

On the maximal operator of Walsh–Marcinkiewicz means

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Abstract. In this paper we prove that the maximal operator $\tilde{\mathcal{M}}^* f := \sup_{n \in \mathbf{P}} \frac{|\mathcal{M}_n f|}{\log^{3/2}(n+1)}$, where $\mathcal{M}_n f$ is the n th Marcinkiewicz–Fejér mean of the 2-dimensional Walsh–Fourier series, is bounded from the Hardy space $H_{2/3}(G^2)$ to the space $L_{2/3}(G^2)$.

1. Introduction

The a.e. convergence of Walsh–Fejér means $\sigma_n f$ was proved by FINE [2]. In 1975 SCHIPP [12] showed that the maximal operator σ^* is of weak type $(1, 1)$ and of type (p, p) for $1 < p \leq \infty$. The boundedness fails to hold for $p = 1$. But, FUJII [3] proved that σ^* is bounded from the dyadic Hardy space H_1 to the space L_1 . The theorem of FUJII was extended by WEISZ [17], he showed that the maximal operator σ^* is bounded from the martingale Hardy space H_p to the space L_p for $p > 1/2$. Simon gave a counterexample [13], which shows that the boundedness does not hold for $0 < p < 1/2$. The counterexample for $p = 1/2$ due to GOGINAVA [6]. In the endpoint case $p = 1/2$ two positive result was showed. WEISZ [18] proved that σ^* is bounded from the Hardy space $H_{1/2}$ to the space weak- $L_{1/2}$. In 2008 GOGINAVA [8] proved that the maximal operator $\tilde{\sigma}^*$ defined by

$$\tilde{\sigma}^* := \sup_{n \in \mathbf{P}} \frac{|\sigma_n f|}{\log^2(n+1)}$$

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is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$. He also proved that for any nondecreasing function $\varphi : \mathbf{P} \rightarrow [1, \infty)$ satisfying the condition

$$\overline{\lim}_{n \rightarrow \infty} \frac{\log^2(n+1)}{\varphi(n)} = +\infty$$

the maximal operator $\sup_{n \in \mathbf{P}} \frac{|\sigma_n f|}{\varphi(n)}$ is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

In 1939 for the two-dimensional trigonometric Fourier partial sums $S_{j,j}(f)$ MARCINKIEWICZ [10] has proved for $f \in L \log L([0, 2\pi]^2)$ that the means

$$\mathcal{M}_n f = \frac{1}{n} \sum_{j=1}^n S_{j,j}(f)$$

converge a.e. to f as $n \rightarrow \infty$. ZHIZHIASHVILI [19] improved this result for $f \in L([0, 2\pi]^2)$.

For the two-dimensional Walsh–Fourier series WEISZ [14] proved that the maximal operator

$$\mathcal{M}^* f = \sup_{n \geq 1} \frac{1}{n} \left| \sum_{j=0}^{n-1} S_{j,j}(f) \right|$$

is bounded from the two-dimensional dyadic martingale Hardy space H_p to the space L_p for $p > 2/3$ and is of weak type $(1, 1)$. GOGINAVA [6] proved that the assumption $p > 2/3$ is essential for the boundedness of the maximal operator $\mathcal{M}^*$ from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$. Namely, in the endpoint case $p = 2/3$ he gave a counterexample for which the boundedness does not hold. In the endpoint case $p = 2/3$, GOGINAVA [7] proved that the maximal operator $\mathcal{M}^*$ of the Walsh–Marcinkiewicz means of double Fourier series is bounded from the Hardy space $H_{2/3}$ to the space weak- $L_{2/3}$.

In the present paper we prove that the maximal operator $\tilde{\mathcal{M}}^*$ defined by

$$\tilde{\mathcal{M}}^* := \sup_{n \in \mathbf{P}} \frac{|\mathcal{M}_n f|}{\log^{3/2}(n+1)}$$

is bounded from the Hardy space $H_{2/3}$ to the space $L_{2/3}$. We also prove that for any nondecreasing function $\varphi : \mathbf{P} \rightarrow [1, \infty)$ satisfying the condition

$$\overline{\lim}_{n \rightarrow \infty} \frac{\log^{3/2}(n+1)}{\varphi(n)} = +\infty$$

the maximal operator $\sup_{n \in \mathbf{P}} \frac{|\mathcal{M}_n f|}{\varphi(n)}$ is not bounded from the Hardy space $H_{2/3}$ to the space $L_{2/3}$. That is, we prove the analogue of the theorems of GOGINAVA mentioned above [8].

