Noncommutative compact totally disconnected groups

R. Toledo

College of Nyíregyháza

February 9, 2011
Las Palmas de Gran Canaria

The Rademacher functions H. A. Rademacher (1922)

$$r(x) = \operatorname{sgn}(\sin(2^{n+1}\pi x)) \qquad x \in [0,1[.$$

The binary expansion of n: $(n_0, n_1, \dots)$

Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k 2^k, \qquad \text{where } n_k = 0 \text{ or } n_k = 1.$$

The Walsh-Paley system

$$\omega_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (x \in [0,1[).$$

The Rademacher functions H. A. Rademacher (1922)

$$r(x) = \operatorname{sgn}(\sin(2^{n+1}\pi x)) \qquad x \in [0, 1[.$$

The binary expansion of n: $(n_0, n_1, \dots)$

Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k 2^k, \qquad \text{where } n_k = 0 \text{ or } n_k = 1.$$

The Walsh-Paley system

$$\omega_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (x \in [0, 1[).$$

The Rademacher functions

H. A. Rademacher (1922)

$$r(x) = \mathrm{sgn}(\sin(2^{n+1}\pi x)) \qquad x \in [0,1[.$$

The binary expansion of n: $(n_0, n_1, \dots)$

Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k 2^k, \qquad \text{where } n_k = 0 \text{ or } n_k = 1.$$

The Walsh-Paley system

$$\omega_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (x \in [0,1[).$$

The characters of the Dyadic group

The Dyadic group $\left(G := \underset{k=0}{\overset{\infty}{\times}} \mathbb{Z}_2\right)$

is the complete product of cyclic groups of order 2, with discrete topology and assign each singleton the measure $\frac{1}{2}$. G has the product topology and measure. (Haar measure)

The system of characters

Define $\varphi(x) = (-1)^x$, $(x \in \mathbb{Z}_2)$. For each $n \in \mathbf{N}$ with binary expansion $(n_0, n_1, \ldots)$ let

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi^{n_k}(x_k), \quad (x = (x_0, x_1, \ldots) \in G).$$

Theorem

The system of characters is an orthonormal and complete system on $L^2(G)$.

The characters of the Dyadic group

The Dyadic group $\left(G := \bigtimes_{k=0}^{\infty} \mathbb{Z}_2\right)$

is the complete product of cyclic groups of order 2, with discrete topology and assign each singleton the measure $\frac{1}{2}$. G has the product topology and measure. (Haar measure)

The system of characters

Define $\varphi(x) = (-1)^x$, $(x \in \mathbb{Z}_2)$. For each $n \in \mathbf{N}$ with binary expansion $(n_0, n_1, \dots)$ let

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi^{n_k}(x_k), \quad (x = (x_0, x_1, \dots) \in G).$$

Theorem

The system of characters is an orthonormal and complete system on $L^2(G)$.

The characters of the Dyadic group

The Dyadic group $\left(G := \mathop{\times}\limits_{k=0}^{\infty} \mathbb{Z}_2\right)$

is the complete product of cyclic groups of order 2, with discrete topology and assign each singleton the measure $\frac{1}{2}$. G has the product topology and measure. (Haar measure)

The system of characters

Define $\varphi(x) = (-1)^x$, $(x \in \mathbb{Z}_2)$. For each $n \in \mathbf{N}$ with binary expansion $(n_0, n_1, \dots)$ let

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi^{n_k}(x_k), \quad (x = (x_0, x_1, \dots) \in G).$$

Theorem

The system of characters is an orthonormal and complete system on $L^2(G)$.

The Fine's map

N. J. Fine (1949)

For any $x \in [0,1[$ there exists a sequence of numbers 0 and 1 such that

$$x := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \qquad ((x_0, x_1, \ldots) \in G),$$

but only the numbers $p/2^n$ have two expressions of this form. In this case we have the one which terminates in 0's. Define Fine's map by

$$\rho(x) = (x_0, x_1, \ldots) \in G.$$

Fine's map gives a natural relation between the new structure of $[0,1[$ and the structure of G (Harmonic analysis).

