ON THE LOGARITHMIC SUMMABILITY OF FOURIER SERIES

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Dedicated to the memory of Professor George Tkebuchava.

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Abstract. In the paper [17] Tkebuchava constructed a set of logarithmic summation methods which in particular cases includes Riesz and Nörlund logarithmic summation methods as limit case. He gave estimation for kernels for trigonometric system and also some consequences with respect to convergence of means of Fourier series. In this article we would like to give such good estimates for the kernels with respect to Walsh and Walsh-Kaczmarz system as Tkebuchava did.

1. Introduction

The n-th Riesz's logarithmic means of a Fourier series is defined by

$$\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{k},$$

where $l_n := \sum_{k=1}^{n-1}\frac{1}{k}$. The Riesz's logarithmic means with respect to the trigonometric system was studied by a lot of authors, e.g. Szász [16] and Yabuta [18], with respect to Walsh, Vilenkin system by Simon [13] and Gát [3].

The n-th Nörlund logarithmic means of an integrable function f is defined by

$$t_n(f) := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{n-k}.$$

The n-th Nörlund logarithmic kernel is

$$F_n := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{D_k}{n-k}.$$

Recently, for the Walsh system Gát, Goginava and Tkebuchava [4, 8, 6, 9] proved some convergence and divergence properties of Nörlund logarithmic means of functions.

More results on this logarithmic means with respect to unbounded Vilenkin system can be found in [2] written by Blahota and Gát.

In one of his last papers [17] Tkebuchava constructed a set of logarithmic summation methods which in particular cases contains both mentioned logarithmic summation methods as limit case. Namely, for any integers m,n such that $0 \le m \le n$ let the Tkebuchava's mean $t_{m,n}$ be defined by

$$t_{m,n}(f) := \frac{1}{l(m,n)} \left(\sum_{k=0}^{m-1} \frac{S_k(f)}{m-k+1} + S_m(f) + \sum_{k=m+1}^{n} \frac{S_k(f)}{k-m+1} \right),$$

where

$$l(m,n) = \sum_{k=0}^{m-1} \frac{1}{m-k+1} + 1 + \sum_{k=m+1}^{n} \frac{1}{k-m+1}.$$

It is clear that $l(m,n) \asymp \ln n$. This summation method includes Riesz (for $m = 0$) and Nörlund (for $m = n$) logarithmic methods too.

Define the kernels of Tkebuchava's means $F_{m,n}$ by

$$F_{m,n} := \frac{1}{l(m,n)} \left(\sum_{k=0}^{m-1} \frac{D_k}{m-k+1} + D_m + \sum_{k=m+1}^{n} \frac{D_k}{k-m+1} \right).$$

Tkebuchava gave estimates of kernels in the trigonometric case. Namely, the following theorem holds.

Theorem A. *(see [17]) There exist positive constants C_1, C_2 $(0 < C_1 \le C_2)$ such that for any integers m and n $(0 \le m \le n)$*

$$C_1 \left(1 + \frac{\ln^2(m+2)}{\ln(n+2)} \right) \le \|F_{m,n}\|_{L^1[-\pi,\pi]} \le C_2 \left(1 + \frac{\ln^2(m+2)}{\ln(n+2)} \right)$$

holds.

Let $K := [0,1)$ denote the unit interval in $\mathbb{R}$. The Rademacher functions are defined by

$$r_n(x) := r_0(2^n x), \quad n \ge 1 \text{ and } x \in K, \text{ where } r_0(x) := \begin{cases} 1, & \text{if } x \in [0,1/2), \\ -1, & \text{if } x \in [1/2,1), \end{cases}$$

and $r_0(x+1) := r_0(x)$. Each natural number n can be uniquely expressed as $n = \sum_{i=0}^{\infty} n_i 2^i$, $n_i \in \{0,1\}$ $(i \in \mathbb{N})$, where only a finite number of n_i's are different from zero. Let the order of $n \ge 1$ be denoted by $|n| := \max\{j \in \mathbb{N} : n_j \ne 0\}$.

