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ON WEIGHTED $(0,2)$-TYPE INTERPOLATION*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. The weighted (0,2)-interpolation with additional Hermite-type conditions is studied in a unified way with respect to the existence, uniqueness and representation (explicit formulae). Sufficient conditions are given on the nodes and the weight function, for the problem to be regular. Examples are presented on the zeros of the classical orthogonal polynomials.

Key words. Birkhoff interpolation, Pál-type interpolation, Hermite interpolation, weighted (0,2)-interpolation, regularity, explicit formulae

AMS subject classification. 41A05

1. Introduction.

P. Turán [17] initiated the study of $(0,2)$-interpolation in order to get an approximate solution to the differential equation

$$y'' + f \cdot y = 0.$$

The *weighted* $(0,2)$*-interpolation* is a generalization of this problem, introduced by J. Balázs [2]. Let the system of nodes

$$-\infty \le a < x_n < x_{n-1} < \cdots < x_1 < b \le \infty \tag{1.1}$$

be given in the possibly infinite open or closed interval (a,b) and let $w \in C^2(a,b)$ be a given function (the *weight function*). Find a polynomial R_n of minimal degree satisfying the conditions

$$R_n(x_k) = y_k; \quad (wR_n)''(x_k) = y_k'', \quad (k = 1,\dots,n; n \in N) \tag{1.2}$$

where y_k, y_k'' are arbitrary given real numbers.

If for any choice of y_k and y_k'' $(k = 1,\dots,n)$ there exists a unique polynomial R_n of degree $\le 2n-1$ satisfying the conditions (1.2), the weighted $(0,2)$-interpolation problem is called *regular* on the given nodes (1.1) with respect to the weight function w. The question is how to choose the nodal points and the weight function w so that the problem is regular, and in the regular case how to find a simple explicit form of R_n.

J. Balázs [2] investigated the above problem on the interval $[-1,1]$, when the nodes are the zeros of the ultraspherical polynomial $P_n^{(\alpha)}$ $(\alpha > -1)$, and the weight function is $w(x) = (1-x^2)^{(\alpha+1)/2}$. He showed, that in this case there does not exist a polynomial of degree $\le 2n-1$ satisfying the requirements (1.2). He proved, that if n is even, then under the condition

$$R_n(0) = \sum_{k=1}^{n} y_k \ell_k^2(0), \tag{1.3}$$

where $\ell_k(x)$ represent the Lagrange-fundamental polynomials corresponding to the nodal points x_k, there exists a unique polynomial of degree $\le 2n$ (if n is odd, then the uniqueness fails). He gave the explicit form of this polynomial and proved convergence theorem.

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Several authors investigated the weighted $(0,2)$-interpolation with the additional Balázs-type condition (1.3) on the zeros of the classical orthogonal polynomials (I. Joó [6], I. Joó and L. Szili [7], J. Prasad [13], [14], [15], [16], L. Szili [19], [20]). The additional condition (1.3) was replaced by interpolatory type conditions in special cases by J. Bajpai [1], S. Datta and P. Mathur [4], S. Eneduanya [5] and J. Balázs [3]. Recently the author ([8], [9], [10]) studied the weighted $(0,2)$-interpolation with one and two additional interpolatory conditions in a unified way with respect to the existence, uniqueness and representation (explicit formulae).

In this paper we study the following generalization of the problem: There are two disjoint sets of the nodes, and weighted $(0,2)$-interpolation is prescribed on the points of one of the sets with Hermite-interpolation on the points of the other one. In the case where the second set is empty, we recover the original problem of J. Balázs [2].

The paper is organized as follows. In Section 2 we describe the problem and we give general forms of the explicit formulae for the fundamental polynomials. In Section 3 we give necessary and sufficient conditions on the nodes and the weight function for the problem to be regular with different additional interpolatory conditions. In Section 4 we present some special cases, Pál-type weighted $(0,2;0)$- and $(0;0,2)$-interpolation problems on the zeros of the classical orthogonal polynomials.

2. The problem. On the finite or infinite interval $[a,b]$ let $\{x_i\}_{i=1}^n$ and $\{\bar{x}_i\}_{i=1}^m$ be disjoint sets of distinct points (*nodes*), and let $w \in C^2(a,b)$ be a given function, called *weight function*. Find a minimal degree polynomial R_N satisfying the weighted $(0,2)$-interpolation conditions

$$R_N(x_i) = y_i, \qquad (wR_N)''(x_i) = y_i'' \qquad (i = 1,\dots,n) \tag{2.1}$$

and the additional Hermite-type interpolation conditions

$$R_N^{(j)}(\bar{x}_i) = \bar{y}_i^{(j)} \qquad (j = 0,\dots,j_i - 1;\ i = 1,\dots,m), \tag{2.2}$$

where y_i, y_i'' and $\bar{y}_i^{(j)}$ are arbitrary given real numbers.

As the number of conditions is $N = 2n + M$, where $M = j_1 + \ldots + j_m$, the problem is *regular*, if for any choice of the values y_i, y_i'' and $\bar{y}_i^{(j)}$ there exists a unique polynomial R_N of degree at most $N-1$. (For $m = 0$ there are no Hermite-type conditions, and we recover the original problem of J. Balázs [2].)

The problem is not regular in general, because such a minimal degree polynomial might not exist ([2]), or if it exists, it might not be unique. Therefore we study the problem with additional interpolatory conditions and find conditions for the problem to be regular. In regular cases we give simple explicit forms for R_N. Finally we present examples on the zeros of the classical orthogonal polynomials.

2.1. The fundamental polynomials. In what follows, let p_n and q be polynomials of degree n and M, respectively, associated with the interpolation conditions, that is

$$p_n(x_i) = 0 \qquad (i = 1,\dots,n), \tag{2.3}$$

$$q^{(j)}(\bar{x}_i) = 0 \qquad (j = 0,\dots,j_i - 1;\ i = 1,\dots,m). \tag{2.4}$$

LEMMA 2.1. *If* $q = q_1q_2$, $w = w_1w_2$ *and*

$$(q_1w_1p_n)''(x_i) = 0 \quad \textit{and} \quad (w_2q_2)'(x_i) = 0 \quad (i = 1,\dots,n), \tag{2.5}$$

then for any $Q \in C^2(a,b)$

$$(qwp_nQ)''(x_i) = 2(qwp_n'Q')(x_i) \qquad (i = 1,\dots,n). \tag{2.6}$$

The general explicit formulae for the fundamental polynomials of first and second kind, associated with the weighted $(0,2)$-interpolation, can be proved in the same way as in [8].

LEMMA 2.2. *If on the nodes* $\{x_i\}_{i=1}^n$ *and* $\{\bar{x}_i\}_{i=1}^m$ *the weight function* w *satisfies* (2.5), $w(x_i) \neq 0$ $(i = 1, \ldots, n)$, *then the fundamental polynomials of first kind, which satisfy the conditions*

$$A_k(x_i) = \delta_{ki}, \quad (wA_k)''(x_i) = 0, \qquad (i = 1, \ldots, n),$$
$$A_k^{(j)}(\bar{x}_i) = 0, \qquad (j = 0, \ldots, j_i - 1; \; i = 1, \ldots, m),$$

can be written in the form

$$A_k(x) = \frac{q(x)}{q(x_k)} \ell_k^2(x) + \frac{(qp_n)(x)}{(qp_n')(x_k)} \left\{ c_k + \int_{x_0}^{x} \frac{\ell_k'(x_k)\ell_k(t) - \ell_k'(t)}{t - x_k} \, dt + a_k \int_{x_0}^{x} \ell_k(t) dt + b_k \int_{x_0}^{x} p_n(t) dt \right\}, \tag{2.7}$$

where

$$a_k = -\frac{(wq)''(x_k)}{2(wq)(x_k)} = -\frac{(w_1q_1)''(x_k)}{2(w_1q_1)(x_k)} - \frac{(w_2q_2)''(x_k)}{2(w_2q_2)(x_k)} \tag{2.8}$$

and

$$\ell_k(x) = \frac{p_n(x)}{p_n'(x_k)(x - x_k)} \qquad (k = 1, \ldots, n). \tag{2.9}$$