- The Haar measure corresponds to the Lebesgue measure.
- The characters of G corresponds to the Walsh-Paley system.

The Fine's map

N. J. Fine (1949)

For any $x \in [0,1[$ there exists a sequence of numbers 0 and 1 such that

$$x := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \qquad ((x_0, x_1, \dots) \in G),$$

but only the numbers $p/2^n$ have two expressions of this form. In this case we have the one which terminates in 0's. Define Fine's map by

$$\rho(x) = (x_0, x_1, \dots) \in G.$$

Fine's map gives a natural relation between the new structure of $[0,1[$ and the structure of G (Harmonic analysis).

- The Haar measure corresponds to the Lebesgue measure.
- The characters of G corresponds to the Walsh-Paley system.

A Vilenkin group $\left(G := \bigtimes_{k=0}^{\infty} \mathbb{Z}_{m_k}\right)$ N. Ja. Vilenkin (1947)

is the complete product of cyclic groups of order m_k ($m_k \geq 2$, $k \in \mathbf{N}$), with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. G has the product topology and measure. (Haar measure)

Bounded Vilenkin group

if the sequence $m = (m_0, m_1, \dots)$ is a bounded sequence.

The generalized Rademacher functions

$$\varphi_k^s(x) = \exp(2\pi \imath s x / m_k) \qquad (s \in \{0, \dots m_k - 1\},\ x \in \mathbb{Z}_{m_k},\ \imath^2 = -1)$$

The generalized Rademacher functions are the characters of cyclic groups.

The Vilenkin groups

A Vilenkin group $\left(G := \underset{k=0}{\overset{\infty}{\times}} \mathbb{Z}_{m_k}\right)$ N. Ja. Vilenkin (1947)

is the complete product of cyclic groups of order m_k ($m_k \geq 2$, $k \in \mathbf{N}$), with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. G has the product topology and measure. (Haar measure)

Bounded Vilenkin group

if the sequence $m = (m_0, m_1, \dots)$ is a bounded sequence.

The generalized Rademacher functions

$$\varphi_k^s(x) = \exp(2\pi\imath sx/m_k) \qquad (s \in \{0, \dots m_k - 1\},\ x \in \mathbb{Z}_{m_k},\ \imath^2 = -1)$$

The generalized Rademacher functions are the characters of cyclic groups.

The Vilenkin groups

A Vilenkin group $\left(G := \bigtimes_{k=0}^{\infty} \mathcal{Z}_{m_k} \right)$ N. Ja. Vilenkin (1947)

is the complete product of cyclic groups of order m_k ($m_k \geq 2$, $k \in \mathbf{N}$), with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. G has the product topology and measure. (Haar measure)

Bounded Vilenkin group

if the sequence $m = (m_0, m_1, \dots)$ is a bounded sequence.

The generalized Rademacher functions

$$\varphi_k^s(x) = \exp(2\pi \imath s x / m_k) \qquad (s \in \{0, \dots m_k - 1\},\ x \in \mathcal{Z}_{m_k},\ \imath^2 = -1)$$

The generalized Rademacher functions are the characters of cyclic groups.

A Vilenkin group $\left(G := \bigtimes_{k=0}^{\infty} \mathcal{Z}_{m_k}\right)$ N. Ja. Vilenkin (1947)

is the complete product of cyclic groups of order m_k ($m_k \geq 2$, $k \in \mathbf{N}$), with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. G has the product topology and measure. (Haar measure)

Bounded Vilenkin group

if the sequence $m = (m_0, m_1, \dots)$ is a bounded sequence.

The generalized Rademacher functions

$$\varphi_k^s(x) = \exp(2\pi\imath sx/m_k) \qquad (s \in \{0, \dots m_k - 1\},\ x \in \mathcal{Z}_{m_k},\ \imath^2 = -1)$$

The generalized Rademacher functions are the characters of cyclic groups.

The m-adic expansion of n: $(n_0, n_1, \ldots)$

Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, $(k \in \mathbf{N})$. Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \leq n_k < m_k).$$

A Vilenkin system

is the product system of φ:

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x = (x_0, x_1, \ldots) \in G).$$

Theorem

The functions of the Vilenkin system are the characters of the Vilenkin group, thus it is an orthonormal and complete system on $L^2(G)$.

