

Recent developments on approximation theory with respect to representative product systems



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Introduction

SEVERAL results in Fourier analysis with respect to Walsh functions are obtained viewing them as the characters of the dyadic group, i.e., the complete product of the discrete cyclic group of order 2 with the product of topologies and measures. Then we often enumerate the Walsh functions in the Paley's sense writing them as the finite product of the Rademacher functions. It is named the Walsh-Paley system. The above structure was generalized by Vilenkin [5] in 1947 studying the complete product of arbitrary cyclic groups. The construction of the Vilenkin system is similar, taking the finite product of the characters of the cyclic groups as it Paley did. In Vilenkin groups the order of the cyclic groups appeared in the product can be unbounded. The methods applied in the study of these cases differ significantly from the bounded cases and in many instances we obtain different results for the same question.

A natural generalization of the Vilenkin group is the complete product of arbitrary groups, non necessarily commutative groups, denoted by G . In this case we use representation theory in order to obtain orthonormal systems. These systems are named representative product systems and Section 1 deals with the structure of them. Representative product systems can be represented on the interval $[0, 1]$, where this systems are also orthonormal under the Lebesgue measure (see [3]).

SOME examples have been showed in Section 2, where we give two simple for non commutative cases. Naturally, the represented systems φ are defined only on the finite groups and we have to take the finite product of them to obtain the system ψ . In the both non commutative cases we suppose the complete product of the same finite group. Notice that representative product systems are not uniformly bounded for non commutative cases. This fact encumbers the study of these systems. Moreover, the values of the system φ depend on the chosen basis if it is not formed only by characters.

DIRICHLET kernels and Lebesgue constants play an important role in the study of Fourier series. Hence we deal with the properties of them in Section 3 and Section 4, illustrating these properties with the two non commutative examples of Section 2. Finally, we compare these illustrations with the Lebesgue constants of the Walsh-Paley system.

1. Representative product systems

LET $m := (m_k, k \in \mathbb{N})$ be a sequence of positive integers such that $m_k \geq 2$ and G_k be a finite group with order m_k , ($k \in \mathbb{N}$). Suppose that each group has discrete topology and normalized Haar measure. Let G be the compact group formed by the complete direct product of G_k with the product of the topologies, operations and measures. Thus each $x \in G$ consist of sequences $x := (x_0, x_1, \dots)$, where $x_k \in G_k$, ($k \in \mathbb{N}$). We call this sequence the expansion of x . The compact totally disconnected group G is called a **bounded group** if the sequence m is bounded.

If $M_0 := 1$ and $M_{k+1} := m_k M_k$, $k \in \mathbb{N}$, then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \leq n_k < m_k$, $n_k \in \mathbb{N}$. This allows us to say that the sequence (n_0, n_1, \dots) is the expansion of n with respect to m . We often use the following notation: $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$.

NOTE by Σ_k the dual object of the finite group G_k ($k \in \mathbb{N}$). Thus each $\sigma \in \Sigma_k$ is a set of continuous irreducible unitary representations of G_k which are equivalent to some fixed representation $U^{(\sigma)}$. Let d_σ be the dimension of its representation space and let $\{\zeta_1, \zeta_2, \dots, \zeta_{d_\sigma}\}$ be a fixed but arbitrary orthonormal basis in the representation space. The functions

$$u_{i,j}^{(\sigma)}(x) := \langle U^{(\sigma)} \zeta_i, \zeta_j \rangle \quad (i, j \in \{1, \dots, d_\sigma\}, x \in G_k)$$

are called the coordinate functions for $U^{(\sigma)}$ and the basis $\{\zeta_1, \zeta_2, \dots, \zeta_{d_\sigma}\}$. In this manner for each $\sigma \in \Sigma_k$ we obtain d_σ^2 number of coordinate functions, in total m_k number of functions for the whole dual object of G_k . The L^2 -norm of these functions is $1/\sqrt{d_\sigma}$. Let $\{\varphi_k^s : 0 \leq s < m_k\}$ be a system of all **normalized coordinate functions** of the group G_k . We do not decide now the enumeration of the system φ , only suppose that φ_k^0 is always the character 1. Thus for every $0 \leq s < m_k$ there exists a $\sigma \in \Sigma_k$, $i, j \in \{1, \dots, d_\sigma\}$ such that

