

Convergence in L^p -norm of Fourier series on representative product systems

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February 2010
Puebla

Walsh functions

The term of **Walsh functions** refers to one of three orthonormal systems which differ only from their enumerations.

- **The original Walsh system** J. L. Walsh (1923)
was generated recursively, it is the Hadamard transform of the Haar system.
- **The Walsh-Paley system** R. E. A. C. Paley (1932)
is the finite products of Rademacher functions.
- **The Walsh-Kaczmarz system** A. A. Šneider (1948)
is also the finite products of Rademacher functions, but in different order.

Theorem

The Walsh system is an orthonormal and complete system on $L^2([0, 1])$, taking on only the values $+1$ and -1 .

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H. A. Rademacher (1922)

$$r(x) = \operatorname{sgn}(\sin(2^{n+1}\pi x)) \quad x \in [0, 1[.$$

The binary expansion of n : (n_0, n_1, \dots)

Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k 2^k, \quad \text{where } n_k = 0 \text{ or } n_k = 1.$$

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$$\omega_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (x \in [0, 1[).$$

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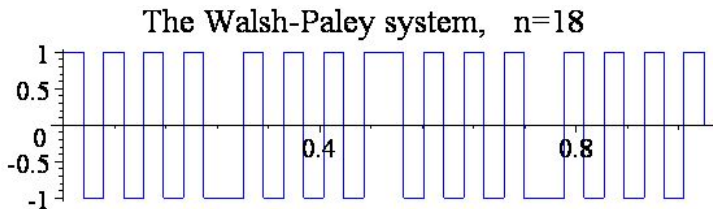
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The Walsh-Paley system

Plotted by Maple



The characters of the Dyadic group

The Dyadic group $\left(G := \prod_{k=0}^{\infty} \mathbb{Z}_2 \right)$

is the complete product of cyclic groups of order 2, with discrete topology and assign each singleton the measure $\frac{1}{2}$. G has the product topology and measure. (Haar measure)

The system of characters

Define $\varphi(x) = (-1)^x$, ($x \in \mathbb{Z}_2$). For each $n \in \mathbf{N}$ with binary expansion (n_0, n_1, \dots) let

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi^{n_k}(x_k), \quad (x = (x_0, x_1, \dots) \in G).$$

Theorem

The system of characters is an orthonormal and complete system on $L^2(G)$.

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The representation of the Dyadic group on $[0, 1[$

The Fine's map

N. J. Fine (1949)

For any $x \in [0, 1[$ there exists a sequence of numbers 0 and 1 such that

$$x := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \quad ((x_0, x_1, \dots) \in G),$$

but only the numbers $p/2^n$ have two expressions of this form. In this case we have the one which terminates in 0's. Define **Fine's map** by

$$\rho(x) = (x_0, x_1, \dots) \in G.$$

Fine's map gives a natural relation between the new structure of $[0, 1[$ and the structure of G (Harmonic analysis).

- The Haar measure corresponds to the Lebesgue measure.
- The characters of G corresponds to the Walsh-Paley system.

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The dyadic group is **metrizable** by the norm

$$|x| := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \in [0, 1[\quad ((x_0, x_1, \dots) \in \mathbf{G}).$$

A **new operation** on the interval $[0, 1[$:

$$x \odot y := |\rho(x)\rho(y)| \quad (x, y \in [0, 1[).$$

The interval $[0, 1[$ is not a group under the new operation.

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Theorem

Let ρ denote the Fine's map. If f is integrable on G then $f \circ \rho$ is Lebesgue integrable and

$$\int_G f d\mu = \int_0^1 (f \circ \rho)(x) dx.$$

Conversely, if g is Lebesgue integrable and f is defined by $f(x) := g(|x|)$ ($x \in G$) then f is integrable on G and

$$\int_0^1 g(x) dx = \int_G f d\mu.$$

The Vilenkin groups

A Vilenkin group $\left(G := \prod_{k=0}^{\infty} \mathbb{Z}_{m_k} \right)$

N. Ja. Vilenkin (1947)

is the complete product of cyclic groups of order m_k ($m_k \geq 2$, $k \in \mathbf{N}$), with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. G has the product topology and measure. (Haar measure)

Bounded Vilenkin group

if the sequence $m = (m_0, m_1, \dots)$ is a bounded sequence.