The Vilenkin systems

The *m*-adic expansion of n: $(n_0, n_1, \dots)$

Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, $(k \in \mathbf{N})$. Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \leq n_k < m_k).$$

A Vilenkin system

is the product system of φ:

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x = (x_0, x_1, \dots) \in G).$$

Theorem

The functions of the Vilenkin system are the characters of the Vilenkin group, thus it is an orthonormal and complete system on $L^2(G)$.

The m-adic expansion of n: $(n_0, n_1, \dots)$

Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, $(k \in \mathbf{N})$. Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \leq n_k < m_k).$$

A Vilenkin system

is the product system of φ:

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x = (x_0, x_1, \dots) \in G).$$

Theorem

The functions of the Vilenkin system are the characters of the Vilenkin group, thus it is an orthonormal and complete system on $L^2(G)$.

The complete product of finite groups

The group $\left(G := \underset{k=0}{\overset{\infty}{\times}} G_k\right)$

Denote by G the complete product of arbitrary finite groups of order m_k ($m_k \geq 2$, $k \in \mathbf{N}$), with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. G has the product topology and measure. (Haar measure)

$\varphi_k^s =?$, $\psi_n =?$ $\qquad \rightarrow \qquad$ Harmonic Analysis

The complete product of finite groups

The group $\left(G := \mathop{\times}\limits_{k=0}^{\infty} G_k\right)$

Denote by G the complete product of arbitrary finite groups of order m_k ($m_k \geq 2$, $k \in \mathbf{N}$), with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. G has the product topology and measure. (Haar measure)

$\varphi_k^s =?$, $\psi_n =?$ $\quad \rightarrow \quad$ Harmonic Analysis

The complete product of finite groups

The group $\left(G := \bigtimes_{k=0}^{\infty} G_k \right)$

Denote by G the complete product of arbitrary finite groups of order m_k ($m_k \geq 2$, $k \in \mathbf{N}$), with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. G has the product topology and measure. (Haar measure)

$\varphi_k^s =?$, $\psi_n =?$ $\rightarrow$ Harmonic Analysis

Orthonormal systems on finite groups

The dual object (Σ_k) of the finite group G_k $(k \in \mathbf{N})$

is the set of all continuous irreducible unitary representations of the group G_k which are not equivalents.

The Coordinate functions

For any $\sigma \in \Sigma_k$, let $\{\xi_1, \ldots, \xi_{d_\sigma}\}$ be a fixed basis of the representation space of a representation $U^{(\sigma)}$ in the class σ having the dimension d_σ. The Coordinate functions:

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j \rangle, \quad i,j \in \{1, ..., d_\sigma\}, \sigma \in \Sigma_k$$

Orthonormal systems on finite groups

The dual object (Σ_k) of the finite group G_k ($k \in \mathbf{N}$)

is the set of all continuous irreducible unitary representations of the group G_k which are not equivalents.

The Coordinate functions

For any $\sigma \in \Sigma_k$, let $\{\xi_1, \ldots, \xi_{d_\sigma}\}$ be a fixed basis of the representation space of a representation $U^{(\sigma)}$ in the class σ having the dimension d_σ. The Coordinate functions:

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j \rangle, \quad i, j \in \{1, ..., d_\sigma\}, \sigma \in \Sigma_k$$

The system φ_k

We order the all normalized coordinate functions of the finite group G_k ($\varphi_k^0(x) = 1$) to obtain exactly m_k number of functions.

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_k,\ s = 0, \ldots, m_k - 1),$$

where $\sigma \in \Sigma_k$, $i, j \in \{1, ..., d_\sigma\}$.

Theorem

The system φ_k is an orthonormal and complete system on $L^2(G_k)$.

The system φ_k

We order the all normalized coordinate functions of the finite group G_k $(\varphi_k^0(x) = 1)$ to obtain exactly m_k number of functions.

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_k,\ s = 0, \ldots, m_k - 1),$$

where $\sigma \in \Sigma_k$, $i, j \in \{1, ..., d_\sigma\}$.

