ON THE MAXIMAL OPERATOR OF WALSH-KACZMARZ-FEJÉR MEANS

Ushangi Goginava, Tbilisi, Károly Nagy, Nyíregyháza.

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Abstract. In this paper we prove that the maximal operator $\tilde{\sigma}^{\kappa,*}f := \sup_{n\in\mathbf{P}} \frac{|\sigma_n^{\kappa} f|}{\log^2(n+1)}$, where $\sigma_n^{\kappa} f$ is the nth Fejér mean of Walsh-Kaczmarz-Fourier series, is bounded from the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$.

Keywords: Walsh-Kaczmarz system, Fejér means, Maximal operator.

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1. Introduction

The a.e. convergence of Walsh-Fejér means $\sigma_n f$ was proved by Fine [2]. In 1975 Schipp [11] showed that the maximal operator σ^* is of weak type $(1,1)$ and of type (p,p) for $1 < p \leq \infty$. The boundedness fails to hold for $p = 1$. But, Fujii [3] proved that σ^* is bounded from the dyadic Hardy space H_1 to the space L_1. The theorem of Fujii was extended by Weisz [21], he showed that the maximal operator σ^* is bounded from the martingale Hardy space H_p to the space L_p for $p > 1/2$. Simon gave a counterexample [13], which showes that the boundedness does not hold for $0 < p < 1/2$. The counterexample for $p = 1/2$ due to Goginava [9]. In the endpoint case $p = 1/2$ two positive result was showed. Weisz [22] proved that σ^* is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$. In 2008 Goginava [8] proved that the maximal operator $\tilde{\sigma}^*$ defined by

$$\tilde{\sigma}^* f := \sup_{n\in\mathbf{P}} \frac{|\sigma_n f|}{\log^2(n+1)}$$

is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$. He also proved that for any nondecreasing function $\varphi : \mathbf{P} \to [1, \infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\log^2(n+1)}{\varphi(n)} = +\infty \tag{1.1}$$

the maximal operator $\sup_{n\in\mathbf{P}} \frac{|\sigma_n f|}{\varphi(n)}$ is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

In 1948 Šneider [16] introduced the Walsh-Kaczmarz system and showed that the inequality

$$\limsup_{n\to\infty} \frac{D_n^\kappa(x)}{\log n} \geq C > 0$$

holds a.e. In 1974 Schipp [12] and Young [18] proved that the Walsh-Kaczmarz system is a convergence system. Skvortsov in 1981 [15] showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to f for any continuous functions f. Gát [4] proved, for any integrable functions, that the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere to the function. He showed that the maximal operator of Walsh-Kaczmarz-Fejér means $\sigma^{\kappa,*}$ is weak type $(1,1)$ and of type (p,p) for all $1 < p \leq \infty$. Gát's result was generalized by Simon [14], who showed that the maximal operator $\sigma^{\kappa,*}$ is of type (H_p, L_p) for $p > 1/2$.

In the endpoint case $p = 1/2$ the first author [6] proved that maximal operator is not of type $(H_{1/2}, L_{1/2})$ and Weisz [22] showed that the maximal operator is of weak type $(H_{1/2}, L_{1/2})$.

In the present paper we prove that the maximal operator $\tilde{\sigma}^{\kappa,*}$ defined by

$$\tilde{\sigma}^{\kappa,*} := \sup_{n\in\mathbf{P}} \frac{|\sigma_n^\kappa f|}{\log^2(n+1)}$$

is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$. We also prove that for any nondecreasing function $\varphi : \mathbf{P} \to [1, \infty)$ satisfying the condition (1.1) the maximal operator $\sup_{n\in\mathbf{P}} \frac{|\sigma_n^\kappa f|}{\varphi(n)}$ is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

2. Definitions and notation

Now, we give a brief introduction to the theory of dyadic analysis [1, 10]. Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Denote $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is $\mathbb{Z}_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $\mathbb{Z}_2$ is given such that the measure of a singleton is 1/2. Let G be the complete direct product of the countable infinite copies of the compact groups $\mathbb{Z}_2$. The elements of G are of the form $x = (x_0, x_1, ..., x_k, ...)$ with $x_k \in \{0,1\}\,(k \in \mathbf{N})$. The group operation on G is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$I_0(x) := G, \quad I_n(x) := I_n(x_0, ..., x_{n-1}) := \{y \in G : \, y = (x_0, ..., x_{n-1}, y_n, y_{n+1}, ...)\},$$