The Walsh-Paley functions are defined by

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k}.$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 := 1$ and for $n \ge 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k}.$$

Each $x \in K = [0,1)$ can be expressed as $x = \sum_{j=0}^{\infty} x_j 2^{-n-1}$, where $x_j \in \{0,1\}$ $(j \in \mathbb{N})$. This expression is unique if x is not a dyadic rational. In other words, if x is not of the form $j/2^n$, where j,n are nonnegative integers. If x is a dyadic rational, then we choose the expansion which terminates in zeros. In this way we have the unicity of this expression for all x.

For $A \in \mathbb{N}$ define the transformation $\tau_A : K \to K$ by

$$\tau_A(x) := \frac{x_{A-1}}{2^1} + \frac{x_{A-2}}{2^2} + \cdots + \frac{x_0}{2^{A-1}} + \sum_{j=A}^{\infty} \frac{x_j}{2^{j+1}}.$$

In other words, if the coordinates of x are $x_0, x_1, \ldots, x_{A-1}, x_A, \ldots$, then the coordinates of $\tau_A(x)$ are $x_{A-1}, x_{A-2}, \ldots, x_1, x_0, x_A, \ldots$. By the definition of τ_A (see [15]), we have

$$\kappa_n(x) = r_{|n|}(x)\omega_n(\tau_{|n|}(x)) \quad (n \in \mathbb{N}, x \in [0,1)).$$

Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels and the Fejér kernels by

$$\hat{f}^{\alpha}(k) := \int_0^1 f(t)\alpha_k(t)dt, \qquad S_n^{\alpha}(f) := \sum_{k=0}^{n-1} \hat{f}^{\alpha}(k)\alpha_k,$$

$$D_n^{\alpha} := \sum_{k=0}^{n-1} \alpha_k, \qquad K_n^{\alpha} := \frac{1}{n}\sum_{k=0}^{n} D_k^{\alpha},$$

where $\alpha = \omega$ or κ.

2. Tkebuchava's means and kernels of Walsh-(Kaczmarz)-Fourier series

To prove our main theorem we need the following Lemma 1 in [4] and the Lemma 2 in [11].

Lemma 1. *Let* $p_A := 2^{2A} + ... + 2^2 + 2^0$, *then* $\|F_{p_A}\|_1 \geq c \ln p_A$.

Lemma 2. *Let* $2^A \leq m < 2^{A+1}$, *then*

$$\begin{aligned} l_m F_m^{\kappa}(x) &= l_{m-2^{A-1}+1} D_{2^A}(x) \\ &- \omega_{2^A-1}(x) \sum_{j=1}^{2^{A-1}-1} \left(\frac{1}{m-2^A+j} - \frac{1}{m-2^A+j+1}\right) j K_j^{\omega}(\tau_{A-1}(x)) \\ &- \omega_{2^A-1}(x) \frac{2^{A-1}}{m-2^{A-1}} K_{2^{A-1}}^{\omega}(\tau_{A-1}(x)) + r_A(x) l_{m-2^A} F_{m-2^A}^{\omega}(\tau_A(x)) \\ &+ \sum_{s=0}^{2^{A-1}-2} \left(\frac{1}{m-s} - \frac{1}{m-s+1}\right) s K_s^{\kappa}(x) \\ &+ \frac{2^{A-1}-1}{m-2^{A-1}+1} K_{2^{A-1}-1}^{\kappa}(x). \end{aligned}$$

It is not possible to give so good lower estimate for Walsh or Walsh-Kaczmarz system as Tkebuchava did, because $\|F_{2^n,2^n}^{\alpha}\|_1 = \|F_{2^n}^{\alpha}\|_1 \leq c$ holds for all $n \in \mathbb{N}$, in case $\alpha = \omega$. It is proved for one dimension in [8] for two dimension in [6], and in case $\alpha = \kappa$ it is proved for one and two dimension in [11] (see Section 3 for the definition). But, the upper estimation can be given in the form of Theorem A. Thus, we state our main theorem in the following form.