Furthermore, the fundamental polynomials of second kind are

$$B_k(x) = \frac{(qp_n)(x)}{2(qwp_n')(x_k)} \left\{ \tilde{c}_k + \int_{x_0}^{x} \ell_k(t) dt + \tilde{b}_k \int_{x_0}^{x} p_n(t) dt \right\} \tag{2.10}$$

for $k = 1, \ldots, n$, *which satisfy the conditions*

$$B_k(x_i) = 0, \quad (wB_k)''(x_i) = \delta_{ki}, \qquad (i = 1, \ldots, n),$$
$$B_k^{(j)}(\bar{x}_i) = 0, \qquad (j = 0, \ldots, j_i - 1; \; i = 1, \ldots, m).$$

LEMMA 2.3. *If on the nodes* $\{x_i\}_{i=1}^n$ *and* $\{\bar{x}_i\}_{i=1}^m$ *the weight function* w *satisfies* (2.5), $w(x_i) \neq 0$ $(i = 1, \ldots, n)$, $q_2(x) = \bar{p}(x) = c(x - \bar{x}_1) \ldots (x - \bar{x}_m)$, $q_1(\bar{x}_k) \neq 0$ $(k = 1, \ldots, m)$, *and* $\frac{q_1 w_2'}{w_2}$ *is a polynomial, then the fundamental polynomials associated with the additional Lagrange-type conditions at* $\{\bar{x}_i\}_{i=1}^m$ *be written in the form*

$$C_k(x) = \frac{(q_1^2 p_n)(x)}{(q_1^2 p_n)(\bar{x}_k)} \bar{\ell}_k^2(x) - \frac{(qp_n)(x)}{(q_1^2 p_n^2 \bar{p}')(\bar{x}_k)} \left\{ \bar{c}_k + \bar{b}_k \int_{x_0}^{x} p_n(t) dt + \int_{x_0}^{x} \frac{p_n(\bar{x}_k) g_k(t) - g_k(\bar{x}_k) p_n(t)}{t - \bar{x}_k} dt \right\}, \tag{2.11}$$

where

$$g_k(x) = \left(q_1' + q_1 \frac{w_2'}{w_2} \right)(x) \bar{\ell}_k(x) + 2q_1(x) \bar{\ell}_k'(x)$$

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with

$$\bar{\ell}_k(x) = \frac{\bar{p}(x)}{\bar{p}'(\bar{x}_k)(x-\bar{x}_k)} \qquad (k=1,\dots,m),$$

and fulfill the conditions

$$\begin{aligned} &C_k(x_i)=0, \quad (wC_k)''(x_i)=0, && (i=1,\dots,n),\\ &C_k(\bar{x}_i)=\delta_{ik} && (i=1,\dots,m), \end{aligned}$$

for $k=1,\dots,m$.

Proof. The polynomials C_k for $k=1,\dots,m$ are to be determined in the form

$$C_k(x) = \frac{\left(q_1^2 p_n\right)(x)}{\left(q_1^2 p_n\right)(\bar{x}_k)} \bar{\ell}_k^2(x) + \left(q p_n Q_k\right)(x),$$

where Q_k is a polynomial. On using (2.6) and

$$\left(wp_n q_1^2 \bar{\ell}_k^2\right)''(x_i) = \left(q_1 w_1 p_n \cdot q_1 w_2 \bar{\ell}_k^2\right)''(x_i) = 2\left(q_1 w_1 p_n\right)'(x_i)\left(q_1 w_2 \bar{\ell}_k^2\right)'(x_i)$$

$$= 2\left(q_1 w_1 p_n' \bar{\ell}_k\right)(x_i)\left((q_1 w_2)'\bar{\ell}_k + 2q_1 w_2 \bar{\ell}_k'\right)(x_i),$$

from the conditions $(wC_k)''(x_i)=0$ $(i=1,\dots,n)$ we obtain

$$Q_k'(x_i) = \frac{-1}{\left(q_1^2 p_n \bar{p}'\right)(\bar{x}_k)} \frac{g_k(x_i)}{x_i-\bar{x}_k}.$$

If $\frac{q_1 w_2'}{w_2}$ is a polynomial, then

$$Q_k'(x) = \frac{-1}{\left(q_1^2 p_n^2 \bar{p}'\right)(\bar{x}_k)} \frac{p_n(\bar{x}_k)g_k(x) - g_k(\bar{x}_k)p_n(x)}{x-\bar{x}_k}$$

is a polynomial, as well, and by integration we get (2.11) for the polynomial C_k. □

3. Additional interpolatory conditions. We study the interpolation problem (2.1)–(2.2) with different additional interpolatory conditions prescribed at the points x_0 and $x_{n+1} \in [a,b]$. We find conditions on the nodes and the weight function for the problem to be regular. The results can be summarized in the following

THEOREM 3.1. *If on the nodes* $\{x_i\}_{i=1}^n$ *and* $\{\bar{x}_i\}_{i=1}^m$ *the weight function* w *satisfies* (2.5), $w(x_i) \neq 0$ $(i=1,\dots,n)$, *then the interpolation problem* (2.1)–(2.2) *is regular under the additional condition(s) (i)-(ix) if and only if the condition(s) in the third column of Table 3.1 is(are) fulfilled.*

Proof. For the sake of simplicity we will prove only the case (v), the other cases are similar. We study the homogeneous problem: Find a polynomial of degree at most $N+1 = 2n+M+1$ for which

$$\begin{aligned} &\bar{R}_N(x_i)=0, \quad (w\bar{R}_N)''(x_i)=0, && (i=0,\dots,n),\\ &\bar{R}_N^{(j)}(\bar{x}_i)=0, && (j=0,\dots,j_i-1;\ i=1,\dots,m). \end{aligned}$$

It is obvious that the polynomial $\bar{R}_N$ can be written in the form $\bar{R}_N(x) = \left(qp_nQ\right)(x)$, where p_n and q are defined in (2.3) and (2.4), respectively, and Q is a polynomial of degree at most $n+1$. On using Lemma 2.1 the weighted (0,2)-interpolatory conditions are

$$\left(w\bar{R}_N\right)''(x_i) = 2\left(qwp_n'Q'\right)(x_i) = 0 \qquad (i=1,\dots,n),$$