Theorem

The system φ_k is an orthonormal and complete system on $L^2(G_k)$.

Example 1: The permutation group of 3 elements, $\mathcal{S}_3$

	e	(12)	(13)	(23)	(123)	(132)	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

$$\max_{s=0\ldots5} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3}$$

Example 1: The permutation group of 3 elements, $\mathcal{S}_3$

	e	(12)	(13)	(23)	(123)	(132)	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

$$\max_{s=0\ldots5} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3}$$

Example 1: The permutation group of 3 elements, $\mathcal{S}_3$

	e	(12)	(13)	(23)	(123)	(132)	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

$$\max_{s=0\ldots5} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3}$$

Example 1: The permutation group of 3 elements, $\mathcal{S}_3$

	e	(12)	(13)	(23)	(123)	(132)	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

$$\max_{s=0\ldots5} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3}$$

Example 2: The quaternion group of order 8: $\mathcal{Q}_2 := \{[a,b] : a^4 = e,\ b^2 = a^2,\ bab^{-1} = a^3\}$

	e	a	a^2	a^3	b	ab	a^2b	a^3b	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1	1	1
φ^2	1	-1	1	-1	1	-1	1	-1	1	1
φ^3	1	-1	1	-1	-1	1	-1	1	1	1
φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}\imath$	$\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$

$$\max_{s=0\ldots7} \|\varphi^s\|_1 \|\varphi^s\|_\infty = 1$$

Example 2: The quaternion group of order 8: $\mathcal{Q}_2 := \{[a,b] : a^4 = e,\ b^2 = a^2,\ bab^{-1} = a^3\}$

	e	a	a^2	a^3	b	ab	a^2b	a^3b	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1	1	1
φ^2	1	-1	1	-1	1	-1	1	-1	1	1
φ^3	1	-1	1	-1	-1	1	-1	1	1	1
φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}\imath$	$\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$

$$\max_{s=0\ldots7} \|\varphi^s\|_1 \|\varphi^s\|_\infty = 1$$

Example 2: The quaternion group of order 8: $\mathcal{Q}_2 := \{[a,b] : a^4 = e,\ b^2 = a^2,\ bab^{-1} = a^3\}$

	e	a	a^2	a^3	b	ab	a^2b	a^3b	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1	1	1
φ^2	1	-1	1	-1	1	-1	1	-1	1	1
φ^3	1	-1	1	-1	-1	1	-1	1	1	1
φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}\imath$	$\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$

$$\max_{s=0\ldots7} \|\varphi^s\|_1 \|\varphi^s\|_\infty = 1$$

Example 2: The quaternion group of order 8: $\mathcal{Q}_2 := \{[a, b] : a^4 = e,\ b^2 = a^2,\ bab^{-1} = a^3\}$

	e	a	a^2	a^3	b	ab	a^2b	a^3b	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1	1	1
φ^2	1	-1	1	-1	1	-1	1	-1	1	1
φ^3	1	-1	1	-1	-1	1	-1	1	1	1
φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}\imath$	$\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$

$$\max_{s=0\ldots 7} \|\varphi^s\|_1 \|\varphi^s\|_\infty = 1$$

The m-adic expansion of n: $(n_0, n_1, \dots)$

Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, $(k \in \mathbf{N})$. Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \leq n_k < m_k).$$

A representative product systems G. Gát and R. Toledo (1996)

is the product system of φ:

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G).$$

The m-adic expansion of n: $(n_0, n_1, \dots)$

Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, $(k \in \mathbf{N})$. Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \leq n_k < m_k).$$

A representative product systems G. Gát and R. Toledo (1996)

is the product system of φ:

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G).$$

Theorem

A representative product system is an orthonormal and complete system on $L^2(G)$.

Characteristics of the system ψ for noncommutative cases:

- It is not uniformly bounded.
- It takes the value 0.

Representative product systems

Gát, G., Toledo, R., Anal. Math., 1996

Theorem

A representative product system is an orthonormal and complete system on $L^2(G)$.