$(x \in G, n \in \mathbf{N})$. These sets are called dyadic intervals. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G, and $I_n := I_n(0)\ (n \in \mathbf{N})$. Set $e_n := (0, ..., 0, 1, 0, ...) \in G$, the nth coordinate of which is 1 and the rest are zeros $(n \in \mathbf{N})$.

For $k \in \mathbf{N}$ and $x \in G$ denote

$$r_k(x) := (-1)^{x_k}$$

the kth Rademacher function. If $n \in \mathbf{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$ can be written, where $n_i \in \{0,1\} \quad (i \in \mathbf{N})$, i. e. n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \qquad (x \in G, n \in \mathbf{P}).$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 = 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The set of Walsh-Kaczmarz functions and the set of Walsh-Paley functions is the same in dyadic blocks. Namely,

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{w_n : 2^k \leq n < 2^{k+1}\}$$

for all $k \in \mathbf{P}$ and $\kappa_0 = w_0$.

V. A. Skvortsov (see [15]) gave a relation between the Walsh-Kaczmarz functions and the Walsh-Paley functions by the help of the transformation $\tau_A : G \to G$ defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, ..., x_1, x_0, x_A, x_{A+1}, ...)$$

for $A \in \mathbf{N}$. By the definition of τ_A, we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbf{N}, x \in G).$$

The Dirichlet kernels are defined by

$$D_n^\alpha := \sum_{k=0}^{n-1} \alpha_k,$$

where $\alpha_n = w_n$ or κ_n $(n \in \mathbf{P})$, $D_0^\alpha := 0$. The 2^nth Dirichlet kernels have a closed form (see e.g. [10])

$$D_{2^n}^w(x) = D_{2^n}^\kappa(x) = D_{2^n}(x) = \begin{cases} 0, & \text{if } x \notin I_n \\ 2^n, & \text{if } x \in I_n. \end{cases} \tag{2.1}$$

The σ-algebra generated by the dyadic intervals of measure 2^{-k} will be denoted by F_k $(k \in \mathbf{N})$. Denote by $f = \left(f^{(n)}, n \in \mathbf{N}\right)$ a martingale with respect to $(F_n, n \in \mathbf{N})$ (for details see, e. g. [20]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbf{N}} \left| f^{(n)} \right|.$$

In case $f \in L_1(G)$, the maximal function can also be given by

$$f^*(x) = \sup_{n \in \mathbf{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right|, \quad x \in G.$$

For $0 < p < \infty$ the Hardy martingale space $H_p(G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(G)$, then it is easy to show that the sequence $(S_{2^n} f : n \in \mathbf{N})$ is a martingale. If f is a martingale, that is $f = (f^{(0)}, f^{(1)}, ...)$ then the Walsh-(Kaczmarz)-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i) = \lim_{k \to \infty} \int_G f^{(k)}(x)\, \alpha_i(x)\, d\mu(x) \quad (\alpha_i = w_i \text{ or } \kappa_i).$$

The Walsh-(Kaczmarz)-Fourier coefficients of $f \in L_1(G)$ are the same as the ones of the martingale $(S_{2^n} f : n \in \mathbf{N})$ obtained from f.

The partial sums of the Walsh-(Kaczmarz)-Fourier series are defined as follows:

$$S_M^\alpha(f; x) := \sum_{i=0}^{M-1} \widehat{f}(i)\, \alpha_i(x) \quad (\alpha = w \text{ or } \kappa).$$

For $n=1,2,...$ and a martingale f the Fejér means of order n of the Walsh-(Kaczmarz)-Fourier series of the function f is given by

$$\sigma_n^{\alpha}(f;x)=\frac{1}{n}\sum_{j=0}^{n-1}S_j^{\alpha}(f;x).$$

The Fejér kernel of order n of the Walsh-(Kaczmarz)-Fourier series defined by

$$K_n^{\alpha}\left(x\right):=\frac{1}{n}\sum_{k=0}^{n-1}D_k^{\alpha}\left(x\right).$$

For the martingale f we consider maximal operators

$$\sigma^{\kappa,*}f=\sup_{n\in\mathbf{P}}|\sigma_n^{\kappa}(f;x)|,$$

$$\widetilde{\sigma}^{\kappa,*}f=\sup_{n\in\mathbf{P}}\frac{|\sigma_n^{\kappa}(f;x)|}{\log^2\left(n+1\right)}.$$

3. Auxiliary propositions and main results

First, we formulate our main theorems.

