Reconstruction of functions via Walsh-Fourier cofficients

György Gát

College of Nyíregyháza, Inst. of Math. and Comp. Sci., Nyíregyháza, P.O.Box 166., H–4400, Hungary, gatgy@nyf.hu

Abstract. It is of main interest in the theory and also in applications of Fourier series that how to reconstruct a function from the partial sums of its Walsh-Fourier series. In 1955 Fine proved the Fejér-Lebesgue theorem, that is for each integrable function we have the almost everywhere convergence of Fejér means $\sigma_n f \to f$. It is also of prior interest that what can be said - with respect to this reconstruction issue - if we have only a subsequence of the partial sums. In this paper we give a brief résumé of the recent results with respect to this issue above also regarding the class of two-variable integrable functions.

Introduction

Let the numbers $n \in \mathbf{N}$ and $x \in I := [0,1)$ be expanded with respect to the binary number system:

$$n = \sum_{k=0}^{\infty} n_k 2^k,$$

$$x = \sum_{k=0}^{\infty} x_k 2^{-k-1},$$

$(n_k, x_k \in \{0,1\})$ where if x is a dyadic rational, that is an element of the set $\{k/2^n : k, n \in \mathbf{N}\}$, then we choose the finite expansion. Define the k -th Rademacher function as $r_k(x) = (-1)^{x_k} (x \in I, k \in \mathbf{N})$. Let $(\omega_n, n \in \mathbf{N})$ represent the Walsh-Paley system. That is, the n-th Walsh-Paley function is

$$\omega_n(x) = \prod_{k=0}^{\infty} (-1)^{n_k x_k}.$$

Consequiently, $\omega_n(x) = \prod_{k=0}^{\infty} r_k(x)$.

What is the Walsh-Paley system good for? The Walsh functions have many possible applications. Just to mention a few:

- Data transmission [29]
- Face, fingerprint identification [14]
- Approximate solution of integral and differential equations [24].

A great advantage of the Walsh functions (comparing the trigonometric ones) that they can take the values $+1$ and -1 only. Besides, it can be determined very efficiently that which Walsh function at what point is one or minus one.

The n-th Walsh-Fourier coefficient of the integrable function $f \in L^1(I)$ is

$$\hat{f}(n) := \int_I f(x)\omega_n(x)dx.$$

The n-th partial sum of the Walsh-Fourier series of the integrable function $f \in L^1(I)$:

$$S_n f(y) := \sum_{k=0}^{n-1} \hat{f}(k)\omega_k(y).$$

It is of main interest in the theory of Fourier series that how to reconstruct the function from the partial sums of its Fourier series. It is quite well known (see e.g. [21]) that for functions belonging the Lebesgue space L^p for $1 < p < \infty$ we have the relation $S_n f \to f$ with respect to almost everywhere convergence and also with respect to the L^p-norm convergence. In the case of $p = 1$ both relation fail to hold for all integrable function. That is, it is of prior interest to discuss this case.

The Cesàro means

The n-th Fejér or $(C, 1)$ mean of the function f is

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f.$$

In 1955 Fine proved [4] for the Walsh-Paley system the well known Fejér-Lebesgue theorem. Namely, for every integrable function f we have the a.e. relation

$$\sigma_n f \to f.$$

Let have a look for the situation with the (C, α) means. What are they? Let $A_n^\alpha := \frac{(1+\alpha)\dots(n+\alpha)}{n!}$, where $n \in \mathbf{N}$ and $\alpha \in \mathbf{R}$ $(-\alpha \notin \mathbf{N})$. It is known, that $A_n^\alpha \sim n^\alpha$.

The n-th (C, α) mean of the function $f \in L^1(I)$:

$$\sigma_{n+1}^\alpha f := \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f.$$

In 1975 Schipp proved [20], that $\sigma_n^\alpha f \to f$ a.e. for each $f \in L^1(I)$ and $\alpha > 0$. In other words, the maximal convergence space of the (C, α) means is the L^1 Lebesgue space. That is, the largest possible.

