

Orthonormal systems on the interval $[0, 1[$

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May 23, 2008
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Walsh functions

The term of **Walsh functions** refers to one of three orthonormal systems which differ only from their enumerations.

- **The original Walsh system** J. L. Walsh (1923)
was generated recursively, it is the Hadamard transform of the Haar system.
- **The Walsh-Paley system** R. E. A. C. Paley (1932)
is the finite products of Rademacher functions.
- **The Walsh-Kaczmarz system** A. A. Šneider (1948)
is also the finite products of Rademacher functions, but in different order.

Theorem

The Walsh system is an orthonormal and complete system on $L^2([0, 1])$, taking on only the values $+1$ and -1 .

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The Rademacher functions

H. A. Rademacher (1922)

$$r(x) = \operatorname{sgn}(\sin(2^{n+1}\pi x)) \quad x \in [0, 1[.$$

The binary expansion of n : (n_0, n_1, \dots)

Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k 2^k, \quad \text{where } n_k = 0 \text{ or } n_k = 1.$$

The Walsh-Paley system

$$\omega_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (x \in [0, 1[).$$

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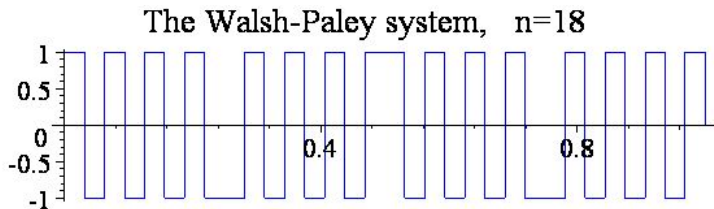
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The Walsh-Paley system

Plotted by Maple



The characters of the Diadic group

The Diadic group $\left(G := \prod_{k=0}^{\infty} \mathbb{Z}_2 \right)$

is the complete product of cyclic groups of order 2, with discrete topology and assign each singleton the measure $\frac{1}{2}$. G has the product topology and measure. (Haar measure)

The system of characters

Define $\varphi(x) = (-1)^x$, ($x \in \mathbb{Z}_2$). For each $n \in \mathbf{N}$ with binary expansion (n_0, n_1, \dots) let

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi^{n_k}(x_k), \quad (x = (x_0, x_1, \dots) \in G).$$

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The system of characters is an orthonormal and complete system on $L^2(G)$.

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The representation of the Diadic group on $[0, 1[$

The Fine's map

N. J. Fine (1949)

For any $x \in [0, 1[$ there exists a sequence of numbers 0 and 1 such that

$$x := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \quad ((x_0, x_1, \dots) \in G),$$

but only the numbers $p/2^n$ have two expressions of this form. In this case we have the one which terminates in 0's. Define **Fine's map** by

$$\rho(x) = (x_0, x_1, \dots) \in G.$$

Theorem

The dyadic group is metrizable by the norm

$$|x| := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \in [0, 1[\quad ((x_0, x_1, \dots) \in G).$$

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The representation of the Diadic group on $[0, 1[$

Fine's map gives a natural relation between the new structure of $[0, 1[$ and the structure of G (Harmonic analysis).

Theorem

Let ρ denote the Fine's map. If f is integrable on G then $f \circ \rho$ is Lebesgue integrable and

$$\int_G f d\mu = \int_0^1 (f \circ \rho)(x) dx.$$

Conversely, if g is Lebesgue integrable and f is defined by $f(x) := g(|x|)$ ($x \in G$) then f is integrable on G and

$$\int_0^1 g(x) dx = \int_G f d\mu.$$

- The Haar measure corresponds to the Lebesgue measure.
- The characters of G corresponds to the Walsh-Paley system.

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A Vilenkin group $\left(G := \prod_{k=0}^{\infty} \mathbb{Z}_{m_k} \right)$

N. Ja. Vilenkin (1947)

is the complete product of cyclic groups of order m_k ($m_k \geq 2$, $k \in \mathbf{N}$), with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. G has the product topology and measure. (Haar measure)

Bounded Vilenkin group

if the sequence $m = (m_0, m_1, \dots)$ is a bounded sequence.

The generalized Rademacher functions

$$\varphi_k^s(x) = \exp(2\pi i s x / m_k) \quad (s \in \{0, \dots, m_k - 1\}, x \in \mathbb{Z}_{m_k}, i^2 = -1)$$

The generalized Rademacher functions are the characters of cyclic groups.

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The functions of the Vilenkin system are the characters of the Vilenkin group, thus it is an orthonormal and complete system on $L^2(G)$.

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The complete product of finite groups

The group $\left(G := \prod_{k=0}^{\infty} G_k \right)$

Denote by G the complete product of arbitrary finite groups of order m_k ($m_k \geq 2$, $k \in \mathbf{N}$), with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. G has the product topology and measure. (Haar measure)

$$\varphi_k^s = ?, \psi_n = ?$$

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Harmonic Analysis

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$\varphi_k^s = ?, \psi_n = ? \quad \rightarrow \quad$ Harmonic Analysis

The dual object (Σ_k) of the finite group G_k ($k \in \mathbf{N}$)

is the set of all continuous irreducible unitary representations of the group G_k which are not equivalent.