For Walsh–Kaczmarz–Marcinkiewicz means the author [9] proved, that it is of weak type (1,1) and of type (p, p) for $1 < p \leq \infty$. This theorem was extended in [4].

2. Definitions and notation

Now, we give a brief introduction to the theory of dyadic analysis [11], [1]. Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Denote $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is $\mathbb{Z}_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $\mathbb{Z}_2$ is given such that the measure of a singleton is $1/2$. Let G be the complete direct product of the countable infinite copies of the compact groups $\mathbb{Z}_2$. The elements of G are of the form $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0, 1\} (k \in \mathbf{N})$. The group operation on G is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$I_0(x) := G,$$

$$I_n(x) := I_n(x_0, \dots, x_{n-1}) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\},$$

($x \in G, n \in \mathbf{N}$). These sets are called dyadic intervals. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G , and $I_n := I_n(0) (n \in \mathbf{N})$. Set $e_n := (0, \dots, 0, 1, 0, \dots) \in G$, the n th coordinate of which is 1 and the rest are zeros ($n \in \mathbf{N}$).

For $k \in \mathbf{N}$ and $x \in G$ denote

$$r_k(x) := (-1)^{x_k}$$

the k th Rademacher function. If $n \in \mathbf{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$ can be written, where $n_i \in \{0, 1\} (i \in \mathbf{N})$, i.e. n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh–Paley system is defined as the sequence of Walsh–Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in G, n \in \mathbf{P}).$$

The σ -algebra generated by the dyadic 2-dimensional cube I_k^2 of measure $2^{-k} \times 2^{-k}$ will be denoted by $\mathcal{F}_k$ ($k \in \mathbf{N}$).

The space $L_p(G^2)$, $0 < p \leq \infty$ with norms or quasi-norms $\|\cdot\|_p$ is defined in the usual way (For details see e.g. WEISZ [15]).

Denote by $f = (f_n, n \in \mathbf{N})$ the one-parameter martingale with respect to $(\mathcal{F}_n, n \in \mathbf{N})$. The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbf{N}} |f_n|.$$

For $0 < p \leq \infty$ the Hardy martingale space $H_p(G^2)$ consists of all martingales for which

$$\|f\|_{H_p} = \|f^*\|_p < \infty.$$

The Dirichlet kernels are defined by

$$D_n(x) := \sum_{k=0}^{n-1} w_k(x).$$

Recall that (see e.g. [11])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n(0), \\ 0, & \text{if } x \notin I_n(0). \end{cases} \quad (1)$$

The Fejér kernels are defined as follows

$$K_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(x).$$

The Kroneker product $(w_{n,m} : n, m \in \mathbf{N})$ of two Walsh system is said to be the two-dimensional Walsh system. Thus,

$$w_{n,m}(x^1, x^2) = w_n(x^1) w_m(x^2).$$

If $f \in L(G^2)$, then the number $\widehat{f}(n, m) := \int_{G^2} f w_{n,m}$ ($n, m \in \mathbf{N}$) is said to be the (n, m) th Walsh-Fourier coefficient of f . We can extend this definition to martingales in the usual way (see WEISZ [15], [16]). Denote by $S_{n,m}$ the (n, m) th rectangular partial sum of the Walsh-Fourier series of a martingale f . Namely,

$$S_{n,m}(f; x^1, x^2) := \sum_{k=0}^{n-1} \sum_{i=0}^{m-1} \widehat{f}(k, i) w_{k,i}(x^1, x^2).$$

The Marcinkiewicz–Fejér means of a martingale f are defined by

$$\mathcal{M}_n(f; x^1, x^2) := \frac{1}{n} \sum_{k=0}^{n-1} S_{k,k}(f; x^1, x^2).$$

The 2-dimensional Dirichlet kernels and Marcinkiewicz–Fejér kernels are defined by

$$D_{k,l}(x^1, x^2) := D_k(x^1)D_l(x^2), \quad K_n(x^1, x^2) := \frac{1}{n} \sum_{k=0}^{n-1} D_{k,k}(x^1, x^2).$$

For the martingale f we consider the maximal operator

$$\mathcal{M}^* f(x^1, x^2) = \sup_{n \in \mathbf{P}} |\mathcal{M}_n(f; x^1, x^2)|.$$

3. Auxiliary propositions and main results

First, we formulate our main theorems. Our theorems are the two-dimensional analogue of the theorems of GOGINAVA [8] for Walsh–Fejér means.