The generalized Rademacher functions

$$\varphi_k^s(x) = \exp(2\pi i s x / m_k) \quad (s \in \{0, \dots, m_k - 1\}, x \in \mathbb{Z}_{m_k}, i^2 = -1)$$

The generalized Rademacher functions are the characters of cyclic groups.

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The complete product of finite groups

Gát, G., Toledo, R., Anal. Math., 1996

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The group G is called **bounded**
if the sequence $m = (m_0, m_1, \dots)$ is a bounded sequence.

$$\varphi_k^s = ?, \psi_n = ?$$

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Harmonic Analysis

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The dual object (Σ_k) of the finite group G_k ($k \in \mathbf{N}$)

is the set of all continuous irreducible unitary representations of the group G_k which are not equivalents.

The Coordinate functions

For any $\sigma \in \Sigma_k$, let $\{\xi_1, \dots, \xi_{d_\sigma}\}$ be a fixed basis of the representation space of a representation $U^{(\sigma)}$ in the class σ having the dimension d_σ .

The Coordinate functions:

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j \rangle, \quad i, j \in \{1, \dots, d_\sigma\}, \sigma \in \Sigma_k$$

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Orthonormal systems on finite groups

The system φ_k

We order the all normalized coordinate functions of the finite group G_k ($\varphi_k^0(x) = 1$) to obtain exactly m_k number of functions.

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \quad (x \in G_k, s = 0, \dots, m_k - 1),$$

where $\sigma \in \Sigma_k$, $i, j \in \{1, \dots, d_\sigma\}$.

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Example 1: The permutation group of 3 elements, S_3

	e	(12)	(13)	(23)	(123)	(132)	$\ \varphi^s\ _1$	$\ \varphi^s\ _\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
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$$\max_{s=0\dots 5} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3}$$

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Example 2: The quaternion group of order 8:

$$\mathcal{Q}_2 := \{[a, b] : a^4 = e, b^2 = a^2, bab^{-1} = a^3\}$$

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φ^2	1	-1	1	-1	1	-1	1	-1	1	1
φ^3	1	-1	1	-1	-1	1	-1	1	1	1
φ^4	$\sqrt{2}$	$\sqrt{2}i$	$-\sqrt{2}$	$-\sqrt{2}i$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}i$	$-\sqrt{2}$	$\sqrt{2}i$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}i$	$-\sqrt{2}$	$\sqrt{2}i$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
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$$\max_{s=0\dots 7} \|\varphi^s\|_1 \|\varphi^s\|_\infty = 1$$

Example 2: The quaternion group of order 8:

$$\mathcal{Q}_2 := \{[a, b] : a^4 = e, b^2 = a^2, bab^{-1} = a^3\}$$

	e	a	a^2	a^3	b	ab	a^2b	a^3b	$\ \varphi^s\ _1$	$\ \varphi^s\ _\infty$
φ^0	1	1	1	1	1	1	1	1	1	1
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Representative product systems

Gát, G., Toledo, R., Anal. Math., 1996

The m -adic expansion of n : (n_0, n_1, \dots)

Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, ($k \in \mathbf{N}$). Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \leq n_k < m_k).$$

A representative product systems

is the product system of φ :

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \quad (x \in G).$$

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A representative product system is an orthonormal and complete system on $L^2(G)$.