TABLE 3.1

	Additional Interpolatory Condition(s)	Condition for Regularity	Degree of R_N
(i)	$R_N(x_0) = y_0$	$(qp_n)(x_0) \neq 0$	$\leq 2n + M$
(ii)	$R'_N(x_0) = y'_0$	$(qp_n)'(x_0) \neq 0$	$\leq 2n + M$
(iii)	$(wR_N)''(x_0) = y''_0$	$p_n(x_0) \neq 0, \quad (qwp_n)''(x_0) \neq 0$	$\leq 2n + M$
(iv)	$R_N(x_0) = y_0,$ $R'_N(x_0) = y'_0$	$(qp_n)(x_0) \neq 0$	$\leq 2n + M + 1$
(v)	$R_N(x_0) = y_0,$ $(wR_N)''(x_0) = y''_0$	$(qp_n)(x_0) \neq 0,$ $\left(qwp_n \int_{x_0}^{x} p_n(t)dt\right)''(x_0) \neq 0$	$\leq 2n + M + 1$
(vi)	$R'_N(x_0) = y'_0,$ $(wR_N)''(x_0) = y''_0$	$p_n(x_0) \neq 0,$ $(qp_n)'(x_0)\left(qwp_n \int_{x_0}^{x} p_n(t)dt\right)''(x_0)$ $-(qwp_n)''(x_0)(qp_n^2)(x_0) \neq 0$	$\leq 2n + M + 1$
(vii)	$R_N(x_0) = y_0,$ $R_N(x_{n+1}) = y_{n+1}$	$(qp_n)(x_0) \neq 0, \quad (qp_n)(x_{n+1}) \neq 0,$ $\int_{x_0}^{x_{n+1}} p_n(t)dt \neq 0$	$\leq 2n + M + 1$
(viii)	$R'_N(x_0) = y'_0,$ $R'_N(x_{n+1}) = y'_{n+1}$	$(qp_n)'(x_0)(qp_n)'(x_{n+1}) \int_{x_0}^{x_{n+1}} p_n(t)dt$ $+(qp_n)'(x_0)(qp_n^2)(x_{n+1})$ $-(qp_n)'(x_{n+1})(qp_n^2)(x_0) \neq 0$	$\leq 2n + M + 1$
(ix)	$(wR_N)''(x_0) = y''_0,$ $(wR_N)''(x_{n+1}) = y''_{n+1}$	$\left(qwp_n \int_{x_0}^{x} p_n(t)dt\right)''(x_0)(qwp_n)''(x_{n+1})$ $-\left(qwp_n \int_{x_0}^{x} p_n(t)dt\right)''(x_{n+1})(qwp_n)''(x_0)$ $\neq 0$	$\leq 2n + M + 1$

where $q(x_i) \neq 0$, $w(x_i) \neq 0$, $p'_n(x_i) \neq 0$. Hence $Q'(x_i) = 0$ for $i = 1, \ldots, n$, that is $Q'(x) = cp_n(x)$ and

$$Q(x) = c \int_{x_0}^{x} p_n(t)dt + d.$$

From $\bar{R}_N(x_0) = (qp_nQ)(x_0)$ it follows $Q(x_0) = 0$ and $d = 0$. Finally,

$$\left(w\bar{R}_N\right)''(x_0) = c\left(qwp_n \int_{x_0}^{x} p_n(t)dt\right)''(x_0) = 0$$

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has the unique solution $c = 0$ if and only if

$$\left(qwp_n \int_{x_0}^{x} p_n(t)dt\right)''(x_0) \neq 0$$

which completes the proof. ∎

4. Special cases.

4.1. Pál-type weighted $(0,2;0,1,\ldots,r-1)$-interpolation. Let us consider the weighted $(0,2)$-interpolation problem on the zeros of

$$p_n(x) = c(x-x_1)(x-x_2)\ldots(x-x_n)$$

with Hermite-type interpolation on the zeros of q_1p_n', where the derivatives up to the $(r-1)$-st order are prescribed.

LEMMA 4.1. *If on the zeros of* p_n

$$(q_1w_1p_n)''(x_i) = 0, \qquad w_1(x_i) \neq 0 \qquad (i = 1,\ldots,n),$$

then the first and second kind fundamental polynomials of the Pál-type weighted $(0,2;0,1,\ldots,r-1)$*-interpolation with respect to the weight function* $w = w_1(q_1w_1)^{2r}$ *are given by* (2.7)–(2.10)*, where* $w_2 = (q_1w_1)^{2r}$ *and* $q = (q_1p_n')^r$*.*

Proof. It is easy to see that

$$\begin{aligned}(w_2q_2)'(x_i) &= \left([(q_1w_1)^2p_n']^r\right)'(x_i)\\ &= r\left\{\left[(q_1w_1)^2p_n'\right](x_i)\right\}^{r-1} q_1(x_i)w_1(x_i)(q_1w_1p_n)''(x_i) = 0\end{aligned}$$

for $i = 1,\ldots,n$ and we can apply Lemma 2.2. ∎

For $r = 1$ we obtain the Pál-type weighted $(0,2;0)$-interpolation problem, which is weighted $(0,2)$-interpolation on the zeros of p_n with Lagrange interpolation on the zeros of q_1p_n'.

4.1.1. Pál-type weighted $(0,2;0)$-interpolation on the zeros of Hermite polynomials. Let $\{x_i\}_{i=1}^n$ and $\{\bar{x}_i\}_{i=1}^{n-1}$ be the zeros of H_n and H_n', respectively, where H_n denotes the Hermite polynomial of degree n, for which

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

with

$$H_n(0) = \begin{cases} 0 & \text{for odd } n,\\ (-1)^{\frac{n}{2}}\frac{n!}{(n/2)!} & \text{for even } n.\end{cases}$$

It is known [18], that

$$H_n'(x) = 2nH_{n-1}(x)$$

and on the zeros of H_n

$$\left(e^{-\frac{x^2}{2}}H_n\right)''(x_i) = 0 \qquad (i = 1,\ldots,n).$$

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Furthermore let

$$\ell_k(x) = \frac{H_n(x)}{H_n'(x_k)(x - x_k)} \qquad (k = 1, \dots, n),$$
$$\bar{\ell}_k(x) = \frac{H_n'(x)}{H_n''(\bar{x}_k)(x - \bar{x}_k)} \qquad (k = 1, \dots, n-1).$$

On using the properties of H_n we get

$$H_n''(x_k) = 2x_k H_n'(x_k), \qquad \ell_k'(x_k) = x_k \qquad (k = 1, \dots, n),$$

$$H_n''(\bar{x}_k) = -2nH_n(\bar{x}_k), \qquad \bar{\ell}_k'(\bar{x}_k) = \bar{x}_k \qquad (k = 1, \dots, n-1).$$

Based on Theorem 3.1 we can state the following result.

THEOREM 4.2. *If $\{x_i\}_{i=1}^n$ and $\{\bar{x}_i\}_{i=1}^{n-1}$ are the zeros of H_n and H_n', respectively, $x_0 = 0$ and the weight function is $w(x) = e^{-\frac{3x^2}{2}}$, then the Pál-type weighted $(0,2;0)$-interpolation is regular with the additional condition (ii) for any n or with (vi) for even n, but it is not regular with the condition (i), (iii), (iv) or (v).*

Proof. We study only the case (ii), the other cases can be discussed in a similar way. In case (ii) the additional condition is

$$R_N'(0) = y_0',$$

where y_0' is arbitrary real number, and the regularity is assured by

$$(qp_n)'(x_0) = (H_n'H_n)'(0) = [H_n'(0)]^2 + H_n(0)H_n''(0) \neq 0.$$

On using the properties of the Hermite polynomials and Lemma 4.1 with $r = 1$, $q_1 = 1$, $q = \bar{p} = H_n'$ and $w_1 = e^{-\frac{x^2}{2}}$, one can verify by (2.7)-(2.11) that the minimal degree interpolational polynomial is

$$R_N(x) = \sum_{k=1}^{n} A_k(x)y_k + \sum_{k=1}^{n} B_k(x)y_k'' + \sum_{k=1}^{n-1} C_k(x)\bar{y}_k + D_0(x)y_0'$$

of degree at most $3n - 1$, where

$$A_k(x) = \frac{H_n'(x)}{H_n'(x_k)}\ell_k^2(x) + \frac{H_n(x)H_n'(x)}{[H_n'(x_k)]^2}\left\{c_k + \int_0^x \frac{x_k\ell_k(t) - \ell_k'(t)}{t - x_k}dt + \frac{2n + 1 - x_k^2}{2}\int_0^x \ell_k(t)dt\right\},$$

$$B_k(x) = \frac{H_n(x)H_n'(x)}{2e^{-\frac{3}{2}x_k^2}[H_n'(x_k)]^2}\int_0^x \ell_k(t)dt$$

for $k = 1, \dots, n$,

$$C_k(x) = \frac{H_n(x)}{H_n(\bar{x}_k)}\bar{\ell}_k^2(x) + \frac{H_n(x)H_n'(x)}{nH_n^2(\bar{x}_k)}\left\{\bar{c}_k + \int_0^x \frac{\bar{\ell}_k'(t) - t\bar{\ell}_k(t)}{t - \bar{x}_k}dt\right\}$$

for $k = 1, \dots, n-1$, and

$$D_0(x) = \frac{H_n(x)H_n'(x)}{(H_n'H_n)'(0)},$$

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where

$$c_k = \begin{cases} 0 & \text{for odd } n, \\ \frac{-H_n(0)}{x_k^2 H_n'(x_k)} & \text{for even } n, \end{cases} \qquad \bar{c}_k = \begin{cases} \frac{-H_n'(0)}{4n\bar{x}_k^2 H_n(\bar{x}_k)} & \text{for odd } n, \\ 0 & \text{for even } n. \end{cases} \qquad \square$$