Theorem 1. *The maximal operator $\tilde{\mathcal{M}}^*$ is bounded from the Hardy space $H_{2/3}(G^2)$ to the space $L_{2/3}(G^2)$.*

Theorem 2. *Let $\varphi : \mathbf{P} \rightarrow [1, \infty)$ be a nondecreasing function satisfying the condition*

$$\lim_{n \rightarrow \infty} \frac{\log^{3/2}(n+1)}{\varphi(n)} = +\infty. \quad (2)$$

Then the maximal operator

$$\sup_{n \in \mathbf{P}} \frac{|\mathcal{M}_n f|}{\varphi(n)}$$

is not bounded from the Hardy space $H_{2/3}(G^2)$ to the space $L_{2/3}(G^2)$.

To prove our Theorem 1 we need the following Lemmas of GOGINAVA [7, Lemma 7, Lemma 9], GLUKHOV [5] and WEISZ [16]:

Lemma 1 (Goginava [7]). *Let $(x^1, x^2) \in (I_{l^1} \setminus I_{l^1+1}) \times (I_{m^2} \setminus I_{m^2+1})$ and $0 \leq l^1 < N, 0 \leq m^2 < N$. Then*

$$\begin{aligned} & \int_{I_N \times I_N} |K_n(x^1 + t^1, x^2 + t^2)| d\mu(t^1, t^2) \\ & \leq \frac{c}{2^{3N}} \left\{ 2^{l^1 - m^2} \sum_{r^1 = l^1 + 1}^{m^2 + 1} 2^{r^1} D_{2^{m^2+1}}(x^1 + e_{l^1} + e_{r^1}) \sum_{s = m^2 + 1}^N D_{2^s}(x^2 + e_{m^2} + x_{m^2+1, s-1}^1) \right\} \end{aligned}$$

$$+ 2^{l^1+m^2} \sum_{s=l^1}^{m^2} \sum_{r^1=l^1+1}^s D_{2^s}(x^1 + e_{l^1} + e_{r^1}) \Big\}, \text{ for } n \geq 2^N,$$

with the notation $x_{i,j} := \sum_{s=i}^j x_s e_s$ ($x_{i,i-1} = 0$).

Lemma 2 (GOGINAVA [7]). *Let $(x^1, x^2) \in I_N \times (I_{m^2} \setminus I_{m^2+1})$ and $0 \leq m^2 < N$. Then*

$$\int_{I_N \times I_N} |K_n(x^1 + t^1, x^2 + t^2)| d\mu(t^1, t^2) \leq c \frac{2^{m^2}}{2^{2N}} \sum_{s=m^2}^{N-1} D_{2^s}(x^2 + e_{m^2}), \text{ for } n > 2^N.$$

Lemma 3 (GLUKHOV [5]). *There exists a constant c such that*

$$\sup_n \int_{G^2} |K_n(x^1, x^2)| d\mu(x^1, x^2) \leq c.$$

A bounded measurable function a is a p -atom, if there exists a dyadic two-dimensional cube I^2 , such that

- a) $\int_{I^2} a d\mu = 0$,
- b) $\|a\|_\infty \leq \mu(I^2)^{-1/p}$,
- c) $\text{supp } a \subset I^2$.

Lemma 4 (WEISZ [16]). *Suppose that the operator T is sublinear and p -quasilocal for any $0 < p \leq 1$. If T is bounded from L_∞ to L_∞ , then*

$$\|Tf\|_p \leq c_p \|f\|_{H_p} \quad \text{for all } f \in H_p.$$

4. Proofs of the theorems

First, we prove Theorem 1.