Characteristics of the system ψ for noncommutative cases:

- It is not uniformly bounded.
- It takes the value 0.

Representative product systems
Gát, G., Toledo, R., Anal. Math., 1996

Theorem

A representative product system is an orthonormal and complete system on $L^2(G)$.

Characteristics of the system ψ for noncommutative cases:

- It is not uniformly bounded.
- It takes the value 0.

The representation of G on $[0, 1[$

Toledo, R., Acta Math. Acad. Paed. Nyregyhziensis, 2003

It is similar to the dyadic group, but first we need to enumerate the elements of all groups G_k, $(k \in \mathbf{N})$ in an arbitrary way but the first is always their identity.

$$G_k \ni x \overset{\text{bijection}}{\to} \overline{x} \in \{0, 1, \ldots, m_k - 1\}, \qquad \overline{e} = 0.$$

Fine's map and norm

With the bijection above we can introduce the Fine's map:

$$\rho(x) = (x_0, x_1, \ldots) \in G.$$

and norm:

$$|x| := \sum_{k=0}^{\infty} \frac{\overline{x_k}}{M_{k+1}} \qquad (x = (x_0, x_1, \ldots) \in G)$$

It is similar to the dyadic group, but first we need to enumerate the elements of all groups G_k, ($k \in \mathbf{N}$) in an arbitrary way but the first is always their identity.

$$G_k \ni x \overset{\text{bijection}}{\rightarrow} \overline{x} \in \{0, 1, \ldots, m_k - 1\}, \qquad \overline{e} = 0.$$

Fine's map and norm

With the bijection above we can introduce the Fine's map:

$$\rho(x) = (x_0, x_1, \ldots) \in G.$$

and norm:

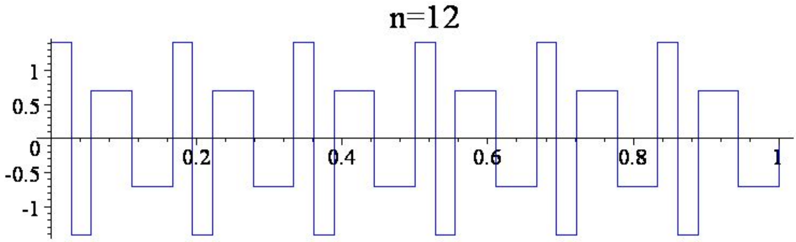
$$|x| := \sum_{k=0}^{\infty} \frac{\overline{x_k}}{M_{k+1}} \qquad (x = (x_0, x_1, \ldots) \in G)$$

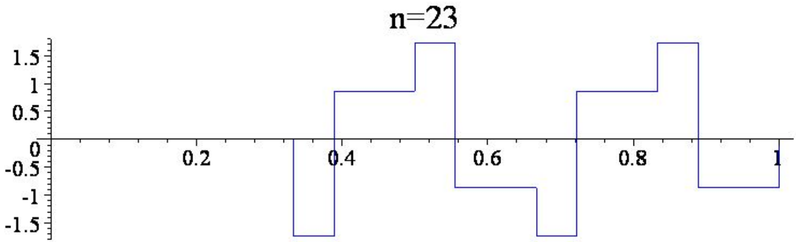
Here Fine's map also gives a natural relation between the new structure of $[0,1[$ and the structure of G (Harmonic analysis).

- The Haar measure corresponds to the Lebesgue measure.
- The new systems $\psi_n \circ \rho$ are orthonomal and complete systems on $[0,1[$, but they are not necessary uniformly bounded.

The complete product of $\mathcal{S}_3$

Plotted by Maple





Convergence in L^p-norm of Fourier series

The problem: Which are the values of p ($1 \leq p < \infty$) such that for all function $f \in L^p(G)$ the sequence of partial sums $S_n f$ of the Fourier series of f converges to the function f in L^p-norm?

For $p = 2$ the answer is affirmative. ($L^2(G)$ is a Hilbert space)

Theorem (P. Simon, F. Schipp and W. S. Young (1976))

Let G be a Vilenkin group and $1 < p < \infty$. Then for all function $f \in L^p(G)$ the sequence of partial sums $S_n f$ of the Fourier series of f converges to the function f in L^p-norm.