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \quad (x \in G_k).$$

ET ψ be the product system of φ_k^s , namely

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \quad (x \in G),$$

where n is of the form $n = \sum_{k=0}^{\infty} n_k M_k$ and $x = (x_0, x_1, \dots)$. Thus we say that ψ is the **representative product system** of φ . The Weyl-Peter's theorem secures that the system ψ is orthonormal and complete on $L^2(G)$. The functions ψ_n ($n \in \mathbb{N}$) are not uniformly bounded if G contains infinitely many non commutative groups in the product, so define

$$\Psi_k = \prod_{i=0}^{k-1} \max_{s < m_i} \|\varphi_i^s\|_1 \|\varphi_i^s\|_\infty \quad (k \in \mathbb{N}).$$

It seems that the boundedness of the sequence Ψ plays an important role in the convergence in L^p -norm of Fourier series.

2. Examples

REPRESENTATIVE product systems are the generalization of the well known **Walsh-Paley system**. Indeed, we obtain this system if $m_k = 2$ and $G_k := \mathbb{Z}_2$, the cyclic group of order 2 for all $k \in \mathbb{N}$. The characters of \mathbb{Z}_2 are the Rademacher functions:

$$\varphi^s(x) = (-1)^{sx} \quad (s \in \{0, 1\}, x \in \mathbb{Z}_2).$$

Moreover, we obtain the **Vilenkin systems** if the sequence m is an arbitrary sequence of integers greater than 1 and $G_k := \mathbb{Z}_{m_k}$, the cyclic group of order m_k for all $k \in \mathbb{N}$. The characters of \mathbb{Z}_{m_k} are the generalized Rademacher functions:

$$\varphi_k^s(x) = \exp(2\pi i s x / m_k) \quad (s \in \{0, \dots, m_k - 1\}, x \in \mathbb{Z}_{m_k}).$$

CONSIDER now the complete product of S_3 where S_3 is the **symmetric group on 3 elements**. Thus $m_k = 6$ for all $k \in \mathbb{N}$. S_3 has two characters and a 2-dimensional representation.

	e	(12)	(13)	(23)	(123)	(132)	$\ \varphi^s\ _1$	$\ \varphi^s\ _\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

Notice that $\max_{0 \leq s < 6} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3} > 1$ implies

$$\Psi_k = \left(\frac{4}{3}\right)^k \rightarrow \infty.$$

LET $m_k = 8$ for all $k \in \mathbb{N}$ and Q_2 be the **quaternion group of order 8**, i.e. $Q_2 := \{[a, b] : a^4 = e, b^2 = a^2, bab^{-1} = a^3\}$. Let $G_k = Q_2$ for all $k \in \mathbb{N}$. Q_2 has four characters and a 2-dimensional representation.

	e	a	a^2	a^3	b	ab	a^2b	a^3b	$\ \varphi^s\ _1$	$\ \varphi^s\ _\infty$
φ^0	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1	1	1
φ^2	1	-1	1	-1	1	-1	1	-1	1	1
φ^3	1	-1	1	-1	-1	1	-1	1	1	1
φ^4	$\sqrt{2}$	$\sqrt{2}i$	$-\sqrt{2}$	$-\sqrt{2}i$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}i$	$-\sqrt{2}$	$\sqrt{2}i$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}i$	$-\sqrt{2}$	$\sqrt{2}i$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}i$	$\sqrt{2}$	$\sqrt{2}i$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$

Notice that $|\varphi^s|$ can only take the values 0 or the square root of the corresponding dimension because this representation is **monomial**. In this case we have $\Psi_k = 1$, since $\max_{0 \leq s < 8} \|\varphi^s\|_1 \|\varphi^s\|_\infty = 1$ for $k \in \mathbb{N}$, but the group G is not abelian.

3. Fourier series and Dirichlet kernels

FOR $f \in L^1(G)$ we define the **Fourier coefficients** \hat{f}_k , the **n -th partial sums of Fourier series** $S_n f$ and the **Dirichlet kernels** D_n by

$$\hat{f}_k := \int_G f \bar{\psi}_k d\mu, \quad S_n f := \sum_{k=0}^{n-1} \hat{f}_k \psi_k, \quad D_n(x, y) := \sum_{k=0}^{n-1} \psi_k(x) \bar{\psi}_k(y)$$

for all $k, n \in \mathbb{N}$. The equality $S_n f(x) = \int_G f(y) D_n(x, y) dy$ shows the importance of the Dirichlet kernels in the study of the convergence of Fourier series.

