C_\beta\lambda\right\} \\
&\leq \operatorname{mes}\left\{y \in \mathbb{T}^2 : E^*_{\beta,\delta} f_0(y) > C_\beta\lambda\right\} + \operatorname{mes}(9F) + \operatorname{mes}\left\{y \in \overline{9F} : E^*_{\beta,\delta}(\sum_{i=1}^{\infty} f_i)(y) > C_\beta\lambda\right\} \\
&=: I + II + III.
\end{aligned}$$

It is quite easy to have that $\|E^*_{\beta,\delta} f_0\|_\infty \leq \|f_0\|_\infty \leq C_\beta\lambda$ and consequently $I = 0$. For III by the Markov inequality and the σ-sublinearity of operator $E^*_{\beta,\delta}$ we have

$$\begin{aligned}
III &\leq \frac{C_\beta}{\lambda} \int_{\overline{9F}} E^*_{\beta,\delta}(\sum_{i=1}^{\infty} f_i) \\
&\leq \frac{C_\beta}{\lambda} \int_{\overline{9F}} \sum_{i=1}^{\infty} E^*_{\beta,\delta} f_i = \frac{C_\beta}{\lambda} \sum_{i=1}^{\infty} \int_{\overline{9F}} E^*_{\beta,\delta} f_i = 0.
\end{aligned}$$

That is, $\operatorname{mes}\left\{E^*_{\beta,\delta} f > 2C_\beta\lambda\right\} \leq \operatorname{mes}(9F) \leq 9\|f\|_1/\lambda$. This completes the proof of Lemma 5. □

The following lemma is concerned with an estimate of the one dimensional Fejér kernel.

Lemma 6. *Let $n, A \in \mathbb{N}$, $2^A \leq n < 2^{A+1}$, that is, $A = \lfloor \log_2 n \rfloor$ $x, y \in \mathbb{T}$. Then we have*

$$K_n(y - x) \leq 2 \sum_{k=0}^{A} 2^{A-2k} 1_{\bigcup_{\delta \in \{-1,0,1\}} I^{\delta}_{A-k}(y)}(x). \tag{1}$$

Proof. First, in order to prove Lemma 6 we prove the following inequality for $0 \leq x$.

$$K_n(x) \leq 2 \sum_{k=0}^{A} 2^{A-2k} 1_{I_{A-k}(0)}(x). \tag{2}$$

If $x \in I_A(0)$, then since the right hand side of (2) for $k = 0$ is $2^A 1_{I_A(0)}(x) = 2^A$ and since the left hand side $K_n(x) \le \frac{n+1}{2} \le 2^A$, then the inequality (2) is proved in this case.

If $x \notin I_A(0)$, then we have a $j \in \{0, \dots, A-1\}$ such that $0 \le x \in I_j(0) \setminus I_{j+1}(0)$, which gives $\frac{\pi}{2^{j+1}} \le x < \frac{\pi}{2^j}$. By the inequality (one can find it in Bary's book [1])

$$0 \le K_n(x) \le \frac{\pi^2}{2(n+1)x^2} \quad (0 < |x| \le \pi, n \in \mathbb{N})$$

we have $K_n(x) \le \frac{2^{2j+1}}{2^A}$. What about the right hand side of (2)? Since $1_{I_j(0)}(x) = 1$, then we have $2\sum_{k=0}^A 2^{A-2k} 1_{I_{A-k}(0)}(x) \ge 2^{2j-A+1} 1_{I_j(0)}(x) = 2^{2j-A+1}$. That is, inequality (2) is proved. Now, we turn our attention to (1). Since the function K_n is even, then $K_n(y-x) = K_n(|y-x|) \le 2\sum_{k=0}^A 2^{A-2k} 1_{I_{A-k}(0)}(|y-x|)$. If $1_{I_{A-k}(0)}(|y-x|) = 1$, that is, $|y-x| \in I_{A-k}(0)$, then $|y-x| < \frac{\pi}{2^{A-k}}$ and this gives that x is an element one of the intervals $I^0_{A-k}(y), I^{-1}_{A-k}(y), I^1_{A-k}(y)$. That is, it is an element of $I_{A-k}(y)$ or one of its two adjacent intervals belonging to $\mathcal{I}_{A-k}$. Thus, $1_{\cup_{\delta\in\{-1,0,1\}} I^\delta_{A-k}(y)}(x) = 1$,