PROOF OF THEOREM 1. Lemma 3 yields the boundedness from the space L_∞ to the space L_∞ . By Lemma 4, the proof will be complete, if we show that the maximal operator $\tilde{\mathcal{M}}^*$ is $2/3$ -quasilocal. That is, there exists a constant c such that

$$\int_{I^2} |\tilde{\mathcal{M}}^* a|^{2/3} d\mu \leq c < \infty$$

for every $2/3$ -atom a . where the dyadic cube I^2 is the support of the $2/3$ -atom a .

Let a be an arbitrary $2/3$ -atom with support I^2 , and $\mu(I^2) = 2^{-2N}$. Without loss of generality, we may assume that $I^2 := I_N \times I_N$. It is evident that $\tilde{\mathcal{M}}_n(a) = 0$ if $n \leq 2^N$ (with the notation $\tilde{\mathcal{M}}_n(f) := \frac{|\mathcal{M}_n f|}{\log^{3/2}(n+1)}$). Therefore, we set $n > 2^N$.

By $\|a\|_\infty \leq 2^{3N}$ we have that

$$\begin{aligned} \frac{|\mathcal{M}_n(a; x^1, x^2)|}{\log^{3/2}(n+1)} &\leq \frac{1}{\log^{3/2}(n+1)} \int_{I_N \times I_N} |a(t^1, t^2)| |K_n(x^1 + t^1, x^2 + t^2)| d\mu(t^1, t^2) \\ &\leq \frac{c2^{3N}}{\log^{3/2}(n+1)} \int_{I_N \times I_N} |K_n(x^1 + t^1, x^2 + t^2)| d\mu(t^1, t^2) \end{aligned}$$

and

$$|\tilde{\mathcal{M}}^* a| \leq \frac{c2^{3N}}{N^{3/2}} \sup_{n > 2^N} \int_{I_N \times I_N} |K_n(x^1 + t^1, x^2 + t^2)| d\mu(t^1, t^2). \quad (3)$$

We write that

$$\begin{aligned} \int_{I_N \times I_N} |\tilde{\mathcal{M}}^* a|^{2/3} d\mu &= \int_{I_N \times \overline{I_N}} |\tilde{\mathcal{M}}^* a|^{2/3} d\mu + \int_{\overline{I_N} \times I_N} |\tilde{\mathcal{M}}^* a|^{2/3} d\mu \\ &\quad + \int_{\overline{I_N} \times \overline{I_N}} |\tilde{\mathcal{M}}^* a|^{2/3} d\mu =: L_1 + L_2 + L_3. \end{aligned}$$

First, we discuss L_1 by the help of Lemma 2 and inequality (3) (the discussion of L_2 goes analogously). We introduce the notation $J_t := I_t \setminus I_{t+1}$ ($t \in \mathbf{N}$).

$$\begin{aligned} L_1 &= \sum_{m^2=0}^{N-1} \int_{I_N \times J_{m^2}} |\tilde{\mathcal{M}}^* a(x^1, x^2)|^{2/3} d\mu(x^1, x^2) \\ &\leq \frac{c}{N} \sum_{m^2=0}^{N-1} \int_{I_N \times J_{m^2}} \left| 2^{3N} \sup_{n > 2^N} \int_{I_N \times I_N} |K_n(x^1 + t^1, x^2 + t^2)| d\mu(t^1, t^2) \right|^{2/3} d\mu(x^1, x^2) \\ &\leq \frac{c}{N} \sum_{m^2=0}^{N-1} \int_{I_N \times J_{m^2}} \left| 2^{m^2+N} \sum_{s=m^2}^{N-1} D_{2^s}(x^2 + e_{m^2}) \right|^{2/3} d\mu(x^1, x^2). \end{aligned}$$

We decompose J_{m^2} as the following disjoint union:

$$J_{m^2} = \bigcup_{q^2=m^2+1}^N I_N^{m^2, q^2},$$

where

$$I_N^{m^2, q^2} := \begin{cases} I_{q^2+1}(0, \dots, 0, x_{m^2} = 1, 0, \dots, 0, x_{q^2} = 1), & \text{for } m^2 < q^2 < N, \\ I_N(0, \dots, 0, x_{m^2} = 1, 0, \dots, 0), & \text{for } q^2 = N. \end{cases}$$