Convergence in L^p-norm of Fourier series

The problem: Which are the values of p ($1 \leq p < \infty$) such that for all function $f \in L^p(G)$ the sequence of partial sums $S_n f$ of the Fourier series of f converges to the function f in L^p-norm?

For $p = 2$ the answer is affirmative. ($L^2(G)$ is a Hilbert space)

Theorem (P. Simon, F. Schipp and W. S. Young (1976))

Let G be a Vilenkin group and $1 < p < \infty$. Then for all function $f \in L^p(G)$ the sequence of partial sums $S_n f$ of the Fourier series of f converges to the function f in L^p-norm.

Convergence in L^p-norm of Fourier series

The problem: Which are the values of p ($1 \leq p < \infty$) such that for all function $f \in L^p(G)$ the sequence of partial sums $S_n f$ of the Fourier series of f converges to the function f in L^p-norm?

For $p = 2$ the answer is affirmative. ($L^2(G)$ is a Hilbert space)

Theorem (P. Simon, F. Schipp and W. S. Young (1976))

Let G be a Vilenkin group and $1 < p < \infty$. Then for all function $f \in L^p(G)$ the sequence of partial sums $S_n f$ of the Fourier series of f converges to the function f in L^p-norm.

Convergence in L^p-norm of Fourier series

Toledo, R., Proceedings of the Alexits Memorial Conference, Budapest, 2002

Theorem

For all G groups there exists a function $f \in L^1(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^1-norm.

For $p = 1$ the answer is negative. (It was a very well known result for commutative groups.)

Theorem

For all G groups there exists a function $f \in L^1(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^1-norm.

For $p = 1$ the answer is negative. (It was a very well known result for commutative groups.)

The sequence Ψ

$$\Psi_k = \prod_{i=0}^{k-1} \max_{s<m_i} \|\varphi_i^s\|_1 \|\varphi_i^s\|_\infty \qquad (k \in \mathbf{N}).$$

Theorem

If G is a bounded group with unbounded sequence Ψ, then for all $p \neq 2$, $1 < p < \infty$ there exists a function $f \in L^p(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^p-norm.

For the complete product of $\mathcal{S}_3$ the answer is negative for all $1 < p < \infty$, except $p = 2$.

The sequence Ψ

$$\Psi_k = \prod_{i=0}^{k-1} \max_{s<m_i} \|\varphi_i^s\|_1 \|\varphi_i^s\|_\infty \qquad (k \in \mathbf{N}).$$

Theorem

If G is a bounded group with unbounded sequence Ψ, then for all $p \neq 2$, $1 < p < \infty$ there exists a function $f \in L^p(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^p-norm.

For the complete product of $\mathcal{S}_3$ the answer is negative for all $1 < p < \infty$, except $p = 2$.

The sequence Ψ

$$\Psi_k = \prod_{i=0}^{k-1} \max_{s<m_i} \|\varphi_i^s\|_1 \|\varphi_i^s\|_\infty \qquad (k \in \mathbf{N}).$$

Theorem

If G is a bounded group with unbounded sequence Ψ, then for all $p \neq 2$, $1 < p < \infty$ there exists a function $f \in L^p(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^p-norm.

For the complete product of $\mathcal{S}_3$ the answer is negative for all $1 < p < \infty$, except $p = 2$.

The problem is open for an arbitrary, but bounded group G with bounded sequence Ψ. However

Theorem

Let G be the complete product of $\mathcal{Q}_2$ having representations and the enumeration appeared in the table before. Thus $S_n f \to f$ in L^p-norm for all $f \in L^p(G)$ and $1 < p < \infty$.

The problem is open for an arbitrary, but bounded group G with bounded sequence Ψ. However

Theorem

Let G be the complete product of $\mathcal{Q}_2$ having representations and the enumeration appeared in the table before. Thus $S_n f \to f$ in L^p-norm for all $f \in L^p(G)$ and $1 < p < \infty$.