Theorem 3.1. *The maximal operator* $\tilde{\sigma}^{\kappa,*}$ *is bounded from the Hardy space* $H_{1/2}(G)$ *to the space* $L_{1/2}(G)$.

Theorem 3.2. *Let* $\varphi:\mathbf{P}\to[1,\infty)$ *be a nondecreasing function satisfying the condition (1.1). Then the maximal operator*

$$\sup_{n\in\mathbf{P}}\frac{|\sigma_n^{\kappa}f|}{\varphi(n)}$$

is not bounded from the Hardy space $H_{1/2}(G)$ *to the space* $L_{1/2}(G)$.

To prove our Theorem 3.1 we need the following Lemmas:

Lemma 3.3 (Skvortsov [15]). *For* $n\in\mathbf{P},x\in G$

$$\begin{aligned} nK_n^{\kappa}(x) &= 1+\sum_{i=0}^{|n|-1}2^iD_{2^i}(x)+\sum_{i=0}^{|n|-1}2^ir_i(x)K_{2^i}^{w}(\tau_i(x)) \\ &+ (n-2^{|n|})(D_{2^{|n|}}(x)+r_{|n|}(x)K_{n-2^{|n|}}^{w}(\tau_{|n|}(x))). \end{aligned}$$

Lemma 3.4 (Gát [4]). *Let $A, t \in \mathbf{N}, A > t$. Suppose that $x \in I_t \backslash I_{t+1}$. Then*

$$K_{2^A}^w(x) = \begin{cases} 0 & \text{if } x - x_t e_t \notin I_A, \\ 2^{t-1} & \text{if } x - x_t e_t \in I_A. \end{cases}$$

If $x \in I_A$, then $K_{2^A}^w(x) = \frac{2^A - 1}{2}$.

A bounded measurable function a is a p-atom, if there exists a dyadic interval I, such that

a) $\int_I a d\mu = 0$,
b) $\|a\|_\infty \leq \mu(I)^{-1/p}$,
c) $\operatorname{supp} a \subset I$.

Lemma 3.5 (Weisz [20]). *Suppose that the operator T is sublinear and p-quasilocal for any $0 < p \leq 1$. If T is bounded from L_∞ to L_∞, then*

$$\|Tf\|_p \leq c_p \|f\|_{H_p} \quad \text{for all } f \in H_p.$$

Lemma 3.6 (Gát, Goginava, Nagy [5]). *Let $n < 2^{A+1}, A > N$ and $x \in I_N(x_0, ..., x_{m-1}, x_m = 1, 0, ..., 0, x_l = 1, 0, ..., 0) =: J_N^{m,l}$, $l = 0, ..., N-1$, $m = -1, 0, ..., l$. Then*

$$\int_{I_N} n \left| K_n^w \left(\tau_A \left(x + t \right) \right) \right| dt \leq c \frac{2^A}{2^{m+l}},$$

where

$$I_N \left(x_0, ..., x_m = 1, 0, ..., 0, x_l = 1, 0, ..., 0 \right) := I_N \left(0, ..., 0, x_l = 1, 0, ..., 0 \right)$$

for $m = -1$.

Lemma 3.7 (Goginava [7]). *Let $2 < A \in \mathbf{P}$ and $q_A := 2^{2A} + 2^{2A-2} + \cdots + 2^2 + 2^0$. Then*

$$q_{A-1} \left| K_{q_{A-1}} \left(x \right) \right| \geq 2^{2m+2s-3}$$

for $x \in I_{2A}(0, ..., 0, x_{2m} = 1, 0, ..., 0, x_{2s} = 1, x_{2s+1}, ..., x_{2A-1})$, $m = 0, 1, ..., A-3$, $s = m+2, m+3, ..., A-1$.