From (1), we get

$$\begin{aligned} L_1 &\leq \frac{c2^{-N/3}}{N} \sum_{m^2=0}^{N-1} \sum_{q^2=m^2+1}^N \int_{I_N^{m^2, q^2}} \left| 2^{m^2} \sum_{s=m^2}^{q^2} 2^s \right|^{2/3} d\mu(x^2) \\ &\leq \frac{c2^{-N/3}}{N} \sum_{m^2=0}^{N-1} \sum_{q^2=m^2+1}^N 2^{2m^2/3+2q^2/3} 2^{-q^2} \leq c. \end{aligned}$$

Now, we discuss L_3 .

$$\begin{aligned} L_3 &= \sum_{l^1=0}^{N-1} \sum_{m^2=0}^{N-1} \int_{J_{l^1} \times J_{m^2}} |\tilde{\mathcal{M}}^* a|^{2/3} d\mu \\ &= \sum_{l^1=0}^{N-1} \sum_{m^2=0}^{l^1-1} \int_{J_{l^1} \times J_{m^2}} |\tilde{\mathcal{M}}^* a|^{2/3} d\mu \\ &\quad + \sum_{l^1=0}^{N-1} \sum_{m^2=l^1}^{N-1} \int_{J_{l^1} \times J_{m^2}} |\tilde{\mathcal{M}}^* a|^{2/3} d\mu = L_{3,1} + L_{3,2}. \end{aligned}$$

We discuss $L_{3,2}$ (the discussion of $L_{3,1}$ goes analogously). By the inequality (3) we have that

$$\begin{aligned} L_{3,2} &\leq \frac{c}{N} \sum_{l^1=0}^{N-1} \sum_{m^2=l^1}^{N-1} \int_{J_{l^1} \times J_{m^2}} \\ &\quad \times \left| 2^{3N} \sup_{n>2^N} \int_{I_N \times I_N} |K_n(x^1+t^1, x^2+t^2)| d\mu(t^1, t^2) \right|^{2/3} d\mu(x^1, x^2) \\ &=: \frac{c}{N} \sum_{l^1=0}^{N-1} \sum_{m^2=l^1}^{N-1} L_{3,2}^{l^1, m^2}. \end{aligned}$$

To discuss $L_{3,2}^{l^1, m^2}$, we write the set J_{l^1} in the form of following disjoint union:

$$J_{l^1} = \bigcup_{k=l^1+1}^{m^2+1} I_{m^2+1}^{l^1, k}.$$

That is,

$$L_{3,2}^{l^1, m^2} = \sum_{k=l^1+1}^{m^2+1} \int_{I_{m^2+1}^{l^1, k} \times J_{m^2}} |\cdot|^{2/3} d\mu(x^1, x^2)$$

By Lemma 1, $\sum_{r^1=l^1+1}^{m^2+1} D_{2^{m^2+1}}(\cdot + e_{l^1} + e_{r^1}) = 0$ and $\sum_{r^1=l^1+1}^{m^2+1} D_{2^{m^2+1}}(\cdot + e_{l^1} + e_{r^1}) \neq 0$ determine two cases. Thus, we write $I_{m^2+1}^{l^1,k} = (I_{m^2+1}^{l^1,k} \cap \overline{\cup_{r^1=l^1+1}^{m^2+1} I_{m^2+1}(e_{l^1} + e_{r^1})}) \cup (I_{m^2+1}^{l^1,k} \cap (\cup_{r^1=l^1+1}^{m^2+1} I_{m^2+1}(e_{l^1} + e_{r^1})))$. Thus, this divide the expression $L_{3,2}^{l^1,m^2}$ into two parts $L_{3,2,1}^{l^1,m^2}$ and $L_{3,2,2}^{l^1,m^2}$. In $L_{3,2,1}^{l^1,m^2}$ we integrate on the set $I_{m^2+1}^{l^1,k} \cap \overline{\cup_{r^1=l^1+1}^{m^2+1} I_{m^2+1}(e_{l^1} + e_{r^1})}$, while in $L_{3,2,2}^{l^1,m^2}$ we integrate on the set $I_{m^2+1}^{l^1,k} \cap (\cup_{r^1=l^1+1}^{m^2+1} I_{m^2+1}(e_{l^1} + e_{r^1})) = I_{m^2+1}(e_{l^1} + e_k)$. Using Lemma 1, we immediately have