REMARK 4.3. *P. Mathur and S. Datta [11] stated the regularity of the weighted* $(0,2;0)$*-interpolation on the zeros of the Hermite polynomials with the condition (ii) at* $x_0 = 0$ *only for odd* n*. For even* n *they presented a construction with the condition (i) but it is not of minimal degree.*

4.1.2. Pál-type weighted $(0,2;0)$-interpolation on the zeros of Laguerre polynomials. Let $L_n^{(\alpha)}$ denote the Laguerre polynomial of degree n, for which

$$xL_n^{(\alpha)''}(x) + (\alpha+1-x)L_n^{(\alpha)'}(x) + nL_n^{(\alpha)}(x) = 0 \qquad (\alpha > -1)$$

with the normalization

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}. \tag{4.1}$$

It is known [18], that

$$L_n^{(\alpha)'}(x) = -L_{n-1}^{(\alpha+1)}(x),$$

and on the zeros of $L_n^{(\alpha)}$

$$\left(e^{-\frac{x}{2}} x^{\frac{\alpha+1}{2}} L_n^{(\alpha)}\right)''(x_i) = 0 \qquad (i = 1, \ldots, n).$$

Based on Theorem 3.1 we can state the following result.

THEOREM 4.4. *If* $\{x_i\}_{i=1}^n$ *and* $\{\bar{x}_i\}_{i=1}^{n-1}$ *are zeros of* $L_n^{(\alpha)}$ *and* $L_n^{(\alpha)'}$*, respectively,* $x_0 = 0$ *and the weight function is* $w(x) = e^{-\frac{3x}{2}} x^{\frac{3(\alpha+1)}{2}}$*, then the Pál-type weighted* $(0,2;0)$*-interpolation is regular with the additional condition (i), (ii) or (iv) for any* $\alpha > -1$*, but it is regular with (v) only for* $\alpha = -\frac{1}{3}$ *and with (iii) or (vi) only for* $\alpha = \pm\frac{1}{3}$*.*

Proof. We discuss in details the case (i) only, when the additional condition is

$$R_N(0) = y_0,$$

where y_0 is arbitrary real number. The regularity of the problem is approved by

$$(qp_n)(x_0) = (L_n^{(\alpha)'} L_n^{(\alpha)})(0) = -L_{n-1}^{(\alpha+1)}(0) L_n^{(\alpha)}(0) \neq 0.$$

With the notations

$$\ell_k(x) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)'}(x_k)(x-x_k)} \qquad (k = 1, \ldots, n),$$

$$\bar{\ell}_k(x) = \frac{L_n^{(\alpha)'}(x)}{L_n^{(\alpha)''}(\bar{x}_k)(x-\bar{x}_k)} \qquad (k = 1, \ldots, n-1),$$

and by the properties of the Laguerre polynomials we have

$$L_n^{(\alpha)''}(x_k) = \frac{x_k - \alpha - 1}{x_k} L_n^{(\alpha)'}(x_k), \quad \ell_k'(x_k) = \frac{x_k - \alpha - 1}{2x_k} \quad (k = 1, \ldots, n),$$

$$L_n^{(\alpha)''}(\bar{x}_k) = -\frac{n}{\bar{x}_k} L_n^{(\alpha)}(\bar{x}_k), \quad \bar{\ell}_k'(\bar{x}_k) = \frac{\bar{x}_k - \alpha - 2}{2\bar{x}_k} \qquad (k = 1, \ldots, n-1).$$

On using the above properties and Lemma 4.1 with $r = 1$, $q_1 = 1$, $q = \bar{p} = L_n^{(\alpha)}$ and $w_1 = e^{-\frac{x}{2}} x^{\frac{\alpha+1}{2}}$, one can easily verify that

$$\begin{aligned} A_k(x) =& \frac{L_n^{(\alpha)'}(x)}{L_n^{(\alpha)'}(x_k)} \ell_k^2(x) - \frac{L_n^{(\alpha)}(x) L_n^{(\alpha)'}(x)}{2x_k [L_n^{(\alpha)'}(x_k)]^2} \left\{ \frac{2L_n^{(\alpha)}(0)}{x_k L_n^{(\alpha)'}(x_k)} \right. \\ & + \int_0^x \frac{2x_k \ell_k'(t) - (x_k - \alpha - 1)\ell_k(t)}{t - x_k} dt \\ & \left. + \frac{x_k^2 - 2(2n+1+\alpha)x_k + \alpha^2 - 1}{4x_k} \int_0^x \ell_k(t)dt \right\}, \\ B_k(x) =& \frac{L_n^{(\alpha)}(x) L_n^{(\alpha)'}(x)}{2e^{-\frac{3x_k}{2}} x_k^{\frac{3(\alpha+1)}{2}} [L_n^{(\alpha)'}(x_k)]^2} \int_0^x \ell_k(t)dt \end{aligned}$$

for $k = 1, \ldots, n$ and

$$A_0(x) = \frac{L_n^{(\alpha)}(x) L_n^{(\alpha)'}(x)}{L_n^{(\alpha)}(0) L_n^{(\alpha)'}(0)}.$$

To apply the construction in Lemma 2.3 with $q_1 = 1$, we need

$$\frac{q_1(x) w_2'(x)}{w_2(x)} = \frac{\alpha + 1 - x}{x}$$

which is not a polynomial, so we have to modify the numerator in the definition of g_k in order to get a polynomial for C_k. Hence let

$$g_k(x) = \frac{L_n^{(\alpha)}(0) \left[(\alpha + 1 - x)\bar{\ell}_k(x) + 2x\bar{\ell}_k'(x) \right] - (\alpha+1)\bar{\ell}_k(0) L_n^{(\alpha)}(x)}{x L_n^{(\alpha)}(0)},$$

and

$$\begin{aligned} C_k(x) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(\bar{x}_k)} \bar{\ell}_k^2(x) + \frac{\bar{x}_k L_n^{(\alpha)'}(x) L_n^{(\alpha)}(x)}{n[L_n^{(\alpha)}(\bar{x}_k)]^3} & \left\{ -\frac{L_n^{(\alpha)'}(0)}{n\bar{x}_k} \right. \\ & \left. + \int_0^x \frac{L_n^{(\alpha)}(\bar{x}_k) g_k(t) - g_k(\bar{x}_k) L_n^{(\alpha)}(t)}{t - \bar{x}_k} dt \right\} \end{aligned}$$

for $k = 1, \ldots, n-1$. Finally, the minimal degree polynomial R_N is of degree at most $3n-1$, and it can be written in the form

$$R_N(x) = \sum_{k=0}^{n} A_k(x) y_k + \sum_{k=1}^{n} B_k(x) y_k'' + \sum_{k=1}^{n-1} C_k(x) \bar{y}_k. \qquad \square$$

In a similar way we obtain the following result.