The Walsh-Paley system means the set of the Walsh functions and their ordering. Another well known, widely investigated (both in theoretical and application point of view) ordering gives the so called Walsh-Kaczmarz system.

This is nothing else, but a rearrangement of the Walsh-Paley system. Introduce it as follows.

If $n > 0$, then let $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$. The n-th Walsh-Kaczmarz function is

$$\kappa_n(x) := r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}},$$

as if $n > 0$, $\kappa_0(x) := 1, x \in I$. The elements of the a Walsh-Kaczmarz system and the Walsh-Paley system is a dyadic blockwise rearrangements of each other. This means as follows:

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{\omega_n : 2^k \leq n < 2^{k+1}\}.$$

We remark that the so called Hadamard matrices and transform are generated by the Walsh-Kaczmarz functions [21].

In 1998 Gát proved [6] the Fejér-Lebesgue theorem for the Walsh-Kaczmarz system. That is, $\sigma_n f \to f$ almost everywhere for each $f \in L^1(I)$. In 2004 Simon generalized [22] the result of Gát above for (C, α) summation methods.

It is also of prior interest that what can be said – with respect to this reconstruction issue – if we have only a subsequence of the partial sums. In 1936 Zalcwasser [28] asked how "rare" can be the sequence of positive integers $a(n)$ such that

$$\frac{1}{N}\sum_{n=1}^{N} S_{a(n)} f \to f. \tag{1}$$

This problem with respect to the trigonometric system was completely solved for continuous functions (uniform convergence) in [19], [27], [1], [3]. That is, if the sequence a is convex, then the condition

$$\sup_n n^{-1/2} \log a(n) < +\infty$$

is necessary and sufficient for the uniform convergence for every continuous function. For the time being, this issue with respect to the Walsh-Paley system has not been solved. Only, a sufficient condition is known, which is the same as in the trigonometric case. The paper about this is written by Glukhov [12]. See the more dimensional case also by Glukhov [13].

With respect to convergence almost everywhere, and integrable functions the situation is more complicated. Belinksky proved [2] for the trigonometric system the existence of a sequence $a(n) \sim \exp(\sqrt[3]{k})$ such that the relation (1) holds a.e. for every integrable function. In this paper Belinksky also conjectured that if the sequence a is convex, then the condition $\sup_n n^{-1/2} \log a(n) < +\infty$ is necessary and sufficient again. So, that would be the answer for the problem of Zalcwasser [28] in this point of view (trigonometric system, a.e. convergence and L^1 functions). The author of this paper proved [10] that this is not the case for the Walsh-Paley system. See below Theorem 1. On the other hand, this difference between the Walsh-Paley and the trigonometric system is not so surprising. Because of the following. Let $v(n) := \sum_{i=0}^{\infty} |n_i - n_{i+1}|, (n = \sum_{i=0}^{\infty} n_i 2^i)$ be the variation of the natural number n expanded in the number system based 2. It is a well-known result in the literature that for each sequence a tending strictly monotone increasing to plus infinity with the property $\sup_n v(a(n)) < +\infty$ we have the a.e. convergence $S_{a(n)}f \to f$ for all integrable function f. Is it also a necessary condition? This question of Balashov was answered by Konyagin [15] in the negative. He gave an example. That is, a sequence a with property $\sup_n v(a(n)) = +\infty$ and he proved that $S_{a(n)}f \to f$ a.e. for all integrable function f.

In paper [10] it is proved (see Theorem 1) that for each lacunary sequence a (that is $a(n+1)/a(n) \geq q > 1$) and each integrable function f the relation (1) holds a.e. This may also be interesting in the following point of view. If the sequence a is lacunary, then the a.e. relation $S_{a(n)}f \to f$ holds for all functions f in the Hardy space H. The trigonometric and the Walsh-Paley case can be found in [31] (trigonometric case) and [16] (Walsh-Paley case). But, the space H is a proper subspace of L^1. Therefore, it is of interest to investigate relation (1) for L^1 functions and lacunary sequence a.