$$L_{3,2,1}^{l^1,m^2} \leq c 2^{-m^2} \sum_{k=l^1+1}^{m^2+1} \int_{I_{m^2+1}^{l^1,k}} \left| 2^{l^1+m^2} \sum_{s=l^1}^{m^2} D_{2^s}(x^1 + e_{l^1} + e_k) \right|^{2/3} d\mu(x^1).$$

Now, we decompose the set $I_{m^2+1}^{l^1,k}$ in the following form of disjoint union:

$$I_{m^2+1}^{l^1,k} = \bigcup_{r=k+1}^{m^2+1} I_{m^2+1}^{l^1,k,r},$$

where $I_{m^2+1}^{l^1,k,r} := I_{r+1}(0, \dots, 0, x_{l^1}^1 = 1, 0, \dots, 0, x_k^1 = 1, 0, \dots, x_r^1 = 1)$ for $k < r \leq m^2$ and $I_{m^2+1}^{l^1,k,r} := I_{m^2+1}(e_{l^1} + e_k)$ for $r = m^2 + 1$. This yields

$$\begin{aligned} L_{3,2,1}^{l^1,m^2} &\leq c 2^{-m^2} \sum_{k=l^1+1}^{m^2+1} \sum_{r=k+1}^{m^2+1} \int_{I_{m^2+1}^{l^1,k,r}} \left| 2^{l^1+m^2} \sum_{s=l^1}^r 2^s \right|^{2/3} d\mu(x^1) \\ &\leq c 2^{-m^2} \sum_{k=l^1+1}^{m^2+1} \sum_{r=k+1}^{m^2+1} 2^{2(l^1+m^2+r)/3} 2^{-r} \leq c 2^{l^1/3-m^2/3}. \end{aligned} \quad (4)$$

Now, we turn our attention to $L_{3,2,2}^{l^1,m^2}$.

$$\begin{aligned} L_{3,2,2}^{l^1,m^2} &\leq c \sum_{k=l^1+1}^{m^2+1} \int_{I_{m^2+1}(e_{l^1}+e_k) \times J_{m^2}} (\quad)^{2/3} d\mu(x^1, x^2) \\ &\leq c \sum_{k=l^1+1}^{m^2+1} \sum_{\substack{x_i^1=0 \\ i \in \{m^2+1, \dots, N-1\} \\ x_j^1=0 \text{ otherwise}}}^1 \int_{I_N(e_{l^1}+e_k+x^1) \times J_{m^2}} (\quad)^{2/3} d\mu(x^1, x^2) \end{aligned}$$

For fixed x_i^1 , $m^2 + 1 \leq i < N$ we decompose the set J_{m^2} in the form of following disjoint union:

$$J_{m^2} = \bigcup_{q^2=m^2+1}^N I_N^{m^2,q^2}(x_{m^2+1,q^2-1}^1),$$

where $I_N^{m^2, q^2}(x_{m^2+1, q^2-1}^1) := I_{q^2+1}(0, \dots, 0, x_{m^2}^2 = 1, x_{m^2+1}^1, \dots, x_{q^2-1}^1, 1 - x_{q^2}^1)$, for $m^2 < q^2 < N$ and $I_N^{m^2, q^2}(x_{m^2+1, q^2-1}^1) := I_N(0, \dots, 0, x_{m^2}^2 = 1, x_{m^2+1}^1, \dots, x_{N-1}^1)$, for $q^2 = N$. That is, by Lemma 1

$$\begin{aligned} L_{3,2,2}^{l^1, m^2} &\leq c \sum_{k=l^1+1}^{m^2+1} \sum_{\substack{x_i^1=0 \\ i \in \{m^2+1, \dots, N-1\} \\ x_j^1=0 \text{ otherwise}}}^1 \sum_{q^2=m^2+1}^N \int_{I_N(e_{l^1}+e_k+x^1) \times I_N^{m^2, q^2}(x_{m^2+1, q^2-1}^1)} \\ &\quad \times \left(2^{l^1+k} \sum_{s=m^2+1}^{q^2} 2^s + 2^{l^1+m^2} \sum_{s=l^1}^{m^2} 2^s \right)^{2/3} d\mu(x^1, x^2) \\ &\leq c \sum_{k=l^1+1}^{m^2+1} \sum_{q^2=m^2+1}^N 2^{2(l^1+m^2+q^2)/3} 2^{-m^2-q^2} \\ &\leq c(m^2+1-l^1)2^{2l^1/3-2m^2/3}. \end{aligned}$$

