Lie Properties of the Group Algebra and the Structure of its Group of Units

DOKTORI (PHD) ÉRTEKEZÉS TÉZISEI

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1. Introduction.

Group rings are rather attractive algebraic structures, used not only in representation theory, but – besides, of course, group theory and ring theory themselves – for instance, also in homology, cohomology, coding theory and algebraic topology. Moreover, they are attractive enough to study them on their own right. Research on these objects is a meeting point of concepts, techniques and problems of group theory and ring theory, since one can examine ring theoretical properties of the group ring and group theoretical properties of its group of units, and in both cases one encounters the need for tools from both theories.

Study of the group of units started in the early 40s with the papers of Higman [14] and Jennings [17]. The most fundamental characterization theorems were proved and the most fundamental books were published during the 70s. Since then research intensified to a great extent and a diverse literature arose.

The scope of this dissertation is the study of certain group theoretical properties – namely, the derived length, the Engel length and the nilpotency class, which features can be viewed as measures of complexity – of the group of units $U(\mathbb{F}G)$ of the modular group algebra $\mathbb{F}G$ of the nonabelian group G over the field $\mathbb{F}$ of prime characteristic p.

As Lie commutators, which measure noncommutativity in the algebra, are considerably easier to calculate than group commutators, which measure noncommutativity in the group of units, one may think of obtaining information on the group of units by looking at the associated Lie algebra. It has turned out that, in most cases, the Lie structure reflects well the characteristics of the group of units and there is a close relationship between properties of these two.

This dissertation consists of five chapters. The first is an introductory one, new results can be found in the next four.

2. Main results.

Group algebras of finite groups with solvable group of units were determined by Bovdi and Khripta et al. (see [40]). We know only few facts concerning the derived length of the group of units. The first result in this direction has been obtained by Shalev [45], who showed that $U(\mathbb{F}G)$ is metabelian if and only if G is abelian provided $p > 3$; moreover, if and only if G is abelian or nilpotent with a commutator subgroup of order 3 provided $p = 3$. In Chapter Two, which covers [22], we solve this problem for $p = 2$:

Theorem 2.1. *Let G be a finite group and $\mathbb{F}$ a field of characteristic 2. The group of units $U(\mathbb{F}G)$ is metabelian if and only if one of the following conditions holds:*

(i) *G is abelian;*

(ii) *G is nilpotent of class 2 and has an elementary abelian commutator subgroup of order 2 or 4;*

(iii) *$\mathbb{F} = \mathrm{GF}(2)$, the field of two elements, and G is an extension of an elementary abelian 3-group H by the group $\langle b \rangle$ of order 2 with $b^{-1}ab = a^{-1}$ for every $a \in H$.*

Coleman and Sandling also obtained the same result independently in [10]. The proof is divided into two steps, the first verifies the theorem for finite 2-groups G, the second, using the previous case and the characterization of group algebras with solvable group of units, for finite groups in general. Most recent related results concerning the Lie derived length of group algebras are the following: in [28] Levin and Rosenberger determined Lie metabelian group rings; in [48] Sharma and Srivastava, and in [26] Külshammer and Sharma described Lie centrally metabelian group rings of characteristic $p > 3$ and $p = 3$, respectively; in [30] Meena Sahai determined group algebras of derived length 3 for $p > 2$ (as to our best knowledge, the previous two problems for $p = 2$ are still open).

The Engel property is of outstanding importance in group theory and also in the theory of Lie algebras. Bounded Lie Engel group algebras were described by Sehgal [40]: the group algebra $\mathbb{F}G$ is bounded Engel if and only if $\mathbb{F}G$ is commutative provided $\mathbb{F}$ is of characteristic 0; if and only if G is nilpotent containing a normal subgroup N such that the commutator subgroup N' and the factorgroup G/N are of p-power orders provided $\mathbb{F}$ is of prime characteristic p. The first result, which gives some information on the Engel length of the group algebra is due to Rips and Shalev [37]: with $\mathbb{F}G$ n-Engel of prime characteristic p if $n < p$ then $\mathbb{F}G$ is commutative, and if $n = p$ then $|G'| \leq p$. In Chapter Three, which covers [23], extending this result we prove

Theorem 5.1. *Let $\mathbb{F}$ be a field of prime characteristic p, G an arbitrary group. Then the group algebra $\mathbb{F}G$ is 3-Engel if and only if one of the following conditions holds:*