Theorem 1. *Let $\alpha := \omega$ or κ. There exist positive constants C_1, C_2 $(0 < C_1 \leq C_2)$ such that for any integers m and n $(0 \leq m \leq n)$*

$$C_1 \leq \sup_{\substack{m,n \in \mathbb{N} \\ 0 \leq m \leq n}} \frac{\|F_{m,n}^{\alpha}\|_1}{\left(1 + \frac{\ln^2 m}{\ln n}\right)} \leq C_2$$

holds.

By standard argument we immediatelly have the following corollary.

Corollary 1. *Let* $m_n = O\left(\exp\sqrt{\ln n}\right)$ *as* $n \to \infty$, *then*

$$\|t_{m_n,n}(f) - f\|_1 \to 0 \quad \text{as } n \to \infty \quad \text{for all } f \in L^1[0,1)$$

and

$$\|t_{m_n,n}(f) - f\|_C \to 0 \quad \text{as } n \to \infty \quad \text{for all } f \in C[0,1).$$

Proof of Theorem 1: First, we prove the upper estimation. To do this set

$$F^{\alpha,1}_{m,n} := \frac{1}{l(m,n)} \sum_{k=0}^{m} \frac{D^\alpha_k}{m-k+1} \quad \text{and} \quad F^{\alpha,2}_{m,n} := \frac{1}{l(m,n)} \sum_{k=m+1}^{n} \frac{D^\alpha_k}{k-m+1}.$$

Discuss $F^{\alpha,1}_{m,n}$.

$$\|F^{\alpha,1}_{m,n}\|_1 \le \frac{1}{l(m,n)} \sum_{j=1}^{m} \frac{\|D^\alpha_j\|_1}{m-j+1} \le \frac{c}{l(m,n)} \sum_{j=1}^{m} \frac{\ln j}{m-j+1} = O\left(\frac{\ln^2 m}{\ln n}\right).$$

Now, we discuss $F^{\alpha,2}_{m,n}$.

If $n \le m^3$, then we have that

$$\begin{aligned} \|F^{\alpha,2}_{m,n}\|_1 &\le \frac{c}{\ln n} \sum_{k=m+1}^{n} \frac{\|D^\alpha_k\|_1}{k-m+1} \le \frac{c}{\ln n} \sum_{k=m+1}^{m^3} \frac{\ln k}{k-m+1} \\ &\le \frac{c\ln m}{\ln n} \sum_{k=m+1}^{m^3} \frac{1}{k-m+1} \le \frac{c\ln^2 m}{\ln n}. \qquad (1) \end{aligned}$$

If $n > m^3$ then we write

$$F^{\alpha,2}_{m,n} = \frac{1}{l(m,n)} \left(\sum_{k=m+1}^{m^2-1} \frac{D^\alpha_k}{k-m+1} + \sum_{k=m^2}^{n} \frac{D^\alpha_k}{k-m+1} \right) =: F^{\alpha,2,1}_{m,n} + F^{\alpha,2,2}_{m,n}.$$

(1) estimates $\|F^{\alpha,2,1}_{m,n}\|_1$ too. For $F^{\alpha,2,2}_{m,n}$ we use Abel's transformation.

$$\begin{aligned} \sum_{k=m^2}^{n} \frac{D^\alpha_k}{k-m+1} &= \sum_{k=m^2}^{n} \frac{1}{(k-m+1)(k-m+2)}[kK^\alpha_k - (m^2-1)K^\alpha_{m^2-1}] \\ &\quad + \frac{1}{n-m+2}[nK^\alpha_n - (m^2-1)K^\alpha_{m^2-1}]. \end{aligned}$$