Characteristics of the system ψ for noncommutative cases:

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The representation of G on $[0, 1[$

Toledo, R., Acta Math. Acad. Paed. Nyregyhziensis, 2003

It is similar to the dyadic group, but first we need to enumerate the elements of all groups G_k , ($k \in \mathbf{N}$) in an arbitrary way but the first is always their identity.

$$G_k \ni x \xrightarrow{\text{bijection}} \bar{x} \in \{0, 1, \dots, m_k - 1\}, \quad \bar{e} = 0.$$

Fine's map and norm

With the bijection above we can introduce the **Fine's map**:

$$\rho(x) = (x_0, x_1, \dots) \in G.$$

and **norm**:

$$|x| := \sum_{k=0}^{\infty} \frac{\bar{x}_k}{M_{k+1}} \quad (x = (x_0, x_1, \dots) \in G)$$

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Fine's map gives a natural relation between the new structure of $[0, 1[$ and the structure of G (Harmonic analysis).

Theorem

Let ρ denote the Fine's map. If f is integrable on G then $f \circ \rho$ is Lebesgue integrable and

$$\int_G f d\mu = \int_0^1 (f \circ \rho)(x) dx.$$

Conversely, if g is Lebesgue integrable and f is defined by $f(x) := g(|x|)$ ($x \in G$) then f is integrable on G and

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- The Haar measure corresponds to the Lebesgue measure.
- The new systems $\psi_n \circ \rho$ are orthonormal and complete systems on $[0, 1[$, but they are not necessary uniformly bounded.

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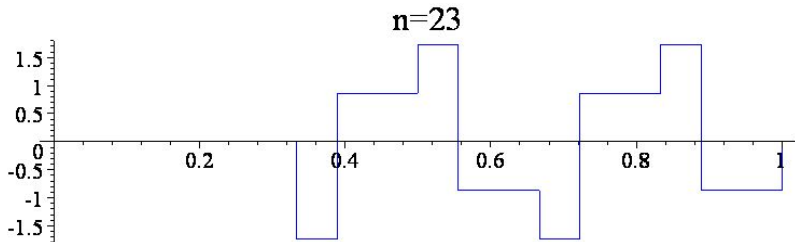
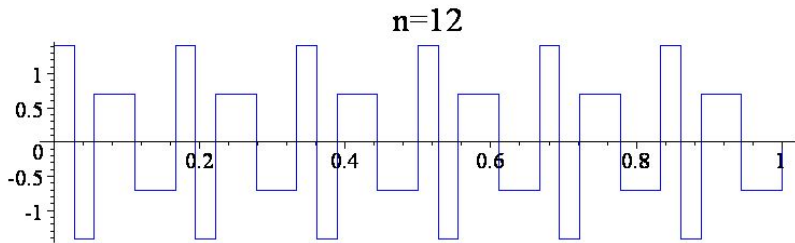
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The complete product of S_3

Plotted by Maple



The Dirichlet kernels

The n -th partial sums of Fourier series

$$S_n f := \sum_{k=0}^{n-1} \hat{f}_k \psi_k \quad (n \in \mathbf{N}), \quad \text{where } \hat{f}_k := \int_G f \bar{\psi}_k d\mu.$$

The Dirichlet kernels

$$D_n(x, y) := \sum_{k=0}^{n-1} \psi_k(x) \bar{\psi}_k(y) \quad (n \in \mathbf{P})$$

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Lemma

If $n \in \mathbf{N}$ and $x, y \in G$, then

$$D_n(x, y) = \sum_{k=0}^{\infty} D_{M_k}(x, y) \left(\sum_{s=0}^{n_k-1} \varphi_k^s(x_k) \overline{\varphi_k^s(y_k)} \right) \psi_{n^{(k+1)}}(x) \overline{\psi_{n^{(k+1)}}(y)},$$

where (n_0, n_1, \dots) is the expansion of n and $x = (x_0, x_1, \dots)$,
 $y = (y_0, y_1, \dots)$.

The Dirichlet kernels

Gát, G., Toledo, R., Anal. Math., 1996

The intervals

$$I_n(x) := \{y \in G : y_k = x_k, \text{ for } 0 \leq k < n\}, \quad I_0(x) := G, \quad I_n := I_n(e)$$

Lemma

If $n \in \mathbf{N}$ and $x, y \in G$, then

$$D_{M_n}(x, y) = \begin{cases} M_n & \text{for } x \in I_n(y), \\ 0 & \text{for } x \notin I_n(y) \end{cases}$$

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For each $f \in L^p(G)$, $p \geq 1$ and $n \in \mathbf{N}$ the $S_{M_n}f$ sums converge to f in L^p -norm and a.e.