(i) *G is abelian;*

(ii) *$p = 2$ and G is nilpotent of class 2 with an elementary abelian commutator subgroup of order 2 or 4;*

(iii) *$p = 2$ and G is nilpotent of class 2 such that its commutator subgroup is an elementary abelian 2-group of either finite order greater than 4 or of infinite order, and there exists an abelian subgroup of index 2 in G;*

(iv) *$p = 3$ and G is nilpotent with a commutator subgroup of order 3.*

Theorem 5.5. *Let $\mathbb{F}$ be a field of prime characteristic $p > 2$, G a nonabelian group. Then the group algebra $\mathbb{F}G$ is $(p+1)$-Engel if and only if G is nilpotent with a commutator subgroup of order p.*

It is well-known that 3-Engel Lie algebras (even the not finitely generated ones) are nilpotent except the characteristic 2 and 5 cases. 3-Engel Lie algebras were studied by Traustason in [49], where an example of a 3-Engel non-nilpotent Lie algebra of characteristic 2 was given. We provide another example. Let $G = A \rtimes \langle b \rangle$, a semidirect product, where A is an infinite direct product of cyclic groups of order 4, and b is of order 2 acting by inversion on A. Then the group algebra of G over the prime field GF(2) is 3-Engel by Theorem 5.1, and not Lie nilpotent by Passi, Passman, Sehgal's characterization theorem [40].

In [42] Shalev showed that the group of units of an associative n-Engel algebra of prime characteristic is m-Engel for some m, depending on n. Let $f(n)$ be the smallest possible such m. For group algebras it is easy to see $f(2) = 2$, and by means of Theorem 5.1 we prove $f(3) = 3$:

Theorem 6.1. *If $\mathbb{F}G$ is a 3-Engel group algebra then the group of units $U(\mathbb{F}G)$ is 3-Engel.*

This strongly suggests that we can tackle the problem of characterizing group algebras with 3-Engel groups of units, which is done in the next chapter.

It is one of Sehgal's problems [40] to find group algebras with Engel or bounded Engel groups of units, which was solved by Bovdi and Khripta [9]. In Chapter Four, which covers [24], first we give a lower bound on the Engel length:

Theorem 8.1. *Let $\mathbb{F}$ be a field of prime characteristic $p > 2$ and G a locally nilpotent nonabelian group with a nontrivial p-Sylow subgroup. Then the group of units $U(\mathbb{F}G)$ is p-Engel if and only if G is nilpotent with a commutator subgroup of order p.*

Theorem 8.2. *Let $\mathbb{F}$ be a field of prime characteristic $p > 2$ and G a locally nilpotent nonabelian group with a nontrivial p-Sylow subgroup. Then $U(\mathbb{F}G)$ is not $(p-2)$-Engel, and $U(\mathbb{F}G)$ is $(p-1)$-Engel if and only if G is nilpotent with $G' = T_p(G)$ of order p.*

We can determine modular group algebras with 2-Engel groups of units easily:

Theorem 9.1. *Let $\mathbb{F}$ be a field of prime characteristic p and G a group with a nontrivial p-Sylow subgroup. Then the group of units $U(\mathbb{F}G)$ is 2-Engel if and only if one of the following conditions holds:*

(i) *G is abelian;*

(ii) *$p = 2$ and G is nilpotent with a commutator subgroup of order 2;*

(iii) *$p = 2$ and G is nilpotent of class 2 with an elementary abelian 2-Sylow subgroup $T_2(G) = G'$ of order 4;*

(iv) *$p = 3$ and G is nilpotent with $G' = T_3(G)$ of order 3.*

However, the proof of our next result requires a sequence of lemmas with – at certain points lengthy – calculations of commutators of both types for the usually most complicated case $p = 2$.

