CONVERGENCE OF WALSH-FOURIER SERIES OF A CLASS $BO\left(p\left(n\right)\uparrow\infty\right)$

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Abstract. We study the pointwise convergence of Walsh-Fourier series of the functions on the class of generalized p-bounded fluctuation.

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P}\bigcup\{0\}$. The concept of the dyadic group is given by the following way. Let Z_2 be the cyclic group of order 2 with discrete topology. We obtain a Haar measure on Z_2 if the measure of all singleton is 1/2. The dyadic group G is the compact group formed by the complete direct product of Z_2 with the product of the topologies, operations $(+)$ and measures (μ). Thus each $x \in G$ consist of sequences $x := (x_0, x_1, \dots)$, where $x_k = 0$ or $x_k = 1$ $(k \in \mathbf{N})$. We call this sequence the expansion of x. The dyadic group is metrizable by the norm:

$$|x| := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \qquad (x \in G).$$

We get a countable base of the topology of G with the sets of dyadic intervals:

$$I_0(x) := G,\ I_n(x) := \{y = (y_i, i \in \mathbf{N}) \in G : y_i = x_i \text{ for } i < n\}$$

for $x \in G$, $n \in \mathbf{P}$. Denote by $0 = (0, i \in \mathbf{N}) \in G$ the zero element of G and $I_n := I_n(0)$, $n \in \mathbf{N}$.

It is clear that each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i 2^i \qquad n_i = 0 \text{ or } n_i = 1,\ i \in \mathbf{N},$$

where only a finite number of n_i's differ from zero. This allows us to say that the sequence $(n_0, n_1, \dots)$ is the dyadic expansion of n. We often use the notation $n^* := (n_0, n_1, \dots) \in G$.

Set

$$r_n(x) := (-1)^{x_n} \qquad (x \in G,\ n \in \mathbf{N})$$

and

$$\psi_n := \prod_{k=1}^{\infty} r_k^{n_k} \qquad (n \in \mathbf{N}).$$

Each ψ_n is a character of G and all the characters of G are of this form. Then the system ψ is a orthonormal and complete system in $L^2(G)$.

The dyadic group can be identified with the interval $[0, 1[$ in such way that the system ψ corresponds to the Walsh-Paley system and the Haar measure μ corresponds to the Lebesgue measure.

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The Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} \psi_k(x) \qquad (x \in G).$$

Recall that [10]

$$D_{2^n}(x) = \begin{cases} 2^n & \text{for } x \in I_n(0), \\ 0 & \text{for } x \notin I_n(0) \end{cases} \tag{1}$$

$$|D_n(x)| < \frac{2}{|x|}. \tag{2}$$

We define the Fourier coefficients $\widehat{f}_k$ of a function $f \in L^1(G)$ by

$$\widehat{f}_k := \int_G f \overline{\psi}_k d\mu \qquad (k \in \mathbf{N}),$$

and the n-th partial sums of Fourier series by

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \qquad (n \in \mathbf{P}).$$

The n-th modulus of continuity of a continuous function f on G is defined by

$$\omega_n(f) = \sup_{h \in I_n} \sup_x |f(x+h) - f(x)|.$$

The notion of function of the bounded variation was introduced by Jordan [6]. Generalizing this notion Wiener [12] has considered the V_p class of functions. Young [15] introduced the notion of functions of Φ-variation. Waterman [11] has studied the class of functions of the bounded Λ-variation, and Chanturia [3] has defined the notion of the modulus of variation of a function. In 1990 Kita and Yoneda [7] introduced the notion of the generalized Wiener's class $BV(p(n) \uparrow p)$. Generalizing the class $BV(p(n) \uparrow p)$ Akhobadze [1, 2] has considered the $BV(p(n) \uparrow p, \varphi)$ and $B\Lambda(p(n) \uparrow p, \varphi)$ classes of functions.

Definition 1. *(see [5]) Let $1 \leq p(n) \uparrow p$ as $n \to \infty$ where $1 \leq p \leq \infty$. We say that a function f belongs to the $BO(p(n) \uparrow p)$ class if*

$$O(f; p(n) \uparrow p) = \sup_n \left\{ \sum_{k=0}^{2^n-1} \sup_{u,t \in I_n(k^*)} |f(u) - f(t)|^{p(n)} \right\}^{1/p(n)} < \infty.$$

When $p(n) = p$ for all n, $BO(p(n) \uparrow p)$ coincides with the class of p-bounded fluctuation ([10] p. 15). Function of bounded fluctuation were introduced by Onneweer and Waterman [9] to investigate uniform convergence of Fourier series.