$$K_n(y-x) = K_n(|y-x|) \le 2\sum_{k=0}^{A} 2^{A-2k} 1_{I_{A-k}(0)}(|y-x|) \le 2\sum_{k=0}^{A} 2^{A-2k} 1_{\bigcup_{\delta\in\{-1,0,1\}} I^\delta_{A-k}(y)}(x).$$

This completes the proof of Lemma 6. $\square$

Now, we are ready to prove Lemma 2.

Proof of Lemma 2. By (1) we prove the following two dimensional inequality for $n \in \mathbb{P}^2$, $A = |n| := \lfloor \log_2 n \rfloor$, $f \in L^1(\mathbb{T}^2)$.

$$|\sigma_n f| \le C \sum_{\delta_1,\delta_2\in\{-1,0,1\}} \sum_{k_1\le A_1, k_2\le A_2} 2^{-k_1-k_2} E_{A-k,\delta}|f|. \tag{3}$$

By the help of Lemma 6 we have

$$\begin{aligned}
|\sigma_n f(y)| &= \left| \frac{1}{\pi^2} \int_{\mathbb{T}^2} f(x) K_n(y-x)\,dx \right| \\
&\le C \sum_{\delta_1,\delta_2\in\{-1,0,1\}} \sum_{k_1\le A_1, k_2\le A_2} 2^{A_1-2k_1} 2^{A_2-2k_2} \int_{\mathbb{T}^2} |f(x)| 1_{I^\delta_{A-k}(y)}(x)\,dx \\
&= C \sum_{\delta_1,\delta_2\in\{-1,0,1\}} \sum_{k_1\le A_1, k_2\le A_2} 2^{-k_1-k_2} E_{A-k,\delta}|f|(y).
\end{aligned}$$

That is, (3) is proved.

For $j=1,2$ the functions $\beta_j(x)=\lfloor\log_2\tilde{a}_j(x)\rfloor$ are monotone increasing on $[1,+\infty)$ ($j=1,2$) and consequently so do the functions $\beta_j(x)-k_j$ for $j=1,2$. We apply Lemma 5.

$$
\begin{aligned}
&\operatorname{mes}\left\{y\in\mathbb{T}^2 : |\sup_n \sigma_{a(n)}f(y)|>\lambda\right\}\\
&\leq \operatorname{mes}\left\{y\in\mathbb{T}^2 : C\sup_{t\in[1,+\infty)}\sum_{\delta_1,\delta_2\in\{-1,0,1\}}\sum_{k_1\leq\beta_1(t),k_2\leq\beta_2(t)}2^{-k_1-k_2}E_{\beta(t)-k,\delta}|f|(y)>\lambda\right\}\\
&\leq \operatorname{mes}\left\{y\in\mathbb{T}^2 : C\sum_{\delta_1,\delta_2\in\{-1,0,1\}}\sum_{k_1,k_2\in\mathbb{N}}2^{-k_1-k_2}E^*_{\beta-k,\delta}|f|(y)>\lambda\right\}\\
&\leq \operatorname{mes}\left(\bigcup_{\delta_1,\delta_2\in\{-1,0,1\}}\bigcup_{k_1,k_2\in\mathbb{N}}\left\{y\in\mathbb{T}^2 : 2^{-k_1-k_2}E^*_{\beta-k,\delta}|f|(y)>\frac{C\lambda}{(|k_1|+1)^2(|k_2|+1)^2}\right\}\right)\\
&\leq C\sum_{\delta_1,\delta_2\in\{-1,0,1\}}\sum_{k_1,k_2\in\mathbb{N}}\operatorname{mes}\left\{y\in\mathbb{T}^2 : 2^{-k_1-k_2}E^*_{\beta-k,\delta}|f|(y)>\frac{C\lambda}{(|k_1|+1)^2(|k_2|+1)^2}\right\}\\
&\leq C_\beta\sum_{\delta_1,\delta_2\in\{-1,0,1\}}\sum_{k_1,k_2\in\mathbb{N}}\frac{(|k_1|+1)^2(|k_2|+1)^2}{\lambda 2^{k_1+k_2}}\|f\|_1\\
&\leq C_\beta\|f\|_1/\lambda.
\end{aligned}
$$