Main ideas of the proof

	e	a	a^2	a^3	b	ab	a^2b	a^3b
φ^0	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1
φ^2	1	-1	1	-1	1	-1	1	-1
φ^3	1	-1	1	-1	-1	1	-1	1
φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}\imath$	$\sqrt{2}$	$\sqrt{2}\imath$

Define by I the set of indexes i of the expansion of n such that $4 \leq n_i \leq 7$ and order the elements of I using the decreasing notation

$$\alpha(1) > \alpha(2) > \cdots > \alpha(L).$$

Main ideas of the proof

	e	a	a^2	a^3	b	ab	a^2b	a^3b
φ^0	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1
φ^2	1	-1	1	-1	1	-1	1	-1
φ^3	1	-1	1	-1	-1	1	-1	1
φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}\imath$	$\sqrt{2}$	$\sqrt{2}\imath$

Define by I the set of indexes i of the expansion of n such that $4 \leq n_i \leq 7$ and order the elements of I using the decreasing notation

$$\alpha(1) > \alpha(2) > \cdots > \alpha(L).$$

Main ideas of the proof

Denote

$$B_i := \begin{cases} \{x \in G \colon x_{\alpha(i)} \in \{e, a, a^2, a^3\} & \text{for } 4 \leq n_{\alpha(i)} \leq 5 \\ \{x \in G \colon x_{\alpha(i)} \in \{b, ab, a^2b, a^3b\} & \text{for } 6 \leq n_{\alpha(i)} \leq 7 \end{cases}$$

and

$$B_0^* := \overline{B_1}$$

$$B_l^* := B_1 \cap B_2 \cap \cdots \cap B_l \cap \overline{B_{l+1}}, \qquad (l = 1, 2, \ldots, L-1)$$

$$B_L^* := B_1 \cap B_2 \cap \cdots \cap B_L$$

Thus the sets $B_0^*, B_1^*, \ldots, B_L^*$ are a disjoint partition of G.

Main ideas of the proof

Lemma

Let $k \in \mathbf{N}$ *and* $x \in B_k^*$. *If* $y \notin B_k^*$, *then* $D_n(x, y) = 0$.

Lemma

For all $k = 0, 1, \ldots, L$ *and* $f \in L^2(G)$

$$\int_{B_k^*} \left| \int_{B_k^*} f(y) D_n(x, y)\, dy \right|^2 dx \leq c \int_{B_k^*} |f(x)|^2\, dx$$

Lemma

For all $k = 0, 1, \ldots, L$ *and* $f \in L^1(G)$

$$\mu(y \in B_k^* : |S_n f(y)| > \lambda) \leq c \int_{B_k^*} |f(x)|\, dx / \lambda$$

Main ideas of the proof

Lemma

Let $k \in \mathbf{N}$ *and* $x \in B_k^*$. *If* $y \notin B_k^*$, *then* $D_n(x,y) = 0$.

Lemma

For all $k = 0, 1, \ldots, L$ *and* $f \in L^2(G)$

$$\int_{B_k^*} \left| \int_{B_k^*} f(y) D_n(x,y)\, dy \right|^2 dx \leq c \int_{B_k^*} |f(x)|^2\, dx$$

Lemma

For all $k = 0, 1, \ldots, L$ *and* $f \in L^1(G)$

$$\mu(y \in B_k^* \colon |S_n f(y)| > \lambda) \leq c \int_{B_k^*} |f(x)|\, dx / \lambda$$

Main ideas of the proof

Lemma

Let $k \in \mathbf{N}$ and $x \in B_k^*$. If $y \notin B_k^*$, then $D_n(x,y) = 0$.

Lemma

For all $k = 0, 1, \ldots, L$ and $f \in L^2(G)$

$$\int_{B_k^*} \left| \int_{B_k^*} f(y) D_n(x,y)\, dy \right|^2 dx \leq c \int_{B_k^*} |f(x)|^2\, dx$$

Lemma

For all $k = 0, 1, \ldots, L$ and $f \in L^1(G)$

$$\mu(y \in B_k^* \colon |S_n f(y)| > \lambda) \leq c \int_{B_k^*} |f(x)|\, dx / \lambda$$

Thank you for your attention!