### Walsh functions

The term of *Walsh functions* refers to one of three orthonormal systems which differ only from their enumerations.

- *The original Walsh system* J. L. Walsh (1923) was generated recursively, it is the Hadamard transform of the Haar system.
- *The Walsh-Paley system* R. E. A. C. Paley (1932) is the finite products of Rademacher functions.
- *The Walsh-Kaczmarz system* A. A. Šneider (1948) is also the finite products of Rademacher functions, but in different order.

**Theorem 1.** *The Walsh system is an orthonormal and complete system on* $L^2([0,1[)$*, taking on only the values* $+1$ *and* $-1$.

### The Walsh-Paley system

The Rademacher functions H. A. Rademacher (1922)

$$r(x) = \operatorname{sgn}(\sin(2^{n+1}\pi x)) \qquad x \in [0,1[.$$

The binary expansion of $n$: $(n_0, n_1, \dots)$ Given $n \in \mathbf{N}$ it is possible to write $n$ uniquely as

$$n = \sum_{k=0}^{\infty} n_k 2^k, \qquad \text{where } n_k = 0 \text{ or } n_k = 1.$$

The Walsh-Paley system

$$\omega_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (x \in [0,1[).$$

### The characters of the Dyadic group

The Dyadic group $\left(G := \mathop{\times}_{k=0}^{\infty} \mathcal{Z}_2\right)$ is the complete product of cyclic groups of order 2, with discrete topology and assign each singleton the measure $\frac{1}{2}$. $G$ has the product topology and measure. (Haar measure)

The system of characters Define $\varphi(x) = (-1)^x$, $(x \in \mathcal{Z}_2)$. For each $n \in \mathbf{N}$ with binary expansion $(n_0, n_1, \dots)$ let

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi^{n_k}(x_k), \quad (x = (x_0, x_1, \dots) \in G).$$

**Theorem 2.** *The system of characters is an orthonormal and complete system on* $L^2(G)$.

### The representation of the Dyadic group on $[0,1[$

The Fine's map N. J. Fine (1949) For any $x \in [0,1[$ there exists a sequence of numbers 0 and 1 such that

$$x := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \qquad ((x_0, x_1, \dots) \in G),$$

but only the numbers $p/2^n$ have two expressions of this form. In this case we have the one which terminates in 0's. Define *Fine's map* by

$$\rho(x) = (x_0, x_1, \dots) \in G.$$

Fine's map gives a natural relation between the new structure of $[0,1[$ and the structure of $G$ (Harmonic analysis).

- The Haar measure corresponds to the Lebesgue measure.
- The characters of $G$ corresponds to the Walsh-Paley system.

### The Vilenkin groups

A Vilenkin group $\left(G := \underset{k=0}{\overset{\infty}{\times}} \mathbb{Z}_{m_k}\right)$ N. Ja. Vilenkin (1947) is the complete product of cyclic groups of order $m_k$ ($m_k \geq 2$, $k \in \mathbf{N}$), with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. $G$ has the product topology and measure. (Haar measure)

Bounded Vilenkin group if the sequence $m = (m_0,\, m_1, \dots)$ is a bounded sequence.

The generalized Rademacher functions

$$\varphi_k^s(x) = \exp(2\pi \imath s x/m_k) \qquad (s \in \{0, \dots m_k - 1\},\ x \in \mathbb{Z}_{m_k},\ \imath^2 = -1)$$

The generalized Rademacher functions are the characters of cyclic groups.

### The Vilenkin systems

The $m$-adic expansion of $n$: $(n_0, n_1, \dots)$ Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, ($k \in \mathbf{N}$). Given $n \in \mathbf{N}$ it is possible to write $n$ uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \leq n_k < m_k).$$

A Vilenkin system is the product system of $\varphi$:

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x = (x_0, x_1, \dots) \in G).$$

**Theorem 3.** *The functions of the Vilenkin system are the characters of the Vilenkin group, thus it is an orthonormal and complete system on $L^2(G)$.*

### The complete product of finite groups

The group $\left(G := \underset{k=0}{\overset{\infty}{\times}} G_k\right)$ Denote by $G$ the complete product of arbitrary finite groups of order $m_k$ ($m_k \geq 2$, $k \in \mathbf{N}$), with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. $G$ has the product topology and measure. (Haar measure)

$\varphi_k^s = ?$, $\psi_n = ?$ → Harmonic Analysis

**Orthonormal systems on finite groups**

The dual object ($\Sigma_k$) of the finite group $G_k$ ($k \in \mathbf{N}$) is the set of all continuous irreducible unitary representations of the group $G_k$ which are not equivalents.