From this we immediatelly get

$$\begin{aligned} \|F^{\alpha,2,2}_{m,n}\|_1 &\le \frac{c}{\ln n} \sum_{k=m^2}^{n} \frac{k\|K^\alpha_k\|_1}{(k-m+1)(k-m+2)} + \frac{c}{\ln n} \sum_{k=m^2}^{n} \frac{(m^2-1)\|K^\alpha_{m^2-1}\|_1}{(k-m+1)(k-m+2)} \\ &\quad + \frac{c}{\ln n} \frac{n\|K^\alpha_n\|_1}{n-m+2} + \frac{c}{\ln n} \frac{(m^2-1)\|K^\alpha_{m^2-1}\|_1}{(n-m+2)}. \end{aligned}$$

For the Walsh system [12] and the Walsh-Kaczmarz system [14] it was proved that

$$\sup_n \|K^\alpha_n\|_1 < \infty. \qquad (2)$$

This and the above written imply

$$\|F_{m,n}^{\alpha,2,2}\|_1 \le c. \tag{3}$$

That is, we showed the upper estimation for $\alpha = \omega$ or κ.

For $\alpha = \omega$ the lower estimation follows from Lemma 1, just we have to choose $m = n = p_A$.

For $\alpha = \kappa$ the lower estimation follows from Lemma 1, Lemma 2 and we have to choose $m = n = p_A$.

$$\begin{aligned}
F_m^{\kappa}(x) &= \frac{l_{m-2^{|m|-1}+1}}{l_m} D_{2^{|m|}}(x) \\
&- \omega_{2^{|m|-1}}(x) \frac{1}{l_m} \sum_{j=1}^{2^{|m|-1}-1} \left(\frac{1}{m-2^{|m|}+j} - \frac{1}{m-2^{|m|}+j+1} \right) j K_j^{\omega}(\tau_{|m|-1}(x)) \\
&- \omega_{2^{|m|-1}}(x) \frac{1}{l_m} \frac{2^{|m|-1}}{m-2^{|m|-1}} K_{2^{|m|-1}}^{\omega}(\tau_{|m|-1}(x)) + r_{|m|}(x) \frac{l_{m-2^{|m|}}}{l_m} F_{m-2^{|m|}}^{\omega}(\tau_{|m|}(x)) \\
&+ \frac{1}{l_m} \sum_{s=0}^{2^{|m|-1}-2} \left(\frac{1}{m-s} - \frac{1}{m-s+1} \right) s K_s^{\kappa}(x) \\
&+ \frac{1}{l_m} \frac{2^{|m|-1}-1}{m-2^{|m|-1}+1} K_{2^{|m|-1}-1}^{\kappa}(x) \\
&=: \sum_{i=1}^{6} F_m^{i,\kappa}(x).
\end{aligned}$$

By (2) we have

$$\|F_m^{1,\kappa}\|_1 \le c,$$

$$\|F_m^{2,\kappa}\|_1 \le \frac{1}{l_m} \sum_{j=1}^{2^{|m|-1}-1} \frac{\|K_j^{\omega} \circ \tau_{|m|-1}\|_1}{j} \le \frac{1}{l_m} \sum_{j=1}^{2^{|m|-1}-1} \frac{\|K_j^{\omega}\|_1}{j} \le c,$$

$$\|F_m^{3,\kappa}\|_1 \le \frac{1}{l_m} \|K_{2^{|m|-1}}^{\omega} \circ \tau_{|m|-1}\|_1 \le \frac{1}{l_m} \|K_{2^{|m|-1}}^{\omega}\|_1 \le c,$$

$$\|F_m^{5,\kappa}\|_1 \le \frac{1}{l_m} \sum_{s=0}^{2^{|m|-1}-2} \frac{\|K_s^{\kappa}\|_1}{m-s} \le c$$

and

$$\|F_m^{6,\kappa}\|_1 \le \frac{1}{l_m} \|K_{2^{|m|-1}-1}^{\kappa}\|_1 \le c.$$

By Lemma (1) for A big enough we get

$$\|F_{m,n}^{\kappa}\|_1 = \|F_{p_A}^{\kappa}\|_1 \ge \|F_{p_A}^{4,\kappa}\|_1 - c \ge c \ln p_{A-1} - c \ge c \ln m + c = c \frac{\ln^2 m}{\ln n} + c.$$

This completes the proof. □

3. Tkebuchava's means and kernels of quadratical partial sums of Walsh-(Kaczmarz)-Fourier series

The two-dimensional Fourier coefficients are defined by

$$\hat{f}^{\alpha}(i,j) := \int_0^1 \int_0^1 f(t,s)\alpha_i(t)\alpha_j(s)dtds,$$

where $\alpha = \omega$ or κ.