That is, we proved that the maximal operator $\sigma_a^*f=\sup_n|\sigma_{a(n)}f|$ is of weak type (L^1,L^1). Since for each trigonometric polynomial P we have the everywhere relation

$\lim_{n\to\infty}\sigma_{(a_1(n),a_2(n))}P=P$, then by the standard density argument (see this principal for instance in [2]) the proof of Lemma 2 is complete. □

Finally, we have to prove Theorem 1. That is, the main result of this paper. The proof comes from Lemma 2 with some easy calculations.

Proof of Theorem 1. Without the loss of generality $a_j(n)\geq 1$ ($j=1,2,n\in\mathbb{N}$) can be supposed. Let L be a positive integer discussed later. For $l,m=0,1,\dots,L-1$ let some disjoint subsets of $\mathbb{N}$ be defined as:

$$
B_{l,m}=\left\{n\in\mathbb{N} : (a_1(n),a_2(n))\in\bigcup_{s,t=0}^{\infty}[2^{sL+l},2^{sL+l+1})\times[2^{tL+m},2^{tL+m+1})\right\}.
$$

It is clear that these sets are pairwise disjoint and their union is $\mathbb{N}$. Denote the elements of $B_{l,m}$ by $n_1^{l,m}<n_2^{l,m}<\dots$. We prove that $\lfloor\log_2 a_j(n_k^{l,m})\rfloor\leq\lfloor\log_2 a_j(n_{k+1}^{l,m})\rfloor$ for every $k\in\mathbb{N},l,m\in\{0,1,\dots L-1\}$ and $j=1,2$. On the contrary, suppose that $\lfloor\log_2 a_j(n_{k+1}^{l,m})\rfloor<\lfloor\log_2 a_j(n_k^{l,m})\rfloor$ for some k,l,m and j. Then the definition of $B_{l,m}$ gives that $\lfloor\log_2 a_j(n_{k+1}^{l,m})\rfloor\leq\lfloor\log_2 a_j(n_k^{l,m})\rfloor-L$. Thus,

$$
\frac{1}{2}a_j(n_{k+1}^{l,m})\leq 2^{\lfloor\log_2 a_j(n_{k+1}^{l,m})\rfloor}\leq 2^{\lfloor\log_2 a_j(n_k^{l,m})\rfloor-L}\leq\frac{1}{2^L}a_j(n_k^{l,m}).
$$

Since, $n_{k+1}^{l,m}>n_k^{l,m}$, then we have $a_j(n_{k+1}^{l,m})\geq\alpha a_j(n_k^{l,m})$ and consequently, also have $\alpha\leq 2^{1-L}$. This is obviously not possible for an L large enough. That is, we proved that $\lfloor\log_2 a_j(n_k^{l,m})\rfloor$

is monotone increasing with respect to $k \in \mathbb{N}$. Lemma 2 gives the a.e. convergence

$$\lim_{k\to\infty} \sigma_{a(n_k^{l,m})} f = f$$

for each integrable function f and $l, m = 0, 1, \ldots, L-1$. Merging the L^2 pieces of subsequences of $\sigma_{a(n)} f$ the proof of Theorem 1 is complete. $\square$

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