THEOREM 4.5. *If $\{x_i\}_{i=1}^n$ and $\{\bar{x}_i\}_{i=0}^{n-1}$ are zeros of $L_n^{(\alpha)}$ and $xL_n^{(\alpha)'}$, respectively, $\bar{x}_0 = x_0 = 0$ and the weight function is $w(x) = e^{-\frac{3x}{2}} x^{\frac{3\alpha+1}{2}}$, then the Pál-type weighted*

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$(0,2;0)$-*interpolation is regular with the additional condition (ii) for any* $\alpha > -1$, *but it is regular with (iii) or (vi) only for* $\alpha = \pm\frac{1}{3}$.

THEOREM 4.6. *If* $\{x_i\}_{i=1}^n$ *and* $\{\bar{x}_i\}_{i=0}^n$ *are zeros of* $L_n^{(1)}$ *and* $xL_n^{(0)}$, *respectively,* $\bar{x}_0 = x_0 = 0$ *and the weight function is* $w(x) = e^{-\frac{3x}{2}}$, *then the Pál-type weighted* $(0,2;0)$-*interpolation is regular with the additional condition (ii), (iii) or (vi).*

Proof. Let $p_n = L_n^{(1)}$, $q_1 = x$, $q_2 = \bar{p} = L_n^{(0)}$, $q = q_1 q_2$, $w_1 = e^{-\frac{x}{2}}$, $w_2 = e^{-x}$. Using (cf. [18])

$$L_n^{(1)}(x) - L_{n-1}^{(1)}(x) = L_n^{(0)}(x)$$

we have

$$\left(w_2 q_2\right)'(x_i) = -e^{-x_i}\left(L_n^{(0)}(x_i) + L_{n-1}^{(1)}(x_i)\right) = -e^{-x_i} L_n^{(1)}(x_i) = 0$$

for $i = 1, \ldots, n$, hence on using Theorem 3.1 with $x_0 = 0$ we can verify the regularity of the $(0,2;0)$-interpolation problem with the additional condition (ii), (iii) or (vi).

Applying (2.7)-(2.11) with

$$\ell_k(x) = \frac{L_n^{(1)}(x)}{L_n^{(1)\prime}(x_k)(x - x_k)}, \qquad \ell_k'(x_k) = \frac{x_k - 2}{2x_k} \qquad (k = 1, \ldots, n),$$
$$\bar{\ell}_k(x) = \frac{L_n^{(0)}(x)}{L_n^{(0)\prime}(\bar{x}_k)(x - \bar{x}_k)}, \qquad \bar{\ell}_k'(\bar{x}_k) = \frac{\bar{x}_k - 1}{2\bar{x}_k} \qquad (k = 1, \ldots, n),$$

we obtain the fundamental polynomials

$$\begin{aligned}
A_k(x) =& \frac{xL_n^{(0)}(x)}{x_k L_n^{(0)}(x_k)}\ell_k^2(x) + \frac{xL_n^{(0)}(x)L_n^{(1)}(x)}{2x_k^2 L_n^{(0)}(x_k)L_n^{(1)\prime}(x_k)}\left\{c_k + b_k\int_0^x L_n^{(1)}(t)dt\right.\\
&\left. + \int_0^x \frac{(x_k - 2)\ell_k(t) - 2x_k\ell_k'(t)}{t - x_k}dt + \left(n + 2 - \frac{x_k}{4}\right)\int_0^x \ell_k(t)dt\right\},\\
B_k(x) =& \frac{xL_n^{(0)}(x)L_n^{(1)}(x)}{2x_k e^{-\frac{3x_k}{2}} L_n^{(0)}(x_k)L_n^{(1)\prime}(x_k)}\left\{\tilde{c}_k + \int_0^x \ell_k(t)dt + \tilde{b}_k\int_0^x L_n^{(1)}(t)dt\right\},\\
C_k(x) =& \frac{x^2 L_n^{(1)}(x)}{\bar{x}_k^2 L_n^{(1)}(\bar{x}_k)}\bar{\ell}_k^2(x) - \frac{xL_n^{(0)}(x)L_n^{(1)}(x)}{\bar{x}_k^2 L_n^{(1)}(\bar{x}_k)L_n^{(0)\prime}(\bar{x}_k)}\\
&\times\left\{\bar{c}_k + \bar{b}_k\int_0^x L_n^{(1)}(t)dt + \int_0^x \frac{(1-t)\bar{\ell}_k(t) + 2t\bar{\ell}_k'(t)}{t - \bar{x}_k}dt\right\}
\end{aligned}$$

for $k = 1, \ldots, n$. The parameters are to be determined from the additional conditions. ☐

Using the fact

$$\left(L_n^{(-1)}\right)'(x) = \left(xL_{n-1}^{(1)}\right)'(x) = nL_{n-1}^{(0)}(x)$$

we can state

COROLLARY 4.7. *On the zeros of* $L_n^{(-1)}$ *the Pál-type weighted* $(0,2;0)$-*interpolation is regular with respect to the weight function* $w(x) = e^{-\frac{3x}{2}}$.

4.1.3. Pál-type weighted $(0,2;0)$-interpolation on the zeros of Jacobi polynomials. Let $y = P_n^{(\alpha,\beta)}$ denote the Jacobi polynomial of degree n, for which

$$(1-x^2)y''(x) + [\beta - \alpha - (\alpha+\beta+2)x]y'(x) + n(n+\alpha+\beta+1)y(x) = 0,$$

where $\alpha, \beta > -1$, with the normalization

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}. \tag{4.2}$$

It is known [18], that

$$P_n^{(\alpha,\beta)'}(x) = \frac{n+\alpha+\beta+1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x),$$

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x),$$

and on the zeros of $P_n^{(\alpha,\beta)}$

$$\left((1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta+1}{2}} P_n^{(\alpha,\beta)}\right)''(x_i) = 0 \qquad (i = 1,\ldots,n).$$

Furthermore let

$$\ell_k(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)'}(x_k)(x-x_k)} \qquad (k = 1,\ldots,n),$$
$$\bar{\ell}_k(x) = \frac{P_n^{(\alpha,\beta)'}(x)}{P_n^{(\alpha,\beta)''}(\bar{x}_k)(x-\bar{x}_k)} \qquad (k = 1,\ldots,n-1),$$

where $\bar{x}_k$ $(k = 1,\ldots,n-1)$ are the zeros of $P_n^{(\alpha,\beta)'}$. On using the properties of the Jacobi polynomials we get

$$P_n^{(\alpha,\beta)''}(x_k) = \frac{\beta-\alpha-(\alpha+\beta+2)x_k}{1-x_k^2} P_n^{(\alpha,\beta)'}(x_k) \qquad (k = 1,\ldots,n),$$

$$P_n^{(\alpha,\beta)''}(\bar{x}_k) = \frac{n(n+\alpha+\beta+1)}{1-\bar{x}_k^2} P_n^{(\alpha,\beta)}(\bar{x}_k) \qquad (k = 1,\ldots,n-1),$$

$$\ell_k'(x_k) = \frac{\beta-\alpha-(\alpha+\beta+2)x_k}{2(1-x_k^2)} \qquad (k = 1,\ldots,n),$$

$$\bar{\ell}_k'(\bar{x}_k) = \frac{\beta-\alpha-(\alpha+\beta+4)\bar{x}_k}{2(1-\bar{x}_k^2)} \qquad (k = 1,\ldots,n-1).$$

Based on Theorem 3.1 and Lemma 4.1 we can state

THEOREM 4.8. *If $\{x_i\}_{i=1}^n$ and $\{\bar{x}_i\}_{i=1}^{n-1}$ are the zeros of $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha,\beta)'}$, respectively, $x_0 = -1$, $x_{n+1} = 1$ and the weight function is*

$$w(x) = (1-x)^{\frac{3(\alpha+1)}{2}}(1+x)^{\frac{3(\beta+1)}{2}},$$

then the Pál-type weighted $(0,2;0)$-interpolation is regular with the additional condition (i), (ii) or (iv) for any $\alpha,\beta>-1$, but it is regular with (v) only for $\beta=-\frac{1}{3}$, and with (iii) or (vi) only for $\beta=\pm\frac{1}{3}$. Furthermore, it is regular with (vii) for any $\alpha,\beta>-1$ if n is even, and for $\alpha\neq\beta$, if n is odd.