Fine [4] has obtained sufficient conditions for the uniform convergence which are in a complete analogy with the Dini-Lipschitz conditions. He has shown that if f is continuous on G and its continuity modulus

$$\omega_n(f) = o\left(\frac{1}{n}\right) \tag{3}$$

then its Walsh-Fourier series converges uniformly on G.

Several times we use the following bit-reversal notation ([10] p. 145). If k is an integer with dyadic expansion $(k_0, k_1, \ldots, k_{n-1}, 0, 0, \ldots)$, then elements $x_k^{(n)}$ are defined by

$$x_k^{(n)} := (k_{n-1}, k_{n-2}, \ldots, k_0, 0, 0, \ldots).$$

Furthermore, denote e_n by the element $e_n = (x_0, x_1, \ldots) \in G$ which satisfies $x_n = 1$ and $x_j = 0$ for all $j \neq n$.

Onnewer [8] has proved that if f is continuous and

$$\sum_{k=1}^{2^n-1} \frac{\left| f\left(x + x_k^{(n)}\right) - f\left(x + x_k^{(n)} + e_n\right) \right|}{k} \to 0 \text{ as } n \to \infty$$

uniformly with respect to x, then partial sums of the Walsh-Fourier series of the function f converge uniformly on G.

From the statement one can derive a number of corollaries such, for example, as the sufficiency of the Dini-Lipschitz condition for the uniform convergence of the Walsh-Fourier series. Then one can prove that it is sufficient for the continuous function to have p-bounded fluctuation in order for its Fourier series to converge to the function uniformly.

It is well-known that, in general, the condition

$$\sup_{h \in I_n} \left| f(x_0 + h) - f(x_0) \right| = o\left(\frac{1}{n}\right) \tag{4}$$

does not guarantee the convergence of $S_m f(x_0)$.

In 1954, Yano [13] proved that if $\widehat{f}(n) = O\left(n^{-\delta}\right)$ for some $\delta > 0$ and the condition (4) holds ,then

$$\lim_{n \to \infty} S_n f(x_0) = f(x_0),$$

Yoneda [14] has shown that if f is a function of V_p, $1 \leq p < \infty$, then

$$\sum_{k=0}^{\infty} \widehat{f}(k) \psi_k(x_0) = \begin{cases} f(x_0), \text{ if } f \text{ is continuous at } x_0, \\ \text{diverge otherwise.} \end{cases}$$

In [5], first author proved that the following are true

Theorem A. Let $p(n) \log p(n) = o(n)$ and $p(2n) \leq cp(n)$ for all $n \geq 1$, where $c > 0$ is a constant

1) If f be a continuous function in the class $BO(p(n) \uparrow \infty)$ and

$$\omega_n(f) = o\left(\frac{1}{p(n) \log p(n)}\right),$$

then the Walsh-Fourier series of function f converges uniformly in G;

2) There exists a function in the class $BO(p(n) \uparrow \infty)$ for which

$$\omega_n(f) = O\left(\frac{1}{p(n) \log p(n)}\right)$$

and the Fourier series diverges at some point.

In this paper we prove that the following is true

Theorem 1. *Let $p(n)\log p(n)=o(n)$ and $p(2n)\leq cp(n)$ for all $n\in\mathbf{P}$, where $c>0$ is a constant. If $f\in BO(p(n)\uparrow\infty)$ and*

$$\sup_{h\in I_n}|f(x_0+h)-f(x_0)|=o\left(\frac{1}{p(n)\log p(n)}\right),$$

then

$$\lim_{m\to\infty}S_mf(x_0)=f(x_0).$$

In order to prove Theorem 1 we need the following lemma.

Lemma 1. *Let $p(n)\log p(n)=o(n)$. If $f\in BO(p(n)\uparrow\infty)$ and*

$$\sup_{h\in I_n}|f(x_0+h)-f(x_0)|=o\left(\frac{1}{n}\right),$$

then

$$\lim_{m\to\infty}S_mf(x_0)=f(x_0).$$

Proof of Lemma 1. Let $a\in(0,1)$ be a fix number. Set $m=2^k+n$ where $0\leq n<2^k$. Thus

$$S_mf=S_{2^k}f+f*(r_kD_n). \tag{5}$$

The condition of the lemma guarantees the continuity of f at x_0, thus $S_{2^k}f(x_0)\to f(x_0)$. Then, it suffices to prove $(f*(r_kD_n))(x_0)=o(1)$.
Write