Theorem 9.2. *Let $\mathbb{F}$ be a field of prime characteristic p and G a group with a nontrivial p-Sylow subgroup. Then the group of units $U(\mathbb{F}G)$ is 3-Engel if and only if one of the following conditions holds:*

(i) *$\mathbb{F}G$ is Lie 3-Engel;*

(ii) *$p = 2$ and G is nilpotent of class 2 such that G' is elementary abelian of order 8, and $T_2(G)$ is of order 8 or 16 and central in G;*

(iii) *$p = 2$ and G is nilpotent of class 2 with G' elementary abelian of order 8, and $T_2(G) = \langle G', a\rangle$ is of order 16 such that $|G : C_G(a)| = 2$ and $C'_G(a) = (a, G)$;*

(iv) *$p = 2$ and G is nilpotent of class 2 with $G' = T_2(G)$ elementary abelian of order 16;*

(v) *$p = 2$ and G is nilpotent with $G' = T_2(G)$ cyclic of order 4;*

(vi) *$p = 2$ and G is nilpotent of class 3 with $G' = T_2(G)$ elementary abelian of order 4.*

In Chapter Five, which covers [25], the nilpotency class of the group of units of a group algebra of prime characteristic p is studied. Modular group algebras with nilpotent groups of units were described by Khripta [19]. Despite the intensive research – just to mention those papers by Baginski [1], Mann-Shalev [29], Shalev [43,44,46], Rao and Sandling [36], certain results of which will be extended or relied upon heavily in this chapter – on the nilpotency class of the group of units, there is no complete description known yet, the best results are for group algebras of finite p-groups of characteristic p with $p > 3$. This research is paralell with studies – just to mention those papers by Bhandari and Passi [2], Du [11], Gupta and Levin [13], Levin and Sehgal [27], Passi and Sehgal [32], Sandling [39], Sharma and Srivastava [47], certain results of which will be extended or relied upon heavily in this chapter – on the Lie nilpotency class or, more precisely, on the lower and upper Lie nilpotency indices of Lie nilpotent group algebras. In the second section of Chapter Five we prove two crucial lemmas, one of which deals with the Lie structure of associative rings in general, improving results of Levin and Sehgal [27], and the other is of purely group-theoretical nature and deals with finite abelian p-groups, using a result of Birkhoff [3]. In the third section in this chapter in certain cases we determine the lower and upper Lie nilpotency indices $t_L(G)$ and $t^L(G)$ of the group algebra $\mathbb{F}G$ in terms of the nilpotency index of the associative ideal $\mathcal{I}(G')$ generated by all elements of form $a-1$ with $a \in G'$. The next theorem was proved by Shalev in [46] for $p \geq 5$:

Theorem 12.1. *Let $\mathbb{F}$ a field of prime characteristic p, G a nilpotent group such that its commutator subgroup G' is of p-power order and $\gamma_3(G) \subseteq G'^p$. Put $I = \mathcal{I}(G')$. Then*

(i) *$\mathbb{F}G^{(k)} = I^{k-1}$ $(k \geq 2)$ and $t^L(G) = t(G') + 1$;*

(ii) *if G' is cyclic of order p^n then $t_L(G) = t^L(G) = p^n + 1$.*

By means of the breakthrough by Du [11], establishing the conncetion betweeen the Lie upper central series of a Jacobson radical ring and the upper central series of its adjoint group, the lower Lie nilpotency index gives immediately the nilpotency class of the group of units provided G is a p-group. We succeeded to do this in two cases: when G' is abelian and the third term of the lower central series of G is contained in G'^p, and when G is of class 3 and $p \neq 3$. Certain parts of the next theorem were proved by Shalev for $p \geq 5$ in [46].

Theorem 12.2. *Let $\mathbb{F}$ be a field of prime characteristic p and G a nilpotent group such that the commutator subgroup G' is a finite abelian p-group with invariants $(p^{m_1}, p^{m_2}, \dots, p^{m_s})$. Then the following statements hold:*

(i) *$t_L(G) \geq t(G') + 1 = 2 + \sum_{i=1}^{s}(p^{m_i} - 1)$;*

(ii) *$t_L(G) = t^L(G) = t(G') + 1$ if $\gamma_3(G) \subseteq G'^p$;*

(iii) *$\mathrm{cl}(U(\mathbb{F}G)) = t(G')$ if G is a p-group and $\gamma_3(G) \subseteq G'^p$.*