This and inequality (4) yield that

$$\begin{aligned} L_{3,2} &\leq \frac{c}{N} \sum_{l^1=0}^{N-1} \sum_{m^2=l^1}^{N-1} 2^{l^1/3-m^2/3} + \frac{c}{N} \sum_{l^1=0}^{N-1} \sum_{m^2=l^1}^{N-1} (m^2+1-l^1)2^{2l^1/3-2m^2/3} \\ &\leq \frac{cN}{N} \leq c. \end{aligned}$$

This completes the proof of Theorem 1. $\square$

Next, we prove Theorem 2.

PROOF OF THEOREM 2. Let

$$f_A(x^1, x^2) := (D_{2^{A+1}}(x^1) - D_{2^A}(x^1))(D_{2^{A+1}}(x^2) - D_{2^A}(x^2)).$$

A simple calculation yields

$$\widehat{f}_A(i, k) = \begin{cases} 1, & \text{if } i, k = 2^A, \dots, 2^{A+1} - 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} &S_{i,j}(f; x^1, x^2) \\ &= \begin{cases} (D_i(x^1) - D_{2^A}(x^1))(D_j(x^2) - D_{2^A}(x^2)), & \text{if } i, j = 2^A + 1, \dots, 2^{A+1} - 1, \\ f_A(x^1, x^2), & \text{if } i, j \geq 2^{A+1}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We can write the n th Dirichlet kernel in the following form:

$$D_n(x) = D_{2^{|n|}}(x) + r_{|n|}(x)D_{n-2^{|n|}}(x)$$

Thus, we have for a nondecreasing function φ that

$$\begin{aligned} \tilde{\mathcal{M}}^* f_A(x^1, x^2) &= \sup_{n \in \mathbf{P}} \frac{|\mathcal{M}_n(f_A; x^1, x^2)|}{\varphi(n)} \geq \max_{t: 1 \leq 2^t \leq 2^A} \frac{|\mathcal{M}_{2^A+2^t}(f_A; x^1, x^2)|}{\varphi(2^A + 2^t)} \\ &\geq \max_{t: 1 \leq 2^t \leq 2^A} \frac{1}{(2^A + 2^t)\varphi(2^A + 2^t)} \left| \sum_{k=0}^{2^A+2^t-1} S_{k,k}(f_A; x^1, x^2) \right| \\ &\geq \max_{t: 1 \leq 2^t \leq 2^A} \frac{1}{2^{A+1}\varphi(2^{A+1})} \left| \sum_{k=2^A+1}^{2^A+2^t-1} (D_k(x^1) - D_{2^A}(x^1))(D_k(x^2) - D_{2^A}(x^2)) \right| \\ &= \max_{t: 1 \leq 2^t \leq 2^A} \frac{1}{2^{A+1}\varphi(2^{A+1})} \left| \sum_{k=2^A+1}^{2^A+2^t-1} r_A(x^1)D_{k-2^A}(x^1)r_A(x^2)D_{k-2^A}(x^2) \right| \\ &= \max_{t: 1 \leq 2^t \leq 2^A} \frac{1}{2^{A+1}\varphi(2^{A+1})} \left| \sum_{l=1}^{2^t-1} D_l(x^1)D_l(x^2) \right| \\ &= \frac{1}{2^{A+1}\varphi(2^{A+1})} \max_{t: 1 \leq 2^t \leq 2^A} 2^t |K_{2^t}(x^1, x^2)|. \end{aligned}$$