In that paper - using the method of the proof of Theorem 1 one can find (Theorem 2) that for any convex sequence a (with $a(+\infty) = +\infty$ - of course) and for each integrable function the Riesz's logarithmic means of the function converges to the function almost everywhere. That is, the Riesz's logarithmic summability method can reconstruct the corresponding integrable function from any (convex) subsequence of the partial sums in the Walsh-Paley situation. For the time being there is no result known with respect to a.e. convergence of logarithmic means of subsequences of partial sums, neither in the trigonometric nor in the Walsh-Kaczmarz case.

The following a.e. convergence theorems with respect to the Fejér and logarithmic means of subsequences of the partial sums of the Walsh-Fourier series of integrable functions are proved by Gát [10].

Theorem 1. *Let $a : \mathbf{N} \to \mathbf{N}$ be a sequence with property $\frac{a(n+1)}{a(n)} \geq q > 1$ ($n \in \mathbf{N}$). Then for all integrable function $f \in L^1(Q)$ we have the a.e. relation*

$$\frac{1}{N}\sum_{n=1}^{N} S_{a(n)}f \to f.$$

Theorem 2. *Let $a : \mathbf{N} \to \mathbf{N}$ be a convex sequence with property $a(+\infty) = +\infty$. Then for each integrable function f we have the a.e. relation*

$$\frac{1}{\log N}\sum_{n=1}^{N} \frac{S_{a(n)}f}{n} \to f.$$

What can be said in the two (more) dimensional situation? This is quite a different story. Define the two-dimensional Walsh-Paley functions in the following way:

$$\omega_n(x) := \omega_{n_1}(x^1)\omega_{n_2}(x^2),$$

where $n = (n_1, n_2) \in \mathbf{N}^2$, $x = (x^1, x^2) \in I^2$. Let f be an integrable function. Its Fourier coefficients, rectangual partial sums of its Fourier series:

$$\hat{f}(n) := \int_{I^2} f(x)\omega_n(x)dx,$$

$$S_{n_1,n_2} f := \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \hat{f}(k_1, k_2)\omega_{k_1,k_2}.$$

Moreover, the two-dimensional Fejér or $(C, 1)$ means of the function $f \in L^1(I^2)$:

$$\sigma_{n_1,n_2} f := \frac{1}{n_1 n_2} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} S_{k_1,k_2} f \quad (n \in \mathbf{P}^2).$$

In 1931 Marczinkiewicz and Zygmund proved for the two-dimensional trigonometric system [17], and in 1992 Móricz, Schipp and Wade verified [18] for the two-dimensional Walsh-Paley system, that for every $f \in L\log^+ L(I^2)$ (which means $|f|\log^+|f|$ is integrable, where $\log^+ x = \max\{1, \log x\}$)

$$\sigma_{n_1,n_2} f \to f$$

a.e. as $\min\{n_1, n_2\} \to \infty$, that is, in the Pringsheim sense.

Since $L\log^+ L(I^2)$ is a proper subspace of $L^1(I^2)$, then it would be interesting to "enlarge" the convergence space, if possible. In 2000 Gát proved [7], that it is impossible. That is:

Theorem 3. *For each measurable function $\delta : [0, +\infty) \to [0, +\infty)$, $\delta(\infty) = 0$, (that is vanishing at plus infinity) there exists a*

$$f \in L\log^+ L\delta(L) \quad \textit{such that} \quad \sigma_{n_1,n_2} f \not\to f$$

a.e. (in the Pringsheim sense).

However, what "positive" can be said for the functions in $L^1(I^2)$ as if the a.e. convergence of the two-dimensional Fejér means in the Pringsheim sense can not be said? That could be the so called restricted convergence. For the two-dimensional trigonometric system Marcinkiewicz and Zygmund proved [30] in 1939, that

$$\sigma_{n_1,n_2} f \to f$$

a.e. for every $f \in L^1(I^2)$ as if $\min\{n_1, n_2\} \to \infty$, provided that

$$2^{-\alpha} \le \frac{n_1}{n_2} \le 2^{\alpha}$$

for some $\alpha \ge 0$. In other words, the set of admissible indices (n_1, n_2) remains in some cone. This theorem for the two-dimensional Walsh-Paley system was verified by Móricz, Schipp and Wade [18] in 1992 in the case when n_1, n_2 both are powers of two. That is,

$$\sigma_{2^{n_1},2^{n_2}} f \to f$$

a.e. for every $f \in L^1(I^2)$ as if $\min\{n_1, n_2\} \to \infty$, provided that $|n_1 - n_2| \leq \alpha$ for some $\alpha \geq 0$.