$$\begin{aligned}(f*(r_kD_n))(x_0)&=\int_G f(x_0+t)r_k(t)D_n(t)\,d\mu(t)\\&=\sum_{j=0}^{2^k-1}\sum_{i=0}^{1}\int_{x_j^{(k)}+ie_k+I_{k+1}}f(x_0+t)r_k(t)D_n(t)\,d\mu(t)\\&=D_n(0)\int_{I_{k+1}}(f(x_0+t)-f(x_0+e_k+t))\,d\mu(t)\\&\quad+\sum_{l=0}^{k-[ka]-1}\sum_{j=2^l}^{2^{l+1}-1}D_n(x_j^{(k)})\int_{I_{k+1}}(f(x_0+x_j^{(k)}+t)-f(x_0+x_j^{(k)}+e_k+t))\,d\mu(t)\\&\quad+\sum_{l=k-[ka]}^{k-1}\sum_{j=2^l}^{2^{l+1}-1}D_n(x_j^{(k)})\int_{I_{k+1}}(f(x_0+x_j^{(k)}+t)-f(x_0+x_j^{(k)}+e_k+t))\,d\mu(t)\\&=J_1+J_2+J_3.\end{aligned}$$

Since $|D_n(0)|=n<2^k$ and $\mu(I_{k+1})=1/2^{k+1}$, we obtain for J_1

$$\begin{aligned}|J_1|&\leq|D_n(0)|\int_{I_{k+1}}(|f(x_0+t)-f(x_0)|+|f(x_0+e_k+t)-f(x_0)|)\,d\mu(t)\\&\leq 2\sup_{h\in I_k}|f(x_0+h)-f(x_0)|=o(1).\end{aligned}$$

To estimate J_2 and J_3, we use the following estimation (see [10], p. 146)

$$|D_n(x_j^{(k)})|<2^{l_0+1}\qquad(x_j^{(k)}\in I_{l_0}\setminus I_{l_0+1}).$$

Consequently,

$$\begin{aligned}
|J_2| \le & \sum_{l=0}^{k-[ka]-1} \sum_{j=2^l}^{2^{l+1}-1} |D_n(x_j^{(k)})| \\
& \times \int_{I_{k+1}} \left(|f(x_0 + x_j^{(k)} + t) - f(x_0)| + |f(x_0 + x_j^{(k)} + e_k + t) - f(x_0)| \right) d\mu(t) \\
\le & \sum_{l=0}^{k-[ka]-1} \sum_{j=2^l}^{2^{l+1}-1} 2^{k-l} \cdot 2 \sup_{h \in I_{k-l-1}} |f(x_0+h) - f(x_0)| \frac{1}{2^{k+1}} \\
\le & \sup_{h \in I_{[ka]}} |f(x_0+h) - f(x_0)| (k - [ka] - 1) = o(1).
\end{aligned}$$

Finally, by Hölder's inequality we obtain

$$\begin{aligned}
|J_3| \le & \sum_{l=k-[ka]}^{k-1} \sum_{j=2^l}^{2^{l+1}-1} |D_n(x_j^{(k)})| \int_{I_{k+1}} |f(x_0 + x_j^{(k)} + t) - f(x_0 + x_j^{(k)} + e_k + t)| \, d\mu(t) \\
\le & \sum_{l=k-[ka]}^{k-1} 2^{k-l} \int_{I_{k+1}} \sum_{j=2^l}^{2^{l+1}-1} |f(x_0 + x_j^{(k)} + t) - f(x_0 + x_j^{(k)} + e_k + t)| \, d\mu(t) \\
\le & \frac{1}{2} \sum_{l=k-[ka]}^{k-1} \frac{1}{2^l} \sup_{t \in I_{k+1}} \sum_{j=2^l}^{2^{l+1}-1} |f(x_0 + x_j^{(k)} + t) - f(x_0 + x_j^{(k)} + e_k + t)| \\
\le & \frac{1}{2} \sum_{l=k-[ka]}^{k-1} \frac{1}{2^l} \sup_{t \in I_{k+1}} \left\{ \sum_{j=2^l}^{2^{l+1}-1} |f(x_0 + x_j^{(k)} + t) - f(x_0 + x_j^{(k)} + e_k + t)|^{p(k)} \right\}^{1/p(k)} \\
& \times 2^{l(1-1/p(k))} \\
\le & c \sum_{l=k-[ka]}^{k-1} \frac{1}{2^{l/p(k)}} O(f; p(n) \uparrow \infty) \\
\le & c \frac{p(k)}{2^{(k-[ka])/p(k)}} \le c \frac{p(k)}{2^{(1-a)\frac{k}{p(k)}}} = c \left(\frac{2^{\frac{p(k)\log p(k)}{k+1}}}{2^{1-a}} \right)^{\frac{k}{p(k)}} \le c \left(\frac{1}{2^{1-a}} \right)^{\frac{k}{p(k)}} = o(1)
\end{aligned}$$

as k tends to infinity. This completes the proof of Lemma 1.