Define the Tkebuchava's means $T_{m,n}$ and kernels $\mathcal{F}_{m,n}$ of quadratical partial sums of two-dimensional Fourier series by

$$T_{m,n}(f) := \frac{1}{l(m,n)} \left(\sum_{k=0}^{m-1} \frac{S^{\alpha}_{k,k}(f)}{m-k+1} + S^{\alpha}_{m,m}(f) + \sum_{k=m+1}^{n} \frac{S^{\alpha}_{k,k}(f)}{k-m+1} \right)$$

and

$$\mathcal{F}_{m,n} := \frac{1}{l(m,n)} \left(\sum_{k=0}^{m-1} \frac{D^{\alpha}_{k,k}}{m-k+1} + D^{\alpha}_{m,m} + \sum_{k=m+1}^{n} \frac{D^{\alpha}_{k,k}}{k-m+1} \right),$$

where

$$S^{\alpha}_{k,k}(f) := \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \hat{f}^{\alpha}(i,j)\alpha_i\alpha_j \quad \text{and} \quad D^{\alpha}_{k,k} := \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \alpha_i\alpha_j = D^{\alpha,1}_k D^{\alpha,2}_k.$$

The n-th Nörlund logarithmic kernel $\mathcal{F}^{\alpha}_n$ and the nth Marcinkiewicz kernel $\mathcal{K}^{\alpha}_n$ of quadratical partial sums of two-dimensional Fourier series is defined by

$$\mathcal{F}^{\alpha}_n := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{D^{\alpha}_{k,k}}{n-k}$$

and

$$\mathcal{K}^{\alpha}_n := \frac{1}{n} \sum_{k=0}^{n} D^{\alpha}_{k,k}.$$

To prove our main theorem we need the following lemma in [5].

Lemma 3. *Let* $p_A := 2^{2A} + ... + 2^2 + 2^0$, *then* $\|\mathcal{F}_{p_A}\|_1 \geq c\ln^2 p_A$.

Theorem 2. *Let* $\alpha := \omega$ *or* κ. *There exist positive constants* C_1, C_2 $(0 < C_1 \leq C_2)$ *such that for any integers* m *and* n $(0 \leq m \leq n)$

$$C_1 \leq \sup_{\substack{m,n\in\mathbb{N} \\ 0\leq m\leq n}} \frac{\|\mathcal{F}^{\alpha}_{m,n}\|_1}{\left(1 + \frac{\ln^3 m}{\ln n}\right)} \leq C_2$$

holds.

By standard argument we immediatelly get the following corollary.

Corollary 2. *Let* $m_n = O\left(\exp \sqrt[3]{\ln n}\right)$ *as* $n \to \infty$, *then*

$$\|T_{m_n,n}(f) - f\|_1 \to 0 \quad \text{as } n \to \infty \quad \text{for all } f \in L^1[0,1)^2$$

and

$$\|T_{m_n,n}(f) - f\|_C \to 0 \quad \text{as } n \to \infty \quad \text{for all } f \in C[0,1)^2.$$

Proof of Theorem 2: First, we prove the upper estimation. To do this set

$$\mathcal{F}^{\alpha,1}_{m,n} := \frac{1}{l(m,n)}\sum_{k=0}^{m}\frac{D^{\alpha}_{k,k}}{m-k+1} \quad \text{and} \quad \mathcal{F}^{\alpha,2}_{m,n} := \frac{1}{l(m,n)}\sum_{k=m+1}^{n}\frac{D^{\alpha}_{k,k}}{k-m+1}.$$