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The maximal value of Dirichlet kernels

$$D_n := \sup_{x, y \in G} |D_n(x, y)| \quad (n \in \mathbf{P})$$

If G is commutative then $D_n = n$, but for an arbitrary group G

Theorem (R. Toledo)

If $n \in \mathbf{P}$ and $A := \max\{k \in \mathbf{N} : n_k \neq 0\}$, then

$$n \leq D_n \leq M_{A+1}.$$

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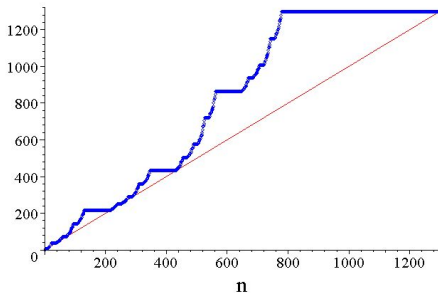
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The Dirichlet kernels

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D_n ($n \leq 6^4$) on the complete product of \mathcal{S}_3

Convergence in L^p -norm of Fourier series

The problem: Which are the values of p ($1 \leq p < \infty$) such that for all function $f \in L^p(G)$ the sequence of partial sums $S_n f$ of the Fourier series of f converges to the function f in L^p -norm?

For $p = 2$ the answer is affirmative. ($L^2(G)$ is a Hilbert space)

Theorem (P. Simon, F. Schipp and W. S. Young (1976))

Let G be a Vilenkin group and $1 < p < \infty$. Then for all function $f \in L^p(G)$ the sequence of partial sums $S_n f$ of the Fourier series of f converges to the function f in L^p -norm.

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Theorem

For all G groups there exists a function $f \in L^1(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^1 -norm.

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$$\Psi_k = \prod_{i=0}^{k-1} \max_{s < m_i} \|\varphi_i^s\|_1 \|\varphi_i^s\|_\infty \quad (k \in \mathbf{N}).$$

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Convergence in L^p -norm of Fejér means

Gát, G., Toledo, R., Anal. Math., 1996

The Fejér means of Fourier series

$$\sigma_n f = \frac{1}{n} \sum_{k=1}^{n-1} S_k f \quad (n \in \mathbf{P})$$

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The Cesàro means of order α

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Theorem

Let G be a bounded group,

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Gát, G., Toledo, R., Acta Math. Hung., 2009

Corollary

If G is a bounded monomial group, $0 < \alpha < 1$ and $f \in L^p(G)$, $1 \leq p < \infty$, then $\sigma_n^\alpha f \rightarrow f$ in L^p -norm.

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Gát, G., Toledo, R., Acta Math. Hung., 2009

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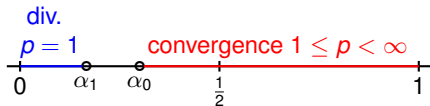
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$$\alpha_1 := \liminf_{k \rightarrow \infty} \log_{m_k} \left(\max_{0 \leq s < m_k} \|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty \right)$$

and $0 < \alpha < \alpha_1$. Then there exists an $f \in L^1(G)$ such that $\sigma_n^\alpha f$ does not converge to the function f in L^1 -norm.

Summary of results

- G is bounded group
- G is the complete product of \mathcal{S}_3
- G is the complete product of \mathcal{Q}_2

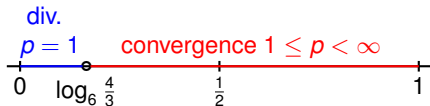


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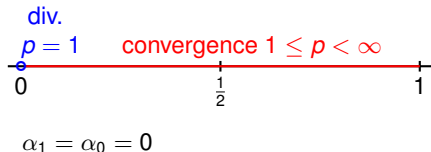
$$\alpha_1 = \alpha_0 = \log_6 \frac{4}{3}$$

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