THEOREM 4.9. *If $\{x_i\}_{i=1}^n$ and $\{\bar{x}_i\}_{i=0}^{n-1}$ are the zeros of $P_n^{(\alpha,\beta)}$ and $(x+1)P_n^{(\alpha,\beta)\prime}$, respectively, $\bar{x}_0=x_0=-1$ and the weight function is*

$$w(x)=(1-x)^{\frac{3\alpha+1}{2}}(1+x)^{\frac{3(\beta+1)}{2}},$$

then the Pál-type weighted $(0,2;0)$-interpolation is regular with the additional condition (i), (ii) and (iv) for any $\alpha,\beta>-1$, but it is regular with (iii) only for $\beta=\pm\frac{1}{3}$, with (v) only for $\beta=-\frac{1}{3}$, and with (vi) for $\beta=\frac{1}{3}$ if $\alpha>-1$ or $\beta=-\frac{1}{3}$ if $\alpha\neq\beta$.

THEOREM 4.10. *If $\{x_i\}_{i=1}^n$ and $\{\bar{x}_i\}_{i=0}^{n-1}$ are the zeros of $P_n^{(\alpha,\beta)}$ and $(1-x)P_n^{(\alpha,\beta)\prime}$, respectively, $\bar{x}_0=1$, $x_0=-1$ and the weight function is*

$$w(x)=(1-x)^{\frac{3(\alpha+1)}{2}}(1+x)^{\frac{3\beta+1}{2}},$$

then the Pál-type weighted $(0,2;0)$-interpolation is regular with the additional condition (ii) for any $\alpha,\beta>-1$, but it is regular with (iii) or (vi) only for $\beta=-\frac{1}{3}$.

4.2. Pál-type weighted $(0,1,\ldots,r-1;0,2)$-interpolation. Let us consider Hermite-type interpolation problem on the zeros of

$$\bar{p}(x)=c(x-\bar{x}_1)\ldots(x-\bar{x}_n),$$

where the derivatives up to the $(r-1)$-st order are prescribed and weighted $(0,2)$-interpolation on the zeros of $\bar{p}'=p_{n-1}$.

LEMMA 4.11. *If on the zeros of $p_{n-1}=\bar{p}'$ the weight function w satisfies*

$$(q_1w\bar{p}')''(x_i)=0,\qquad w(x_i)\neq 0,\qquad (i=1,\ldots,n-1),\tag{4.3}$$

then the first and second kind fundamental polynomials of the Pál-type weighted $(0,1,\ldots,r-1;0,2)$-interpolation are given by (2.7)–(2.10), where $q=q_1\bar{p}^r$ and $w_2=1$.

Proof. With $w_2=1$ and $q_2=\bar{p}^r$

$$(w_2q_2)'(x_i)=(\bar{p}^r)'(x_i)=r\left(\bar{p}^{r-1}\right)(x_i)p_{n-1}(x_i)=0\qquad(i=1,\ldots,n-1),$$

and we can apply Lemma 2.2. ◻

For $r=1$ we get the Pál-type weighted $(0;0,2)$-interpolation problem, where Lagrange interpolation is prescribed at the zeros of $\bar{p}$ and weighted $(0,2)$-interpolation at the zeros of $\bar{p}'$. Based on Lemma 4.11 we can apply (2.7)-(2.11) for the fundamental polynomials.

THEOREM 4.12. *If on the zeros of $\bar{p}'$ the weight function w satisfies (4.3) and $q_1(x_i)\neq 0$ $(i=1,\ldots,n-1)$, then for the Pál-type weighted $(0;0,2)$-interpolation we obtain*

$$\begin{aligned}A_k(x)=\frac{(q_1\bar{p})(x)}{(q_1\bar{p})(x_k)}\ell_k^2(x)+\frac{(q_1\bar{p}\bar{p}')(x)}{(q_1\bar{p}\bar{p}'')(x_k)}\Bigg\{c_k+\int_{x_0}^{x}\frac{\ell_k'(x_k)\ell_k(t)-\ell_k'(t)}{t-x_k}dt\\+b_k\bar{p}(x)-\left(\frac{(wq_1)''(x_k)}{2(wq_1)(x_k)}+\frac{\bar{p}''(x_k)}{2\bar{p}(x_k)}\right)\int_{x_0}^{x}\ell_k(t)dt\Bigg\},\\B_k(x)=\frac{(q_1\bar{p}\bar{p}')(x)}{2(wq_1\bar{p}\bar{p}'')(x_k)}\left\{\tilde{c}_k+\tilde{b}_k\bar{p}(x)+\int_{x_0}^{x}\ell_k(t)dt\right\}\end{aligned}$$

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for $k = 1, \ldots, n-1$, *where*

$$\ell_k(x) = \frac{\bar{p}'(x)}{\bar{p}''(x_k)(x - x_k)},$$

and

$$C_k(x) = \frac{(q_1^2\bar{p}')(x)}{(q_1^2\bar{p}')(\bar{x}_k)}\bar{\ell}_k^2(x) - \frac{(q_1\bar{p}\,\bar{p}')(x)}{(q_1^2[\bar{p}']^3)\,(\bar{x}_k)}\Big\{\bar{c}_k + \bar{b}_k\bar{p}(x) + \int_{x_0}^{x}\frac{\bar{p}'(\bar{x}_k)g_k(t) - g_k(\bar{x}_k)\bar{p}'(t)}{t - \bar{x}_k}dt\Big\}$$

for $k = 1, \ldots, n$, *where*

$$g_k(x) = q_1'(x)\bar{\ell}_k(x) + 2q_1(x)\bar{\ell}_k'(x),$$

$$\bar{\ell}_k(x) = \frac{\bar{p}(x)}{\bar{p}'(\bar{x}_k)(x - \bar{x}_k)}.$$

4.2.1. Pál-type weighted $(0;0,2)$-interpolation on the zeros of Hermite polynomials. Let $\{\bar{x}_i\}_{i=1}^{n}$ and $\{x_i\}_{i=1}^{n-1}$ be the zeros of H_n and H_n', respectively, where H_n denotes the Hermite polynomial of degree n. Now

$$\left(e^{-\frac{x^2}{2}}H_n'\right)''(x_i) = 0 \qquad (i = 1, \ldots, n-1),$$

hence with $q = \bar{p} = H_n$, $p_{n-1} = \bar{p}'$ and $w = e^{-\frac{x^2}{2}}$ we can apply Theorem 4.12 to get the fundamental polynomials A_k, B_k $(k = 1, \ldots, n-1)$, and C_k $(k = 1, \ldots, n)$.