4. Proofs of the theorems

First, we prove Theorem 3.1.

Proof of Theorem 3.1: Lemma 3.3 yields that

$$\begin{aligned}\tilde{\sigma}_n^{\kappa} f &= \frac{|f * K_n^{\kappa}|}{\log^2(n+1)} \leq \left| f * \frac{1}{n|n|^2} \left(1 + \sum_{i=0}^{|n|-1} 2^i D_{2^i} \right) \right| \\ &+ \left| f * \frac{1}{n|n|^2} \sum_{i=0}^{|n|-1} 2^i r_i K_{2^i}^w \circ \tau_i \right| + \left| f * \frac{n - 2^{|n|}}{n|n|^2} \left(D_{2^{|n|}} + r_{|n|} K_{n-2^{|n|}}^w \circ \tau_{|n|} \right) \right| \\ &=: \sum_{i=1}^{3} |f * L_n^i|\end{aligned}$$

and

$$\tilde{\sigma}^{\kappa,*} f \leq \sup_{n \in \mathbf{P}} |f * L_n^1| + \sup_{n \in \mathbf{P}} |f * L_n^2| + \sup_{n \in \mathbf{P}} |f * L_n^3| =: R^1 f + R^2 f + R^3 f.$$

By the help of Lemma 3.5 we show that the operators R^i $(i = 1, 2, 3)$ are of type $(H_{1/2}, L_{1/2})$. The boundedness from the space L_∞ to the space L_∞ follows from (2.1) and

$$\|K_n^w \circ \tau_i\|_1 = \|K_n^w\|_1 \leq 2$$

for $i \leq |n|, n \in \mathbf{P}$ (see Yano [17]). By Lemma 3.5, the proof will be complete, if we show that the maximal operators R^i $(i = 1, 2, 3)$ are $1/2$-quasilocal. That is, there exists a constant c such that

$$\int_{\overline{I}} |R^i a|^{1/2} d\mu \leq c < \infty$$

for every $1/2$-atom a, where the dyadic interval I is the support of the $1/2$-atom a.

Let a be an arbitrary $1/2$-atom with support I, and $\mu(I) = 2^{-N}$. Without loss of generality, we may assume that $I := I_N$.

It is evident that $\tilde{\sigma}_n^{\kappa}(a) = 0$ and $a * L_n^i = 0$ $(i = 1, 2, 3)$ if $n \leq 2^N$. Therefore, we set $n > 2^N$.

By $\|a\|_\infty \leq c 2^{2N}$ we have that

$$|a * L_n^i| \leq \int_{I_N} |a(s)| |L_n^i(x+s)| d\mu(s) \leq c 2^{2N} \int_{I_N} |L_n^i(x+s)| d\mu(s)$$

and

$$|R^i a| \leq c 2^{2N} \sup_{n > 2^N} \int_{I_N} |L_n^i(x+s)| d\mu(s). \tag{4.1}$$

Now, we write

$$\overline{I_N} = \bigcup_{j=0}^{N-1} (I_j \backslash I_{j+1}).$$

Set $x \in I_j \backslash I_{j+1}$ and $s \in I_N$, then $x+s \in I_j \backslash I_{j+1}$ for $j = 0, ..., N-1$. Thus, we have

$$\begin{aligned}
\sup_{n>2^N} \int_{I_N} |L_n^1(x+s)| d\mu(s) &\leq \sup_{n>2^N} \int_{I_N} \frac{1}{n|n|^2} \left(1 + \sum_{i=0}^{j} 2^i D_{2^i}(x+s) \right) d\mu(s) \\
&\leq \frac{c}{2^N N^2} 2^{2j} 2^{-N} \leq \frac{c 2^{2j}}{N^2 2^{2N}}
\end{aligned}$$

and

$$\begin{aligned}
\int_{\overline{I_N}} |R^1 a(x)|^{1/2} d\mu(x) &= \sum_{j=0}^{N-1} \int_{I_j \backslash I_{j+1}} |R^1 a(x)|^{1/2} d\mu(x) \leq \\
&\leq c 2^N \sum_{j=0}^{N-1} \int_{I_j \backslash I_{j+1}} \left(\frac{2^{2j}}{N^2 2^{2N}} \right)^{1/2} d\mu(x) \leq \frac{cN}{N} \leq c.
\end{aligned}$$