Proof of Theorem 1. Let

$$\sup_{h \in I_{[n/2]}} |f(x_0+h) - f(x_0)| \frac{n^2}{p(n) \log p(n)} \le 4.$$

Since

$$\sup_{h \in I_{[n/2]}} |f(x_0+h) - f(x_0)| \, n \le \frac{4p(n)\log p(n)}{n} = o(1) \quad as \quad n \to \infty$$

the validity of Theorem 1 follows immediately from Lemma 1. Hence we can suppose that

$$\sup_{h \in I_{[n/2]}} |f(x_0+h) - f(x_0)| \frac{n^2}{p(n) \log p(n)} > 4,$$

then it is clear that

$$n-\frac{\sqrt{p(n)\log p(n)}}{\sqrt{\sup\limits_{h\in I_{[n/2]}}\left|f\left(x_0+h\right)-f\left(x_0\right)\right|}}>\frac{n}{2}\ .$$

Define

$$m_0\left(n\right)=\left[\sqrt{\frac{p\left(n\right)\log p\left(n\right)}{\sup\limits_{h\in I_{[n/2]}}\left|f\left(x_0+h\right)-f\left(x_0\right)\right|}}\right].$$

We use the same decomposition applied in the proof of Lemma 1. Set $m=2^k+n$ where $0\le n<2^k$. By (5) it suffices to prove $(f*(r_kD_n))(x_0)=o(1)$.
Write

$$\begin{aligned}
(f*(r_kD_n))(x_0) &= \int_G f(x_0+t)r_k(t)D_n(t)\,d\mu(t)\\
&= \sum_{j=0}^{2^k-1}\sum_{i=0}^{1}\int_{x_j^{(k)}+ie_k+I_{k+1}} f(x_0+t)r_k(t)D_n(t)\,d\mu(t)\\
&= D_n(0)\int_{I_{k+1}}(f(x_0+t)-f(x_0+e_k+t))\,d\mu(t)\\
&\quad+\sum_{l=0}^{m_0(k)-1}\sum_{j=2^l}^{2^{l+1}-1}D_n(x_j^{(k)})\int_{I_{k+1}}(f(x_0+x_j^{(k)}+t)-f(x_0+x_j^{(k)}+e_k+t))\,d\mu(t)\\
&\quad+\sum_{l=m_0(k)}^{k-1}\sum_{j=2^l}^{2^{l+1}-1}D_n(x_j^{(k)})\int_{I_{k+1}}(f(x_0+x_j^{(k)}+t)-f(x_0+x_j^{(k)}+e_k+t))\,d\mu(t)\\
&= J_4+J_5+J_6.
\end{aligned}$$

Analogously as above we have $J_4=o(1)$ as $m\to\infty$,

$$\begin{aligned}
|J_5| &\le \sup_{h\in I_{k-m_0(k)}}\left|f\left(x_0+h\right)-f\left(x_0\right)\right|m_0(k)\\
&\le c\sqrt{\sup_{h\in I_{[k/2]}}\left|f\left(x_0+h\right)-f\left(x_0\right)\right|p(k)\log p(k)}=o(1).
\end{aligned}$$

and

$$\begin{aligned}
|J_5| &\le \sum_{j=m_0(k)}^{n-1}\frac{1}{2^{j/p(k)}}O\left(f;p(n)\uparrow\infty\right)\le c\,\frac{p(k)}{2^{\frac{m_0(k)}{p(k)}}}\\
&= c\left(\frac{2}{\exp_2\left\{c\sqrt{\frac{1}{\sup\limits_{h\in I_{[k/2]}}|f(x_0+h)-f(x_0)|p(k)\log p(k)}}\right\}}\right)^{\log p(k)}=o(1)\quad\text{as}\quad k\to\infty.
\end{aligned}$$

This completes the proof of Theorem 1.

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