Theorem 12.3. *Let $\mathbb{F}$ be a field of prime characteristic $p \neq 3$ and G a group nilpotent of class greater than 2 such that the commutator subgroup G' is a finite abelian p-group*

$$G' = \langle a_1 \rangle \times \langle a_2 \rangle \times \dots \times \langle a_s \rangle, \quad |a_i| = p^{m_i}, \quad m_1 \geq m_2 \geq \dots \geq m_s,$$

and $|\gamma_3(G)G'^p/G'^p| = p^r$, $0 < r < s$. Furthermore, let $A = \langle a_{u_1} \rangle \times \langle a_{u_2} \rangle \times \dots \times \langle a_{u_{s-r}} \rangle$, $u_1 < u_2 < \dots < u_{s-r}$ be the weak complement of $\gamma_3(G)$ in G' relative to the basis $\{a_i\}$, and let $\{1, 2, \dots, s\} = \{u_1, u_2, \dots, u_{s-r}, v_1, v_2, \dots, v_r\}$, $v_1 < v_2 < \dots < v_r$. Then

(i) *$t_L(G) \geq t(G') + t(G'/A) = 2 + \sum_{i=1}^{s}(p^{m_i} - 1) + \sum_{j=1}^{r}(p^{m_{v_j}} - 1)$;*

(ii) *$t_L(G) = t^L(G) = t(G') + t(G'/A)$ if G is of class 3;*

(iii) *$\mathrm{cl}(U(\mathbb{F}G)) = t(G') + t(G'/A) - 1$ if G is a p-group of class 3.*

In the fourth section in this chapter the nilpotency class is given in case the commutator subgroup of G is either cyclic or of order p^2.

Theorem 13.3. *Let G be a nilpotent group with a cyclic commutator subgroup G' of order $p^n > 2$ and $\mathbb{F}$ a field of prime characteristic p. Then $U(\mathbb{F}G)$ is nilpotent of class $p^n - 1$ if $\mathrm{Syl}_p(G) = G'$, and of class p^n if $\mathrm{Syl}_p(G) \neq G'$.*

Theorem 13.4. *Let G be a nilpotent group with an elementary abelian commutator subgroup G' of order p^2, and $\mathbb{F}$ a field of prime characteristic p. Then the following statements hold:*

(i) *if G is of class 2 then $\mathrm{cl}(U(\mathbb{F}G)) = 2p - 1$ provided $\mathrm{Syl}_p(G) \neq G'$, and $\mathrm{cl}(U(\mathbb{F}G)) = 2p - 2$ provided $\mathrm{Syl}_p(G) = G'$;*

(ii) *if G is of class 3 then $\mathrm{cl}(U(\mathbb{F}G)) = 3p - 2$ provided $\mathrm{Syl}_p(G) \neq G'$, and $\mathrm{cl}(U(\mathbb{F}G)) = 3p - 3$ provided $\mathrm{Syl}_p(G) = G'$.*

These two results for finite p-groups G with $p > 2$ were obtained also by Mann and Shalev in [29]. A lower bound is established on the difference between the nilpotency classes of G and the group of units in the next

Theorem 13.6. *If $\mathbb{F}$ is a field of prime characteristic $p > 2$ and G is a nilpotent group of class greater than 2 with G' of p-power order then $\mathrm{cl}(U(\mathbb{F}G)) - \mathrm{cl}(G) \geq p$.*

In the last section we determine modular group algebras with groups of units nilpotent of class 3, extending the result of Rao and Sandling [36].

Theorem 14.2. *Let $\mathbb{F}$ be a field of prime characteristic p, G a nilpotent group with G' of p-power order. Then $U(\mathbb{F}G)$ is nilpotent of class 3 if and only if one of the following conditions holds:*

(i) $p = 2$, $\mathrm{cl}(G) = 2$, G' *is elementary abelian of order 4,* $G' \neq \mathrm{Syl}_2(G)$;

(ii) $p = 2$, $\mathrm{cl}(G) = 2$, $G' = \mathrm{Syl}_2(G)$ *is elementary abelian of order 8;*

(iii) $|G'| = 8$, $\mathrm{Syl}_2(G)$ *is of order 16 and central in* G, *and the orders of the conjugacy classes in* G *do not exceed 4;*

(iv) $p = 2$, $G' = \mathrm{Syl}_2(G)$ *is cyclic of order 4;*

(v) $p = 2$, $\mathrm{cl}(G) = 3$, $G' = \mathrm{Syl}_2(G)$ *is elementary abelian of order 4;*

(vi) $p = 3$, G' *is of order 3,* $G' \neq \mathrm{Syl}_3(G)$.

In the last two sections we do not assume G to be a p-group, which makes things more complicated, as in this generality we do not have a satisfactory description of the units themselves.

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