Now, we discuss $\int_{\overline{I_N}} |R^2 a|^{1/2} d\mu$. We use the disjoint decomposition of $\overline{I_N}$ above. That is, $\overline{I_N} = \bigcup_{t=0}^{N-1} (I_t \backslash I_{t+1})$ and we decompose the sets $I_t \backslash I_{t+1}$ as the following disjoint union:

$$I_t \backslash I_{t+1} = \bigcup_{l=t+1}^{N} J_t^l,$$

where $J_t^l := I_N(0, ..., 0, x_t = 1, 0, ..., 0, x_l = 1, x_{l+1}, ..., x_{N-1})$ for $t < l < N$ and $J_t^l := I_N(e_t)$ for $l = N$.

$$\begin{aligned}
&\int_{\overline{I_N}} |R^2 a(x)|^{1/2} d\mu(x) = \sum_{t=0}^{N-1} \sum_{l=t+1}^{N} \int_{J_t^l} |R^2 a(x)|^{1/2} d\mu(x) \\
= &\sum_{t=0}^{N-1} \sum_{l=t+1}^{N-1} \int_{J_t^l} |R^2 a(x)|^{1/2} d\mu(x) + \sum_{t=0}^{N-1} \int_{J_t^N} |R^2 a(x)|^{1/2} d\mu(x) =: \sum_1 + \sum_2 .
\end{aligned}$$

Let $x \in J_t^l$ and $s \in I_N$, then $x+s \in J_t^l$ $(0 \leq t < N,\ t < l \leq N)$.

For $0 \le t < l < N$, Lemma 3.4 and $K^w_{2^i}(\tau_i(x+s)) \neq 0$ imply that $i \le l$, $K^w_{2^i}(\tau_i(x+s)) = 2^{i-t-2}$ for $l > i > t$ and $K^w_{2^i}(\tau_i(x+s)) = \frac{2^i-1}{2}$ for $i \le t$. Thus,

$$\begin{aligned}
\sup_{n>2^N} \int_{I_N} |L_n^2(x+s)| d\mu(s) &\le \sup_{n>2^N} \frac{1}{n|n|^2} \int_{I_N} \sum_{i=0}^{l} 2^i |K^w_{2^i}(\tau_i(x+s))| d\mu(s) \\
&\le \sup_{n>2^N} \frac{c}{n|n|^2} \int_{I_N} \left(\sum_{i=0}^{t} 2^{2i} + \sum_{i=t+1}^{l} 2^i 2^{i-t} \right) d\mu(s) \\
&\le \frac{c(2^{2t} + 2^{2l-t})}{2^{2N} N^2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_1 &\le c \sum_{t=0}^{N-1} \sum_{l=t+1}^{N-1} \int_{J_t^l} 2^N \left(\frac{(2^{2t} + 2^{2l-t})}{2^{2N} N^2} \right)^{1/2} d\mu(x) \\
&\le c \sum_{t=0}^{N-1} \sum_{l=t+1}^{[3t/2]} \int_{J_t^l} \frac{2^t}{N} d\mu(x) + c \sum_{t=0}^{N-1} \sum_{l=[3t/2]+1}^{N-1} \int_{J_t^l} \frac{2^{l-t/2}}{N} d\mu(x) \\
&\le c \sum_{t=0}^{N-1} \sum_{l=t+1}^{[3t/2]} \sum_{\substack{y_i=0 \\ i \in \{l+1,\ldots,N-1\}}}^{1} \int_{I_N(y+e_t+e_l)} \frac{2^t}{N} d\mu(x) \\
&\quad + c \sum_{t=0}^{N-1} \sum_{l=[3t/2]+1}^{N-1} \sum_{\substack{y_i=0 \\ i \in \{l+1,\ldots,N-1\}}}^{1} \int_{I_N(y+e_t+e_l)} \frac{2^{l-t/2}}{N} d\mu(x) \\
&\le c \sum_{t=0}^{N-1} \sum_{l=t+1}^{[3t/2]} \frac{2^t}{N} 2^{-l} + c \sum_{t=0}^{N-1} \sum_{l=[3t/2]+1}^{N-1} \frac{2^{l-t/2}}{N} 2^{-l} \le c.
\end{aligned}$$