THEOREM 4.13. *If* $\{\bar{x}_i\}_{i=1}^{n}$ *and* $\{x_i\}_{i=1}^{n-1}$ *are zeros of* H_n *and* H_n', *respectively,* $x_0 = 0$ *and the weight function is* $w(x) = e^{-\frac{x^2}{2}}$, *then the Pál-type weighted* $(0;0,2)$*-interpolation is regular with the additional condition (ii) for any* n *or with (vi) for odd* n, *but it is not regular with (i), (iii), (iv) or (v).*

Proof. We study only the case (ii), the other cases can be discussed in a similar way. In case (ii) the additional condition is

$$R_N'(0) = y_0'$$

where y_0' is arbitrary real number, and the regularity is assured by

$$(qp_{n-1})\,(x_0) = (H_nH_n')'\,(0) = [H_n'(0)]^2 + H_n(0)H_n''(0) \neq 0.$$

On using the properties of the Hermite-polynomials and Theorem 4.11 with $q_1 = 1$ and $\bar{p} = H_n$ one can verify that the minimal degree polynomial R_N is

$$R_N(x) = \sum_{k=1}^{n-1}A_k(x)y_k + \sum_{k=1}^{n-1}B_k(x)y_k'' + \sum_{k=1}^{n}C_k(x)\bar{y}_k + D_0(x)y_0'$$

of degree at most $3n-2$, where

$$A_k(x) = \frac{H_n(x)}{H_n(x_k)}\ell_k^2(x) - \frac{H_n(x)H_n'(x)}{2nH_n^2(x_k)}\Big\{c_k + \int_0^x\frac{x_k\ell_k(t) - \ell_k'(t)}{t - x_k}dt + \frac{2n+1-x_k^2}{2}\int_0^x\ell_k(t)dt\Big\},$$

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$$B_k(x) = -\frac{H_n(x)H_n'(x)}{4ne^{-x_k^2/2}H_n^2(x_k)}\int_0^x \ell_k(t)dt$$

for $k = 1, \dots, n-1$,

$$C_k(x) = \frac{H_n'(x)}{H_n'(\bar{x}_k)}\bar{\ell}_k^2(x) - \frac{2H_n(x)H_n'(x)}{[H_n'(\bar{x}_k)]^3}\left\{\bar{c}_k + \int_0^x \frac{H_n'(\bar{x}_k)\bar{\ell}_k'(t) - \bar{x}_k H_n'(t)}{t - \bar{x}_k}dt\right\}$$

for $k = 1, \dots, n$, and

$$D_0(x) = \frac{H_n(x)H_n'(x)}{(H_n'H_n)'(0)},$$

where

$$\ell_k(x) = \frac{H_n'(x)}{H_n''(x_k)(x - x_k)} \qquad (k = 1, \dots, n-1),$$
$$\bar{\ell}_k(x) = \frac{H_n(x)}{H_n'(\bar{x}_k)(x - \bar{x}_k)} \qquad (k = 1, \dots, n)$$

and

$$c_k = \begin{cases} \frac{H_n'(0)}{2nx_k^2 H_n(x_k)} & \text{for odd } n, \\ 0 & \text{for even } n, \end{cases} \qquad \bar{c}_k = \begin{cases} 0 & \text{for odd } n, \\ \frac{H_n(0)}{2\bar{x}_k^2} & \text{for even } n. \end{cases} \qquad \square$$

REMARK 4.14. *P. Mathur and S. Datta [12] stated the regularity of the weighted $(0;0,2)$-interpolation on the zeros of the Hermite polynomials with the condition (ii) at $x_0 = 0$ only for odd n. For even n they presented a construction with the condition (i) but it is not of minimal degree.*

4.2.2. Pál-type weighted $(0;0,2)$-interpolation on the zeros of Laguerre polynomials. Let $\{\bar{x}_i\}_{i=1}^n$ and $\{x_i\}_{i=1}^{n-1}$ be the zeros of $L_n^{(\alpha)}$ and $L_n^{(\alpha)'}$, respectively, where $L_n^{(\alpha)}$ is the Laguerre polynomial of degree n with the normalization (4.1). Now

$$\left(e^{-\frac{x}{2}}x^{\frac{\alpha}{2}+1}L_n^{(\alpha)'}\right)''(x_i) = 0 \qquad (i = 1, \dots, n-1),$$

hence we can apply Theorem 4.12 in order to get the fundamental polynomials A_k, B_k $(k = 1, \dots, n-1)$, and C_k $(k = 1, \dots, n)$.

Let us use the notations

$$\ell_k(x) = \frac{L_n^{(\alpha)'}(x)}{L_n^{(\alpha)''}(x_k)(x - x_k)} \qquad (k = 1, \dots, n-1),$$
$$\bar{\ell}_k(x) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)'}(\bar{x}_k)(x - \bar{x}_k)} \qquad (k = 1, \dots, n).$$

THEOREM 4.15. *If $\{\bar{x}_i\}_{i=1}^n$ and $\{x_i\}_{i=1}^{n-1}$ are zeros of $L_n^{(\alpha)}$ and $L_n^{(\alpha)'}$, respectively, $x_0 = 0$ and the weight function is $w(x) = e^{-\frac{x}{2}}x^{\frac{\alpha}{2}+1}$, then the Pál-type weighted $(0;0,2)$-interpolation is regular with the additional condition (i), (ii) or (iv) for any $\alpha > -1$, but it is regular with (v) only for $\alpha = 0$ and with (iii) or (vi) only for $\alpha = 0$ and 2.*

Proof. For the regularity we apply Theorem 3.1 with $q = L_n^{(\alpha)}$, $p_{n-1} = L_n^{(\alpha)'}$, $x_0 = 0$ and $w = e^{-\frac{x}{2}}x^{\frac{\alpha}{2}+1}$. In regular cases we get the fundamental polynomials from Theorem 4.12 with $q_1 = 1$, $w_2 = 1$.

For the sake of simplicity we discuss in details only the case (i), when the additional condition is

$$R_N(0) = y_0,$$

where y_0 is arbitrary real number. The regularity of the problem is approved by

$$(qp_{n-1})(0) = \left(L_n^{(\alpha)} L_n^{(\alpha)'}\right)(0) = -L_n^{(\alpha)}(0) L_{n-1}^{(\alpha+1)}(0) \neq 0.$$

On using the properties of the Laguerre-polynomials and Theorem 4.11 with $q_1 = 1$ and $\bar{p} = L_n^{(\alpha)}$ one can verify that the minimal degree polynomial R_N is

$$R_N(x) = \sum_{k=0}^{n-1} A_k(x) y_k + \sum_{k=1}^{n-1} B_k(x) y_k'' + \sum_{k=1}^{n} C_k(x) \bar{y}_k$$

of degree at most $3n-2$, where

$$\begin{aligned} A_k(x) =& \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(x_k)} \ell_k^2(x) - \frac{L_n^{(\alpha)}(x) L_n^{(\alpha)'}(x)}{2n[L_n^{(\alpha)}(x_k)]^2} \left\{ \frac{2L_n^{(\alpha)'}(0)}{nL_n^{(\alpha)}(x_k)} \right. \\ &+ \int_0^x \frac{(x_k - \alpha - 2)\ell_k(t) - 2x_k \ell_k'(t)}{t - x_k} dt \\ &\left. - \frac{x_k^2 - 2(2n+2+\alpha)x_k + \alpha(\alpha+2)}{4x_k} \int_0^x \ell_k(t) dt \right\}, \\ B_k(x) =& \frac{-L_n^{(\alpha)}(x) L_n^{(\alpha)'}(x)}{2n e^{-\frac{x_k}{2}} x_k^{\frac{\alpha}{2}} [L_n^{(\alpha)}(x_k)]^2} \int_0^x \ell_k(t) dt \end{aligned}$$

for $k = 1, \ldots, n-1$,

$$A_0(x) = \frac{L_n^{(\alpha)}(x) L_n^{(\alpha)'}(x)}{L_n^{(\alpha)}(0) L_n^{(\alpha)'}(0)},$$

and

$$\begin{aligned} C_k(x) = & \frac{L_n^{(\alpha)'}(x)}{L_n^{(\alpha)'}(\bar{x}_k)} \bar{\ell}_k^2(x) - \frac{L_n^{(\alpha)}(x) L_n^{(\alpha)'}(x)}{\bar{x}_k [L_n^{(\alpha)'}(\bar{x}_k)]^3} \left\{ \frac{L_n^{(\alpha)}(0)}{\bar{x}_k} \right. \\ & \left. + \int_0^x \frac{2\bar{x}_k L_n^{(\alpha)'}(\bar{x}_k) \bar{\ell}_k'(t) - (\bar{x}_k - \alpha - 1) L_n^{(\alpha)'}(t)}{t - \bar{x}_k} dt \right\} \end{aligned}$$

for $k = 1, \ldots, n$. □

In a similar way we obtain the following result.