Discuss $\mathcal{F}^{\alpha,1}_{m,n}$.

$$\|\mathcal{F}^{\alpha,1}_{m,n}\|_1 \le \frac{1}{l(m,n)}\sum_{j=1}^{m}\frac{\|D^{\alpha}_{j,j}\|_1}{m-j+1} \le \frac{c}{l(m,n)}\sum_{j=1}^{m}\frac{\ln^2 j}{m-j+1} = O\left(\frac{\ln^3 m}{\ln n}\right).$$

Following the proof of Theorem 1, we get for $n \le m^3$

$$\|\mathcal{F}^{\alpha,2}_{m,n}\|_1 \le c\frac{\ln^3 m}{\ln n}$$

and for $n > m^3$ we get

$$\|\mathcal{F}^{\alpha,2,1}_{m,n}\|_1 \le c\frac{\ln^3 m}{\ln n}$$

again. For the Walsh-Marcinkiewicz kernels [10] and Walsh-Kaczmarz-Marcinkiewicz kernels [7] hold

$$\sup_n \|\mathcal{K}^{\alpha}_n\|_1 < \infty \tag{4}$$

which implies

$$\|\mathcal{F}^{\alpha,2,2}_{m,n}\|_1 \le c.$$

This completes the proof of the upper estimation.

What about the lower estimation?

For $\alpha = \omega$ the lower estimation follows from Lemma 3, just we have to choose $m = n = p_A$.

For $\alpha = \kappa$ we choose $m = n$ and decompose the kernel $\mathcal{F}^{\kappa}_{m,m} = \mathcal{F}^{\kappa}_{m}$ in the following way.

Let $|m| = A$.

$$l_m\mathcal{F}^{\kappa}_{m}(x^1,x^2) = \sum_{j=1}^{2^A}\frac{D^{\kappa}_j(x^1)D^{\kappa}_j(x^2)}{m-j} + \sum_{j=2^A+1}^{m-1}\frac{D^{\kappa}_j(x^1)D^{\kappa}_j(x^2)}{m-j} =: I + II.$$

We will use the notation $D^{\kappa,i}_j(x^1,x^2) := D^{\kappa}_j(x^i)$, $r^i_A(x^1,x^2) := r_A(x^i)$ and $F^{\kappa,i}_n(x^1,x^2) := F^{\kappa}_n(x^i)$ for $i = 1,2$.

To discuss II, we use the following equation

$$D^{\kappa}_{2^A+j}(x) = D_{2^A}(x) + r_A(x)D^{\omega}_j(\tau_A(x)), \quad j = 0,1,...,2^A-1. \tag{5}$$

This immediately gives

$$\begin{aligned}
II &= \sum_{j=1}^{m-2^A-1} \frac{D^{\kappa,1}_{2^A+j} D^{\kappa,2}_{2^A+j}}{m-2^A-j} \\
&= \sum_{j=1}^{m-2^A-1} \frac{D^1_{2^A} D^2_{2^A}}{m-2^A-j} + r^1_A D^1_{2^A} \sum_{j=1}^{m-2^A-1} \frac{D^{\omega,2}_j \circ \tau_A}{m-2^A-j} \\
&\quad + r^2_A D^2_{2^A} \sum_{j=1}^{m-2^A-1} \frac{D^{\omega,1}_j \circ \tau_A}{m-2^A-j} + r^1_A r^2_A \sum_{j=1}^{m-2^A-1} \frac{D^{\omega}_{j,j} \circ (\tau_A \times \tau_A)}{m-2^A-j} \\
&= l_{m-2^A} D^1_{2^A} D^2_{2^A} + r^1_A D^1_{2^A} l_{m-2^A} F^{\omega,2}_{m-2^A} \circ \tau_A + r^2_A D^2_{2^A} l_{m-2^A} F^{\omega,1}_{m-2^A} \circ \tau_A \\
&\quad + r^1_A r^2_A l_{m-2^A} \mathcal{F}^{\omega}_{m-2^A} \circ (\tau_A \times \tau_A) =: II_1 + II_2 + II_3 + II_4.
\end{aligned}$$