For $l = N$, let $x \in J_t^N$. Lemma 3.4 yields

$$\begin{aligned}
&\sup_{n>2^N} \int_{I_N} |L_n^2(x+s)| d\mu(s) \le \sup_{n>2^N} \frac{1}{n|n|^2} \int_{I_N} \sum_{i=0}^{|n|-1} 2^i |K^w_{2^i}(\tau_i(x+s))| d\mu(s) \\
\le\;& c \sup_{n>2^N} \frac{1}{n|n|^2} \left(\int_{I_N} \left(\sum_{i=0}^{t} 2^{2i} + \sum_{i=t+1}^{N} 2^i 2^{i-t} \right) d\mu(s) + \sum_{i=N+1}^{|n|-1} \int_{I_i(x_{N,i-1})} 2^i 2^{i-t} d\mu(s) \right) \\
\le\;& c \sup_{n>2^N} \frac{2^{2t-N} + 2^{N-t} + 2^{|n|-t}}{n|n|^2} \le \frac{c 2^{2t}}{2^{2N} N^2} + \frac{c}{2^t N^2},
\end{aligned}$$

where $x_{N,i-1} := \sum_{j=N}^{i-1} x_j e_j$.

$$
\begin{aligned}
\sum_{2} &\leq c\sum_{t=0}^{N-1}\int_{J_t^N} 2^N\left(\frac{2^{2t}}{2^{2N}N^2}+\frac{1}{2^tN^2}\right)^{1/2} d\mu(x) \\
&\leq c\sum_{t=0}^{[2N/3]}\int_{J_t^N}\frac{2^N}{2^{t/2}N}d\mu(x)+c\sum_{t=[2N/3]+1}^{N-1}\int_{J_t^N}2^N\frac{2^t}{2^NN}d\mu(x) \\
&\leq c\sum_{t=0}^{[2N/3]}\frac{1}{2^{t/2}N}+c\sum_{t=[2N/3]+1}^{N-1}\frac{2^t}{2^NN}\leq c.
\end{aligned}
$$

To discuss $\int_{\overline{I_N}}|R^3a|^{1/2}d\mu$ we use Lemma 3.6 and the following disjoint decomposition of $\overline{I_N}$:

$$
\overline{I_N}=\bigcup_{l=0}^{N-1}\bigcup_{m=-1}^{l}J_N^{l,m},
$$

where the set $J_N^{l,m}$ is defined in Lemma 3.6.

If $x\in\overline{I_N}$ and $s\in I_N$, then $x+s\in\overline{I_N}$ and $D_{2^{|n|}}(x+s)=0$. Moreover, if $x\in J_N^{l,m}$, then $x+s\in J_N^{l,m}$ and by Lemma 4 we have

$$
\begin{aligned}
\sup_{n>2^N}\int_{I_N}|L_n^3(x+s)|d\mu(s) &\leq \sup_{n>2^N}\int_{I_N}\frac{n-2^{|n|}}{n|n|^2}|K^w_{n-2^{|n|}}(\tau_{|n|}(x+s))|d\mu(s) \\
&\leq c\sup_{n>2^N}\frac{1}{n|n|^2}\frac{2^{|n|}}{2^{l+m}} \\
&\leq \frac{c}{2^{l+m}N^2}.
\end{aligned}
$$