The Coordinate functions For any $\sigma \in \Sigma_k$, let $\{\xi_1, \dots, \xi_{d_\sigma}\}$ be a fixed basis of the representation space of a representation $U^{(\sigma)}$ in the class $\sigma$ having the dimension $d_\sigma$. The *Coordinate functions*:

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j \rangle, \quad i, j \in \{1, ..., d_\sigma\}, \sigma \in \Sigma_k$$

**Orthonormal systems on finite groups**

The system $\varphi_k$ We order the all normalized coordinate functions of the finite group $G_k$ ($\varphi_k^0(x) = 1$) to obtain exactly $m_k$ number of functions.

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_k,\ s = 0, \dots, m_k - 1),$$

where $\sigma \in \varSigma_k$, $i, j \in \{1, ..., d_\sigma\}$.

**Theorem 4.** *The system $\varphi_k$ is an orthonormal and complete system on $L^2(G_k)$.*

**Example 1: The permutation group of 3 elements,** $\mathcal{S}_3$

| | $e$ | $(12)$ | $(13)$ | $(23)$ | $(123)$ | $(132)$ | $\|\varphi^s\|_1$ | $\|\varphi^s\|_\infty$ |
|---|---|---|---|---|---|---|---|---|
| $\varphi^0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $\varphi^1$ | $1$ | $-1$ | $-1$ | $-1$ | $1$ | $1$ | $1$ | $1$ |
| $\varphi^2$ | $\sqrt{2}$ | $-\sqrt{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $\frac{2\sqrt{2}}{3}$ | $\sqrt{2}$ |
| $\varphi^3$ | $\sqrt{2}$ | $\sqrt{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $\frac{2\sqrt{2}}{3}$ | $\sqrt{2}$ |
| $\varphi^4$ | $0$ | $0$ | $-\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{2}$ | $-\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{3}$ | $\frac{\sqrt{6}}{2}$ |
| $\varphi^5$ | $0$ | $0$ | $-\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{2}$ | $-\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{3}$ | $\frac{\sqrt{6}}{2}$ |

$$\max_{s=0\dots5} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3}$$

**Example 2: The quaternion group of order 8:** $\mathcal{Q}_2 := \{[a, b] : a^4 = e,\ b^2 = a^2,\ bab^{-1} = a^3\}$

| | $e$ | $a$ | $a^2$ | $a^3$ | $b$ | $ab$ | $a^2b$ | $a^3b$ | $\|\varphi^s\|_1$ | $\|\varphi^s\|_\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\varphi^0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $\varphi^1$ | $1$ | $1$ | $1$ | $1$ | $-1$ | $-1$ | $-1$ | $-1$ | $1$ | $1$ |
| $\varphi^2$ | $1$ | $-1$ | $1$ | $-1$ | $1$ | $-1$ | $1$ | $-1$ | $1$ | $1$ |
| $\varphi^3$ | $1$ | $-1$ | $1$ | $-1$ | $-1$ | $1$ | $-1$ | $1$ | $1$ | $1$ |
| $\varphi^4$ | $\sqrt{2}$ | $\sqrt{2}\imath$ | $-\sqrt{2}$ | $-\sqrt{2}\imath$ | $0$ | $0$ | $0$ | $0$ | $\frac{\sqrt{2}}{2}$ | $\sqrt{2}$ |
| $\varphi^5$ | $\sqrt{2}$ | $-\sqrt{2}\imath$ | $-\sqrt{2}$ | $\sqrt{2}\imath$ | $0$ | $0$ | $0$ | $0$ | $\frac{\sqrt{2}}{2}$ | $\sqrt{2}$ |
| $\varphi^6$ | $0$ | $0$ | $0$ | $0$ | $\sqrt{2}$ | $-\sqrt{2}\imath$ | $-\sqrt{2}$ | $\sqrt{2}\imath$ | $\frac{\sqrt{2}}{2}$ | $\sqrt{2}$ |
| $\varphi^7$ | $0$ | $0$ | $0$ | $0$ | $-\sqrt{2}$ | $-\sqrt{2}\imath$ | $\sqrt{2}$ | $\sqrt{2}\imath$ | $\frac{\sqrt{2}}{2}$ | $\sqrt{2}$ |

$$\max_{s=0\dots7} \|\varphi^s\|_1 \|\varphi^s\|_\infty = 1$$

### Representative product systems

The $m$-adic expansion of $n$: $(n_0, n_1, \dots)$ Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, ($k \in \mathbf{N}$). Given $n \in \mathbf{N}$ it is possible to write $n$ uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \le n_k < m_k).$$

A representative product systems G. Gát and R. Toledo (1996) is the product system of $\varphi$:

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G).$$

### Representative product systems

**Theorem 5.** *A representative product system is an orthonormal and complete system on $L^2(G)$.*

Characteristics of the system $\psi$ for noncommutative cases:

- It is not uniformly bounded.
- It takes the value 0.