The **intervals** $I_n(x) := \{y \in G : y_k = x_k, \text{ for } 0 \leq k < n\}$ form a countable neighborhood base of the product topology.

Lemma 1 (Paley lemma) If $n \in \mathbb{N}$ and $x, y \in G$, then

$$D_{M_n}(x, y) = \begin{cases} M_n & \text{for } x \in I_n(y), \\ 0 & \text{for } x \notin I_n(y) \end{cases}$$

The Paley lemma is used to prove that the $S_{M_n} f$ partial sequence of Fourier sums converge to f in L^p -norm and a.e., if $f \in L^p(G)$, $p \geq 1$. This property mark a essential difference between the comportment of these systems and the trigonometric system.

DEFINE the **maximum of the Dirichlet kernels** by

$$D_n := \sup_{x, y \in G} |D_n(x, y)| \quad (n \in \mathbb{P}).$$

For abelian cases $D_n = n$ for all $n \in \mathbb{P}$, but the general case is a bit more different.

Theorem 1 If $n \in \mathbb{P}$ and $A := \max\{k \in \mathbb{N} : n_k \neq 0\}$, then $n \leq D_n \leq M_{A+1}$.

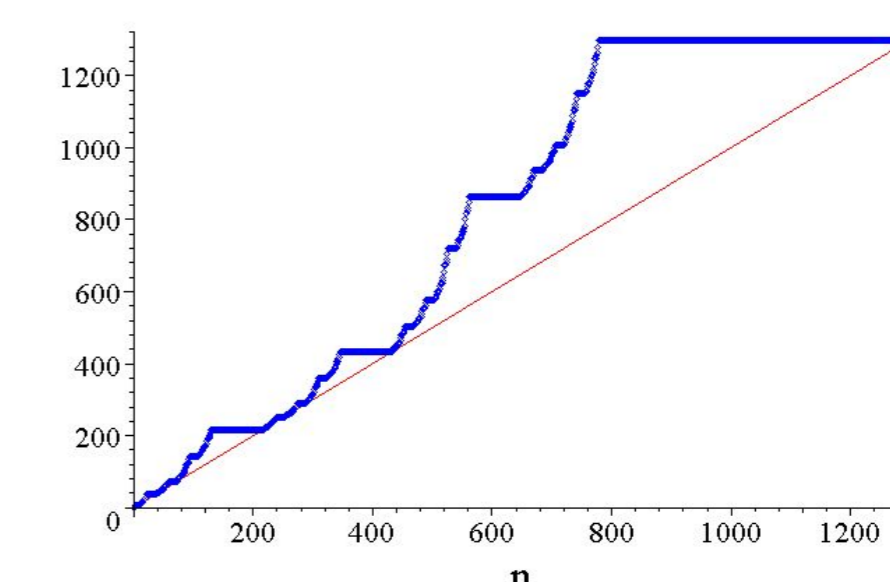


Figure 1: The maximum of the Dirichlet kernels on the complete product of S_3 .

4. Lebesgue constants

DEFINE the **Lebesgue constants** by

$$L_n := \sup_{x \in G} \int_G \sup_{k \leq n} |D_k(x, y)| d\mu(y) \quad (n \in \mathbb{N})$$

Theorem 2 Let G be a bounded group.

- If the sequence Ψ is bounded then $L_n = O(\log n)$.
- If the sequence Ψ is not bounded and $M_k \leq n < M_{k+1}$ then $L_n = O(\Psi_k)$.