We have

$$\|\frac{1}{l_m} II_1\|_1 \le c, \quad \|\frac{1}{l_m} II_2\|_1 \le c \ln m \quad \text{and} \quad \|\frac{1}{l_m} II_3\|_1 \le c \ln m.$$

Now, we discuss I. Abel's transformation gives that

$$I = \sum_{j=1}^{2^A-1} \left(\frac{1}{m-j} - \frac{1}{m-j+1} \right) j \mathcal{K}^{\kappa}_j + \frac{2^A}{m-2^A} \mathcal{K}^{\kappa}_{2^A} =: I_1 + I_2.$$

We choose $m = n = p_A$ (see Lemma 3), this and (4) imply that

$$\|\frac{1}{l_m} I_1\|_1 \le \frac{1}{l_m} \sum_{j=1}^{2^{2A}-1} \frac{j \|\mathcal{K}^{\kappa}_j\|_1}{(m-j)(m-j+1)} \le \frac{c}{l_m} \sum_{j=1}^{2^{2A}-1} \frac{1}{m-j} \le c,$$

$$\|\frac{1}{l_m} I_2\|_1 = \frac{1}{l_m} \frac{2^{2A}}{m-2^{2A}} \|\mathcal{K}^{\kappa}_{2^{2A}}\|_1 \le c$$

and

$$\|\frac{1}{l_m} II_4\|_1 \ge c \|\mathcal{F}^{\omega}_{m-2^{2A}} \circ (\tau_{2A} \times \tau_{2A})\|_1 \ge c \|\mathcal{F}^{\omega}_{m-2^{2A}}\|_1 \ge c \ln^2 p_{A-1} \ge c \ln^2 m.$$

That is,

$$\|\mathcal{F}^{\kappa}_{p_A,p_A}\|_1 = \|\mathcal{F}^{\kappa}_{p_A}\|_1 \ge \|\frac{1}{l_m} II_4\|_1 - c \ge c \ln^2 m + c.$$

□

4. Tkebuchava's means and kernels of quadratical partial sums of trigonometric Fourier series

Define the Tkebuchava's means $T_{m,n}$ the kernels $\mathcal{F}_{m,n}$ and the Nörlund logarithmic kernel $\mathcal{F}_n$ of quadratical partial sums by the same way as in the previous section. But, we use the two-dimensional trigonometric system on the set $[-\pi, \pi] \times [-\pi, \pi]$.

Lemma 4. *There exist a positive constant C such that for any integers m and n $(0 \le m \le n)$*

$$\|\mathcal{F}_{m,n}\|_1 \le C \left(1 + \frac{\ln^3 m}{\ln n} \right)$$

holds.

Proof. For the Marcinkiewicz kernels [19] the inequality

$$\sup_n \|\mathcal{K}_n\|_1 < \infty$$

holds. This and the method of Theorem 2 imply the upper estimation. □

This gives the following corollary.

Corollary 3. *Let* $m_n = O\left(\exp \sqrt[3]{\ln n}\right)$ *as* $n \to \infty$, *then*

$$\|T_{m_n,n}(f) - f\|_1 \to 0 \quad \text{as } n \to \infty \quad \text{for all } f \in L^1[-\pi,\pi]^2$$

and

$$\|T_{m_n,n}(f) - f\|_C \to 0 \quad \text{as } n \to \infty \quad \text{for all } f \in C[-\pi,\pi]^2.$$

We have the following conjecture.

Conjecture 1. *There exist positive constants* C_1, C_2 $(0 < C_1 \leq C_2)$ *such that for any integers* m *and* n $(0 \leq m \leq n)$

$$C_1\left(1 + \frac{\ln^3 m}{\ln n}\right) \leq \|\mathcal{F}_{m,n}\|_1 \leq C_2\left(1 + \frac{\ln^3 m}{\ln n}\right)$$

holds.

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