By the above written

$$
\begin{aligned}
\int_{\overline{I_N}}|R^3a(x)|^{1/2}d\mu(x) &= \sum_{l=0}^{N-1}\sum_{m=-1}^{l}\int_{J_N^{l,m}}|R^3a(x)|^{1/2}d\mu(x) \\
&\leq c\sum_{l=0}^{N-1}\sum_{m=-1}^{l}\int_{J_N^{l,m}}\frac{2^N}{2^{(l+m)/2}N}d\mu(x) \\
&\leq c\sum_{l=0}^{N-1}\sum_{m=-1}^{l}\sum_{\substack{y_i=0\\ i\in\{0,\dots,m-1\}}}^{1}\int_{I_N(y+e_m+e_l)}\frac{2^N}{2^{(l+m)/2}N}d\mu(x) \\
&\leq c\sum_{l=0}^{N-1}\sum_{m=-1}^{l}\frac{2^N}{2^{(l+m)/2}N}2^{-N+m}\leq c.
\end{aligned}
$$

This completes the proof of Theorem 3.1. □

Next, we prove Theorem 3.2.

Proof of Theorem 3.2: Let $\{n_k : k \in \mathbf{P}\}$ be an increasing sequence of positive integers such that

$$\lim_{k\to\infty} \frac{\log^2 n_k}{\varphi(n_k)} = +\infty.$$

It is evident that for every n_k there exists a positive integer m'_k such that

$$q_{m'_k} \leq n_k < q_{m'_k+1} < 5q_{m'_k}.$$

Since, $\varphi(n)$ is nondecreasing function we have

$$\overline{\lim_{k\to\infty}} \frac{(m'_k)^2}{\varphi\left(q_{m'_k}\right)} \geq c \lim_{k\to\infty} \frac{\log^2 n_k}{\varphi(n_k)} = +\infty.$$

Let $\{m_k : k \in \mathbf{P}\} \subset \{m'_k : k \in \mathbf{P}\}$ such that

$$\lim_{k\to\infty} \frac{(m_k)^2}{\varphi(q_{m_k})} = +\infty.$$

Let

$$f_{m_k}(x) := D_{2^{2m_k+1}}(x) - D_{2^{2m_k}}(x).$$

It is evident that

$$\widehat{f}^{\kappa}_{m_k}(i) = \begin{cases} 1, \text{ if } i = 2^{2m_k}, ..., 2^{2m_k+1} - 1, \\ 0, \text{ otherwise.} \end{cases}$$

Then we can write that

$$S^{\kappa}_i(f_{m_k}; x) = \begin{cases} D^{\kappa}_i(x) - D_{2^{2m_k}}(x), & i = 2^{2m_k} + 1, ..., 2^{2m_k+1} - 1, \\ f_{m_k}(x), & i \geq 2^{2m_k+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Since,

$$f^*_{m_k}(x) = \sup_{n\in\mathbf{N}} |S_{2^n}(f_{m_k}; x)| = |f_{m_k}(x)|,$$

from (2.1) we get

$$\|f_{m_k}\|_{H_p} = \left\|f^*_{m_k}\right\|_p = \|D_{2^{2m_k}}\|_p = 2^{2m_k(1-1/p)}. \tag{4.3}$$

Since, we have

$$D^{\kappa}_n(x) = D_{2^{|n|}}(x) + r_{|n|}(x) D^w_{n-2^{|n|}}(\tau_{|n|}(x))$$

from (4.2) we can write

$$\begin{aligned}
\sup_{n\in\mathbf{P}}\frac{|\sigma_n^\kappa(f_{m_k};x)|}{\varphi(n)} &\geq \frac{\left|\sigma_{q_{m_k}}^\kappa\left(f_{m_k};x\right)\right|}{\varphi\left(q_{m_k}\right)} \\
&= \frac{1}{\varphi\left(q_{m_k}\right)}\frac{1}{q_{m_k}}\left|\sum_{j=0}^{q_{m_k}-1}S_j^\kappa\left(f_{m_k};x\right)\right| \\
&= \frac{1}{\varphi\left(q_{m_k}\right)}\frac{1}{q_{m_k}}\left|\sum_{j=2^{2m_k}}^{q_{m_k}-1}S_j^\kappa\left(f_{m_k};x\right)\right| \\
&= \frac{1}{\varphi\left(q_{m_k}\right)}\frac{1}{q_{m_k}}\left|\sum_{i=2^{2m_k}}^{q_{m_k}-1}\left(D_i^\kappa\left(x\right)-D_{2^{2m_k}}\left(x\right)\right)\right| \\
&= \frac{1}{\varphi\left(q_{m_k}\right)}\frac{1}{q_{m_k}}\left|\sum_{i=0}^{q_{m_k-1}-1}\left(D_{i+2^{2m_k}}^\kappa\left(x\right)-D_{2^{2m_k}}\left(x\right)\right)\right| \\
&= \frac{1}{\varphi\left(q_{m_k}\right)}\frac{1}{q_{m_k}}\left|\sum_{i=0}^{q_{m_k-1}-1}D_i^w\left(\tau_{2m_k}\left(x\right)\right)\right| \\
&= \frac{1}{\varphi\left(q_{m_k}\right)}\frac{q_{m_k-1}}{q_{m_k}}\left|K_{q_{m_k-1}}^w\left(\tau_{2m_k}\left(x\right)\right)\right|
\end{aligned}$$