### The representation of $G$ on $[0, 1[$

It is similar to the dyadic group, but first we need to enumerate the elements of all groups $G_k$, $(k \in \mathbf{N})$ in an arbitrary way but the first is always their identity.

$$G_k \ni x \overset{\textit{bijection}}{\longrightarrow} \overline{x} \in \{0,\, 1, \dots,\, m_k - 1\}, \qquad \overline{e} = 0.$$

Fine's map and norm With the bijection above we can introduce the *Fine's map*:

$$\rho(x) = (x_0, x_1, \dots) \in G.$$

and *norm*:

$$|x| := \sum_{k=0}^{\infty} \frac{\overline{x_k}}{M_{k+1}} \qquad (x = (x_0, x_1, \dots) \in G)$$

### The representation of $G$ on $[0, 1[$

Here Fine's map also gives a natural relation between the new structure of $[0, 1[$ and the structure of $G$ (Harmonic analysis).

- The Haar measure corresponds to the Lebesgue measure.
- The new systems $\psi_n \circ \rho$ are orthonomal and complete systems on $[0, 1[$, but they are not necessary uniformly bounded.

### The complete product of $\mathbb{S}_3$ Plotted by Maple

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**Divergence in $L^p$-norm of Fourier series**

The n-th partial sums of Fourier series

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \qquad (n \in \mathbf{N}), \qquad \text{where } \widehat{f}_k := \int_G f \overline{\psi}_k d\mu.$$

**Theorem 6** ( R. Toledo)**.** *For all G groups there exists a function $f \in L^1(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of $f$ does not converge to the function $f$ in $L^1$-norm.*

**Divergence in $L^p$-norm of Fourier series**

The sequence $\Psi$

$$\Psi_k = \prod_{i=0}^{k-1} \max_{s<m_i} \|\varphi_i^s\|_1 \|\varphi_i^s\|_\infty \qquad (k \in \mathbf{N}).$$

**Theorem 7** ( R. Toledo)**.** *If G is a bounded group with unbounded sequence $\Psi$, then for all $p \neq 2$, $1 < p < \infty$ there exists a function $f \in L^p(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of $f$ does not converge to the function $f$ in $L^p$-norm.*

**Convergence in $L^p$-norm of Fejér means**

The Fejér means of Fourier series

$$\sigma_n f = \frac{1}{n} \sum_{k=1}^{n-1} S_k f \qquad (n \in \mathbf{P})$$

**Theorem 8** (G. Gát and R. Toledo (1996))**.** *If G is a bounded group and $f \in L^p(G)$, $1 \leq p < \infty$, then $\sigma_n f \to f$ in $L^p$-norm.*

**Convergence in $L^p$-norm of Cesàro means**

The Cesàro means of order $\alpha$

$$\sigma_n^\alpha f = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f, \quad \text{where } A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!}, \quad (n \in \mathbf{P}).$$

$$\sigma_n^0 f = S_n f, \qquad \sigma_n^1 f = \sigma_n f$$

**Theorem 9** ( G. Gát and R. Toledo (2009))**.** *Let G be a bounded group,*

$$\alpha_0 := \limsup_{k\to\infty} \log_{m_k} \left( \max_{0 \leq s < m_k} \|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty \right)$$

*$\alpha_0 < \alpha < 1$ and $f \in L^p(G)$ for $1 \leq p < \infty$. Then $\sigma_n^\alpha f \to f$ in $L^p$-norm.*

**Convergence in $L^p$-norm of Cesàro means**

**Corollary 10.** *If G is a bounded monomial group, $0 < \alpha < 1$ and $f \in L^p(G)$, $1 \leq p < \infty$, then $\sigma_n^\alpha f \to f$ in $L^p$-norm.*

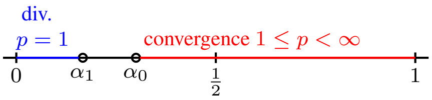
**Theorem 11** ( G. Gát and R. Toledo (2009))**.** *Let G be a bounded group,*

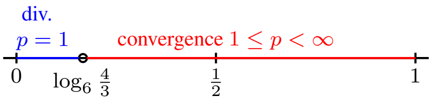
$$\alpha_1 := \liminf_{k\to\infty} \log_{m_k} \left( \max_{0 \leq s < m_k} \|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty \right)$$

*and $0 < \alpha < \alpha_1$. Then there exists an $f \in L^1(G)$ such that $\sigma_n^\alpha f$ does not converge to the function $f$ in $L^1$-norm.*

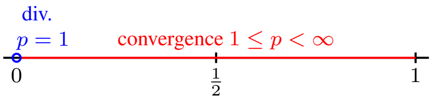
### Summary of results

- $G$ is bounded group
- $G$ is the complete product of $\mathcal{S}_3$
- $G$ is the complete product of $\mathcal{Q}_2$





$\alpha_1 = \alpha_0 = \log_6 \frac{4}{3}$



$\alpha_1 = \alpha_0 = 0$

$$\alpha_0 := \limsup_{k\to\infty} \log_{m_k} \left( \max_{0\leq s<m_k} \|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty \right)$$

$$\alpha_1 := \liminf_{k\to\infty} \log_{m_k} \left( \max_{0\leq s<m_k} \|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty \right)$$