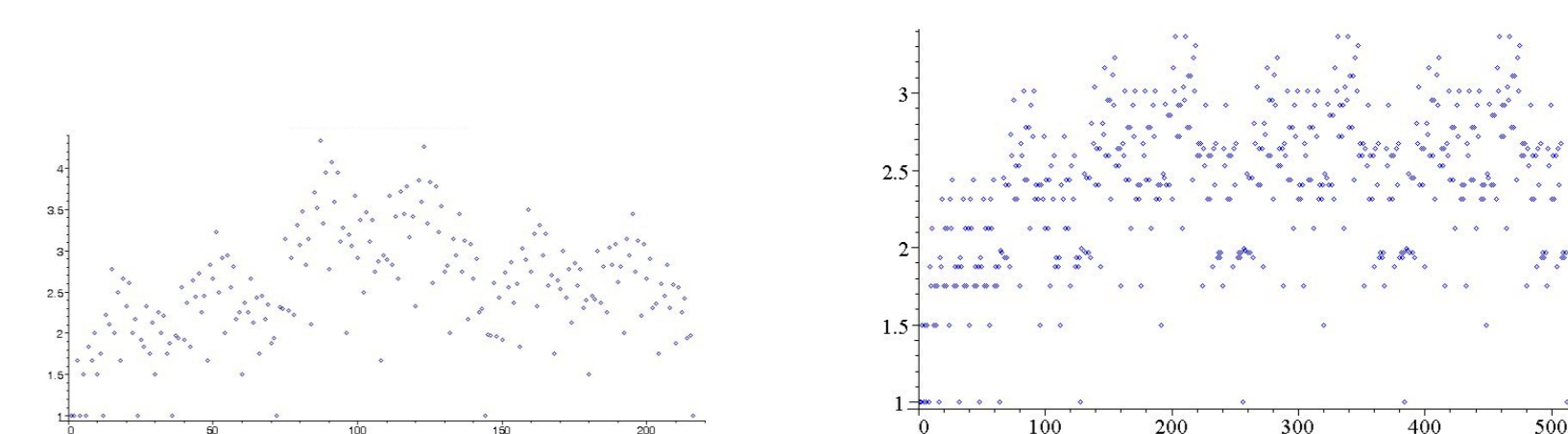


Figure 2: Lebesgue constants for the complete product of (left) S_3 with $L_n = O(\Psi_k)$; (right) Q_2 with $L_n = O(\log n)$.

FINALLY, we illustrate the Lebesgue constants for the classical Walsh-Paley system. This system is commutative, so it has the property $L_n = O(\log n)$.

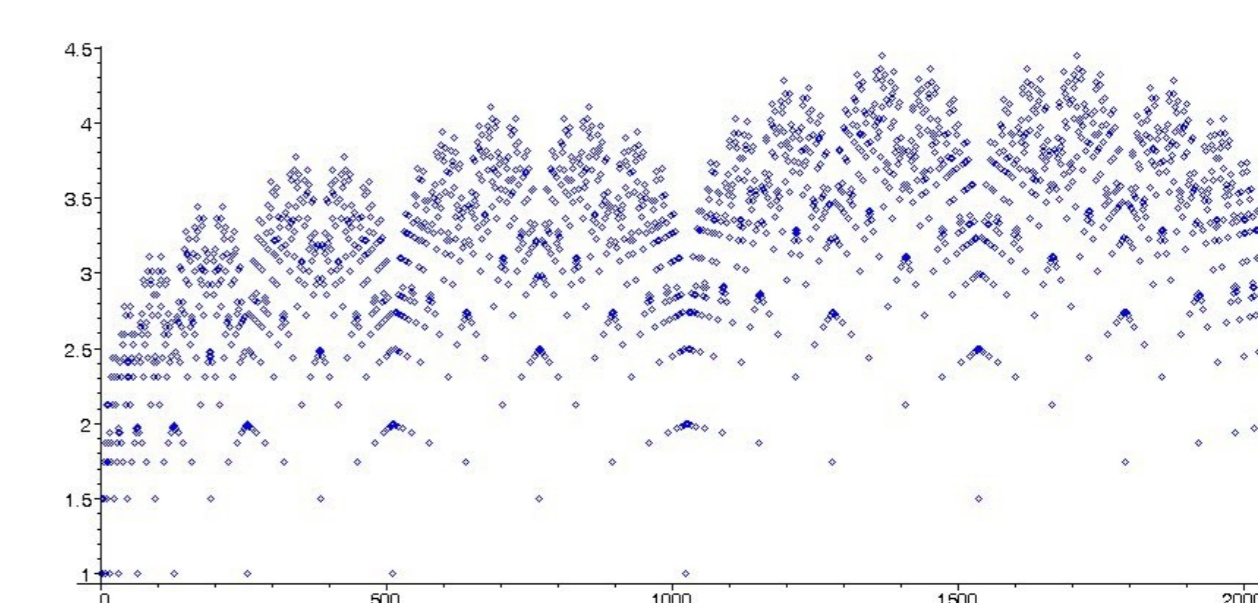


Figure 3: Lebesgue constants for the Walsh-Paley system.

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