THEOREM 4.16. *If $\{\bar{x}_i\}_{i=0}^n$ and $\{x_i\}_{i=1}^{n-1}$ are zeros of $xL_n^{(\alpha)}$ and $L_n^{(\alpha)'}$, respectively, $\bar{x}_0 = x_0 = 0$ and the weight function is $w(x) = e^{-\frac{x}{2}} x^{\frac{\alpha}{2}}$, then the Pál-type weighted $(0; 0, 2)$-interpolation is regular with the additional condition (ii) for any $\alpha > -1$, but it is regular with (vi) only for $\alpha = 0$ and with (iii) only for $\alpha = 0$ and 2.*

Proof. On using Theorem 4.12 with $q_1 = x$ and $q_2 = \bar{p} = L_n^{(\alpha)}$ one can easily verify that

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$$\begin{aligned} A_k(x) =& \frac{xL_n^{(\alpha)}(x)}{x_kL_n^{(\alpha)}(x_k)}\ell_k^2(x) \\ & -\frac{xL_n^{(\alpha)}(x)L_n^{(\alpha)'}(x)}{2nx_k[L_n^{(\alpha)}(x_k)]^2}\left\{c_k+\int_0^x\frac{(x_k-\alpha-2)\ell_k(t)-2x_k\ell_k'(t)}{t-x_k}dt\right. \\ & \left.\qquad -\frac{x_k^2-2(2n+2+\alpha)x_k+\alpha(\alpha+2)}{4x_k}\int_0^x\ell_k(t)dt\right\}, \\ B_k(x) =& \frac{-xL_n^{(\alpha)}(x)L_n^{(\alpha)'}(x)}{2ne^{-\frac{x_k}{2}}x_k^{\frac{\alpha}{2}}[L_n^{(\alpha)}(x_k)]^2}\left\{\tilde{c}_k+\int_0^x\ell_k(t)dt\right\} \end{aligned}$$

for $k=1,\dots,n-1$, and

$$\begin{aligned} C_k(x) =& \frac{x^2L_n^{(\alpha)'}(x)}{\bar{x}_k^2L_n^{(\alpha)'}(\bar{x}_k)}\bar{\ell}_k^2(x)-\frac{xL_n^{(\alpha)}(x)L_n^{(\alpha)'}(x)}{\bar{x}_k^2[L_n^{(\alpha)'}(\bar{x}_k)]^3} \\ & \times\left\{\bar{c}_k+\int_0^x\frac{[\bar{\ell}_k(t)+2t\bar{\ell}_k'(t)]L_n^{(\alpha)'}(\bar{x}_k)-(\bar{x}_k-\alpha)L_n^{(\alpha)'}(t)}{t-\bar{x}_k}dt\right\} \end{aligned}$$

for $k=1,\dots,n$, where the parameters are to be determined by the additional conditions. □

4.2.3. Pál-type weighted $(0;0,2)$-interpolation on the zeros of Jacobi polynomials. Let $\{\bar{x}_i\}_{i=1}^n$ and $\{x_i\}_{i=1}^{n-1}$ be the zeros of $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha,\beta)'}$, respectively, where $P_n^{(\alpha,\beta)}$ denotes the Jacobi polynomial of degree n with the normalization (4.2). Now

$$\left((1-x)^{\frac{\alpha}{2}+1}(1+x)^{\frac{\beta}{2}+1}P_n^{(\alpha,\beta)'}\right)''(x_i)=0 \qquad (i=1,\dots,n-1),$$

hence we can obtain similar results to those in Section 4.1.3. Here we give details of the case $\alpha=\beta=0$ only.

THEOREM 4.17. *Let $\{\bar{x}_i\}_{i=1}^n$ and $\{x_i\}_{i=1}^{n-1}$ be the zeros of P_n and P_n', respectively, where P_n denotes the Legendre polynomial of degree n, and $\bar{x}_0=-1$, $\bar{x}_{n+1}=1$. If $x_0=-1$, $x_n=1$, and the weight function is $w(x)=1$, then on the nodes $\{\bar{x}_i\}_{i=0}^{n+1}$ and $\{x_i\}_{i=1}^{n-1}$ the Pál-type weighted $(0;0,2)$-interpolation is regular with the additional condition (ii), (iii) or (vi) for any n, but it is regular with (viii) or (ix) only for odd n.*

Proof. Let $n:=n-1$, $p_{n-1}=P_n'$, $q_1=1-x^2$, $q_2=\bar{p}=P_n$, $q=q_1q_2$, $w_1=w_2=1$. As (2.5) is fulfilled, on using Theorem 3.1 with $x_0=-1$ and $x_n=1$ we can verify the regularity of the $(0,2;0)$-interpolation problem with the additional conditions.

For the fundamental polynomials applying (2.7)-(2.11) with

$$\begin{aligned} \ell_k(x) &= \frac{P_n'(x)}{P_n''(x_k)(x-x_k)}, \quad \ell_k'(x_k)=\frac{2x_k}{1-x_k^2} \quad (k=1,\dots,n-1) \\ \bar{\ell}_k(x) &= \frac{P_n(x)}{P_n'(\bar{x}_k)(x-\bar{x}_k)}, \quad \bar{\ell}_k'(\bar{x}_k)=\frac{\bar{x}_k}{1-\bar{x}_k^2} \quad (k=1,\dots,n) \end{aligned}$$

we obtain

$$\begin{aligned} A_k(x) =& \frac{(1-x^2)P_n(x)}{(1-x_k^2)P_n(x_k)}\ell_k^2(x)-\frac{(1-x^2)P_n(x)P_n'(x)}{n(n+1)(1-x_k^2)P_n^2(x_k)}\left\{c_k+b_kP_n(x)\right. \\ & \left.+\frac{n^2+n+2}{2}\int_{-1}^x\ell_k(t)dt+\int_{-1}^x\frac{2x_k\ell_k(t)-(1-x_k^2)\ell_k'(t)}{t-x_k}dt\right\}, \end{aligned}$$

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$$B_k(x) = \frac{(x^2-1)P_n(x)P_n'(x)}{2n(n+1)P_n^2(x_k)} \left\{ \tilde{c}_k + \tilde{b}_k P_n(x) + \int_{-1}^{x} \ell_k(t)dt \right\},$$

for $k = 1, \ldots, n-1$,

$$C_k(x) = \frac{(1-x^2)^2 P_n'(x)}{(1-\bar{x}_k^2)^2 P_n'(\bar{x}_k)} \bar{\ell}_k^2(x) + \frac{2(1-x^2)P_n(x)P_n'(x)}{(1-\bar{x}_k^2)^2 [P_n'(\bar{x}_k)]^2} \Big\{ \bar{c}_k + \bar{b}_k P_n(x) + \int_{-1}^{x} \frac{t\bar{\ell}_k(t) - (1-t^2)\bar{\ell}_k'(t)}{t - \bar{x}_k} dt \Big\}$$

for $k = 1, \ldots, n$. The parameters are to be determined from the additional conditions. □

Using the facts

$$\frac{d}{dx}\left[(1-x^2)P_{n-1}'(x)\right] = -n(n-1)P_{n-1}(x),$$

and

$$P_n^{(-1)}(x) = (1-x^2)P_{n-1}'(x) = -n(n-1)\int_{-1}^{x} P_{n-1}(t)dt \qquad (4.4)$$

we can state

COROLLARY 4.18. *On the zeros of the integrated Legendre polynomial* (4.4) *the weighted Pál-type* $(0, 2; 0)$*-interpolation is regular with respect to the weight function* $w(x) = 1$ *if and only if* n *is even.*

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