Since, we have

$$f_A^*(x^1, x^2) = \sup_{n \in \mathbf{N}} |S_{2^n, 2^n}(f_A; x^1, x^2)| = |f_A(x^1, x^2)|$$

and

$$\|f_A\|_{H_{2/3}} = \|f_A^*\|_{2/3} = c2^{-A}.$$

We obtain

$$\frac{\|\tilde{\mathcal{M}}^* f_A\|_{2/3}}{\|f_A\|_{H_{2/3}}} \geq \frac{c}{2^A \varphi(2^{A+1}) 2^{-A}} \left(\int_{G^2} \max_{t: 1 \leq 2^t \leq 2^A} (2^t |K_{2^t}(x^1, x^2)|)^{2/3} d\mu(x^1, x^2) \right)^{3/2}.$$

To investigate the integral $\int_{G^2} \max_{t: 1 \leq 2^t \leq 2^A} (2^t |K_{2^t}(x^1, x^2)|)^{2/3} d\mu(x^1, x^2)$, we decompose the set G as the following disjoint union

$$G = I_A \cup \bigcup_{s=0}^{A-1} (I_s \setminus I_{s+1}).$$

It is easy to show that, for $(x^1, x^2) \in I_s \times I_s$

$$K_{2^s}(x^1, x^2) = \frac{(2^s - 1)(2^{s+1} - 1)}{6}.$$

Therefore,

$$\begin{aligned} & \int_{G \times G} \max_{t: 1 \leq 2^t \leq 2^A} (2^t |K_{2^t}(x^1, x^2)|)^{2/3} d\mu(x^1, x^2) \\ & \geq \sum_{s=1}^{A-1} \int_{(I_s \setminus I_{s+1}) \times (I_s \setminus I_{s+1})} \max_{t: 1 \leq 2^t \leq 2^A} (2^t |K_{2^t}(x^1, x^2)|)^{2/3} d\mu(x^1, x^2) \\ & \geq \sum_{s=1}^{A-1} \int_{(I_s \setminus I_{s+1}) \times (I_s \setminus I_{s+1})} (2^s |K_{2^s}^w(x^1, x^2)|)^{2/3} d\mu(x^1, x^2) \\ & = \sum_{s=1}^{A-1} \int_{(I_s \setminus I_{s+1}) \times (I_s \setminus I_{s+1})} \left(2^s \frac{(2^s - 1)(2^{s+1} - 1)}{6} \right)^{2/3} d\mu(x^1, x^2) \\ & \geq c \sum_{s=1}^{A-1} \int_{(I_s \setminus I_{s+1}) \times (I_s \setminus I_{s+1})} (2^{3s})^{2/3} d\mu(x^1, x^2) \geq c(A-2). \end{aligned}$$

That is,

$$\frac{\|\tilde{\mathcal{M}}^* f_A\|_{2/3}}{\|f_A\|_{H_{2/3}}} \geq \frac{c(A+1)^{3/2}}{\varphi(2^{A+1})}$$

for A big enough.

Now, let $\{n_k : k \in \mathbf{P}\}$ be an increasing sequence of positive integers such that

$$\lim_{k \rightarrow \infty} \frac{\log^{3/2} n_k}{\varphi(n_k)} = +\infty.$$

There exists a positive integer m'_k such that $2^{m'_k} \leq n_k < 2 \cdot 2^{m'_k}$. φ is a nondecreasing function, then we have

$$\overline{\lim}_{k \rightarrow \infty} \frac{(m'_k)^{3/2}}{\varphi(2^{m'_k})} \geq c \lim_{k \rightarrow \infty} \frac{\log^{3/2} n_k}{\varphi(n_k)} = +\infty.$$

Let $\{m_k : k \in \mathbf{P}\} \subset \{m'_k : k \in \mathbf{P}\}$ be such that

$$\lim_{k \rightarrow \infty} \frac{(m_k)^{3/2}}{\varphi(2^{m_k})} = +\infty.$$

This yields

$$\frac{\|\tilde{\mathcal{M}}^* f_{m_k-1}\|_{2/3}}{\|f_{m_k-1}\|_{H_{2/3}}} \geq \frac{c(m_k)^{3/2}}{\varphi(2^{m_k})}.$$

$k \rightarrow \infty$ completes the proof of this theorem. $\square$

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