The proof of the Marcinkiewicz-Zygmund theorem [30] (with respect to the Walsh-Paley system) for arbitrary set of indices remaining in some cone is due to Gát and Weisz [5], [25], separately in 1996.

It is an interesting question that it is possible to weaken somehow the "cone restriction" in a way that a.e. convergence remains for each function in L^1. Maybe for some "interim space" – if not for space L^1. The answer is negative both in the point of view of space and in the point of view of restriction. Namely, in 2001 Gát proved [8] the theorem below:

Theorem 4. *Let $\delta : [0, +\infty) \to [0, +\infty)$ measurable, $\delta(+\infty) = 0$ and let $w : \mathbf{N} \to [1, +\infty)$ be an arbitrary increasing function such that*

$$\sup_{x \in \mathbf{N}} w(x) = +\infty.$$

Moreover, $\vee n := \max\{n_1, n_2\}$, $\wedge n := \min\{n_1, n_2\}$. Then, there exists a function $f \in L \log^+ L\delta(L)$ such that

$$\sigma_{n_1,n_2} f \not\to f$$

a.e. as $\wedge n \to \infty$ such that the restriction condition $\frac{\vee n}{\wedge n} \leq w(\wedge n)$ is also fulfilled.

That is there is no "interim" space. Either we have space $L \log^+ L$ and "no restriction at all", or the "cone restriction" and then the maximal convergence space is L^1. As a consequence of this we have that

$$\sigma_{n_1,n_2} f \to f$$

a.e. for each $f \in L(I^2)$ as $\min\{n_1, n_2\} \to \infty$, provided that

$$\frac{\vee n}{\wedge n} \leq w(\wedge n)$$

if and only if

$$\sup w(x) < \infty.$$

Another question. What is the situation with the (C, α) summation of 2-dimensional Walsh-Fourier series? What is this?

$$\sigma^{\alpha}_{n_1+1,n_2+1} f = \frac{1}{A^{\alpha}_{n_1} A^{\alpha}_{n_2}} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} A^{\alpha-1}_{n_1-k_1} A^{\alpha-1}_{n_2-k_2} S_{k_1,k_2} f.$$

In 1999 Weisz proved [26], that

$$\sigma^{\alpha}_{n_1,n_2} f \to f$$

a.e. as $\min\{n_1, n_2\} \to \infty$ for each $f \in L \log^+ L(I^2)$ and $\alpha > 0$.

The question is that it is possible to give a "larger" convergence space for the (C, α) summability method $(\alpha > 0)$? Is there such an α? If $\alpha \leq 1$, then not. Because for the $(C, 1)$ method one can not give such a "larger" space.

Problem 1. What is the situation with the (C, α) methods, for $\alpha > 1$ in the two (more) dimensional case?

Problem 2. What is the situation with the Cesàro summation of the 2-dimensional Walsh-Kaczmarz system? There is no divergence results with respect to this system and Cesàro means.

Only we have positive results. In 2001 Simon proved [23], that $\sigma_{n_1,n_2} f \to f$ a.e. as if $\min\{n_1, n_2\} \to \infty$ (in the Pringsheim sense) for every $f \in L \log^+ L(I^2)$. He also proved the restricted "cone" convergence for functions belonging to $L^1(I^2)$.

In 2007 Gát generalized [9] the notion of cone restriction of two dimensional Fejér means used by Marcinkiewicz and Zygmund in a way that instead of straight lines he used curved lines. In this paper one can find necessary and sufficient conditions for the almost everywhere convergence of the two dimensional trigonometric Fejér means of integrable functions. Finally, I mention that the author of this paper and Nagy verified [11] the Walsh-Paley version of these results.

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