Let $x\in J_{2m_k}^{2A-2s-1,2A-2l-1}$ for some $l<s<m_k$. Then from Lemma 3.7 we have

$$\frac{\sigma_{q_{m_k}}^\kappa\left(f_{m_k};x\right)}{\varphi(q_{m_k})}\geq c\frac{2^{2s+2l-2m_k}}{\varphi\left(q_{m_k}\right)}.$$

Hence, we can write

$$\begin{aligned}
&\int_G\left(\sup_{n\in\mathbf{P}}\frac{\left|\sigma_n^\kappa\left(f_{m_k};x\right)\right|}{\varphi\left(n\right)}\right)^{1/2}d\mu\left(x\right) \\
\geq\ & c\sum_{l=0}^{m_k-3}\sum_{s=l+2}^{m_k-1}\int_{J_{2m_k}^{2A-2s-1,2A-2l-1}}\left(\sup_{n\in\mathbf{P}}\frac{\left|\sigma_n^\kappa\left(f_{m_k};x\right)\right|}{\varphi\left(n\right)}\right)^{1/2}d\mu\left(x\right) \\
\geq\ & c\sum_{l=0}^{m_k-3}\sum_{s=l+2}^{m_k-1}\sum_{\substack{y_i=0\\ i\in\{0,\dots,2m_k-2s-2\}}}^{1}\int_{I_{2m_k}\left(y+e_{2A-2s-1}+e_{2A-2l-1}\right)}\left(\frac{\left|\sigma_{q_{m_k}}^\kappa\left(f_{m_k};x\right)\right|}{\varphi\left(q_{m_k}\right)}\right)^{1/2}d\mu\left(x\right) \\
\geq\ & c\sum_{l=0}^{m_k-3}\sum_{s=l+2}^{m_k-1}\frac{2^{2m_k-2s}}{2^{2m_k}}\frac{2^{s+l-m_k}}{\sqrt{\varphi\left(q_{m_k}\right)}} \\
\geq\ & \frac{c}{\sqrt{\varphi\left(q_{m_k}\right)}}\frac{m_k}{2^{m_k}}.
\end{aligned}$$

Then from (4.3) we obtain

$$\frac{\left\{\int\limits_G \left(\sup\limits_{n\in\mathbf{P}} \frac{\left|\sigma_n^{\kappa}\left(f_{m_k};x\right)\right|}{\varphi(n)}\right)^{1/2} d\mu\left(x\right)\right\}^2}{\left\|f_{m_k}\right\|_{H_{1/2}}}$$

$$\geq \frac{cm_k^2}{\varphi\left(q_{m_k}\right)2^{-2m_k}2^{2m_k}} = \frac{cm_k^2}{\varphi\left(q_{m_k}\right)} \to \infty \quad \text{as } k \to \infty.$$

Theorem 3.2 is proved. □

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Authors' addresses:

Ushangi Goginava, Institute of Mathematics, Faculty of Exact and Natural Sciences, Tbilisi State University, Chavchavadze str. 1, Tbilisi 0128, Georgia, e-mail: `z_goginava@hotmail.com`.

Károly Nagy, Institute of Mathematics and Computer Sciences, College of Nyíregyháza, P.O. Box 166, Nyíregyháza, H-4400 Hungary, e-mail: `nkaroly@nyf.hu`.