The Coordinate functions

For any $\sigma \in \Sigma_k$, let $\{\xi_1, \dots, \xi_{d_\sigma}\}$ be a fixed basis of the representation space of a representation $U^{(\sigma)}$ in the class σ having the dimension d_σ .

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Orthonormal systems on finite groups

The system φ_k

We order the all normalized coordinate functions of the finite group G_k ($\varphi_k^0(x) = 1$) to obtain exactly m_k number of functions.

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \quad (x \in G_k, s = 0, \dots, m_k - 1),$$

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Example 1: The permutation group of 3 elements, S_3

| | e | (12) | (13) | (23) | (123) | (132) | $\ \varphi^s\ _1$ | $\ \varphi^s\ _\infty$ |
|-------------|------------|-------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|------------------------|
| φ^0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| φ^1 | 1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 |
| φ^2 | $\sqrt{2}$ | $-\sqrt{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $\frac{2\sqrt{2}}{3}$ | $\sqrt{2}$ |
| φ^3 | $\sqrt{2}$ | $\sqrt{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $\frac{2\sqrt{2}}{3}$ | $\sqrt{2}$ |
| φ^4 | 0 | 0 | $-\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{2}$ | $-\frac{\sqrt{6}}{2}$ | $\frac{\sqrt{6}}{3}$ | $\frac{\sqrt{6}}{2}$ |
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$$\max_{s=0\dots 5} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3}$$

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Example 2: The quaternion group of order 8:

$$\mathcal{Q}_2 := \{[a, b] : a^4 = e, b^2 = a^2, bab^{-1} = a^3\}$$

| | e | a | a^2 | a^3 | b | ab | a^2b | a^3b | $\ \varphi^s\ _1$ | $\ \varphi^s\ _\infty$ |
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| φ^3 | 1 | -1 | 1 | -1 | -1 | 1 | -1 | 1 | 1 | 1 |
| φ^4 | $\sqrt{2}$ | $\sqrt{2}i$ | $-\sqrt{2}$ | $-\sqrt{2}i$ | 0 | 0 | 0 | 0 | $\frac{\sqrt{2}}{2}$ | $\sqrt{2}$ |
| φ^5 | $\sqrt{2}$ | $-\sqrt{2}i$ | $-\sqrt{2}$ | $\sqrt{2}i$ | 0 | 0 | 0 | 0 | $\frac{\sqrt{2}}{2}$ | $\sqrt{2}$ |
| φ^6 | 0 | 0 | 0 | 0 | $\sqrt{2}$ | $-\sqrt{2}i$ | $-\sqrt{2}$ | $\sqrt{2}i$ | $\frac{\sqrt{2}}{2}$ | $\sqrt{2}$ |
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$$\max_{s=0\dots 7} \|\varphi^s\|_1 \|\varphi^s\|_\infty = 1$$

Example 2: The quaternion group of order 8:

$$\mathcal{Q}_2 := \{[a, b] : a^4 = e, b^2 = a^2, bab^{-1} = a^3\}$$

| | e | a | a^2 | a^3 | b | ab | a^2b | a^3b | $\ \varphi^s\ _1$ | $\ \varphi^s\ _\infty$ |
|-------------|------------|--------------|-------------|--------------|-------------|--------------|-------------|-------------|----------------------|------------------------|
| φ^0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| φ^1 | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 |
| φ^2 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | 1 |
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Representative product systems

The m -adic expansion of n : (n_0, n_1, \dots)

Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, ($k \in \mathbf{N}$). Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \leq n_k < m_k).$$

A representative product systems

G. Gát and R. Toledo (1996)

is the product system of φ :

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A representative product system is an orthonormal and complete system on $L^2(G)$.

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The representation of G on $[0, 1[$

It is similar to the dyadic group, but first we need to enumerate the elements of all groups G_k , ($k \in \mathbf{N}$) in an arbitrary way but the first is always their identity.

$$G_k \ni x \xrightarrow{\text{bijection}} \bar{x} \in \{0, 1, \dots, m_k - 1\}, \quad \bar{e} = 0.$$

Fine's map and norm

With the bijection above we can introduce the **Fine's map**:

$$\rho(x) = (x_0, x_1, \dots) \in G.$$

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Fine's map gives a natural relation between the new structure of $[0, 1[$ and the structure of G (Harmonic analysis).

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Let ρ denote the Fine's map. If f is integrable on G then $f \circ \rho$ is Lebesgue integrable and

$$\int_G f d\mu = \int_0^1 (f \circ \rho)(x) dx.$$

Conversely, if g is Lebesgue integrable and f is defined by $f(x) := g(|x|)$ ($x \in G$) then f is integrable on G and

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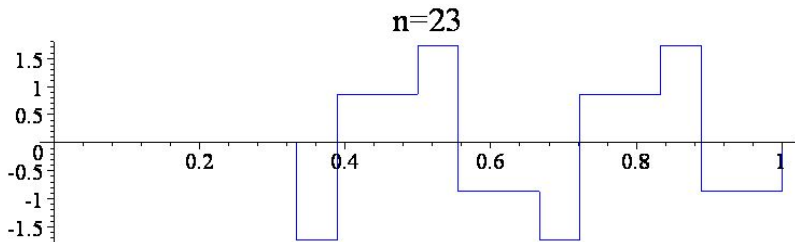
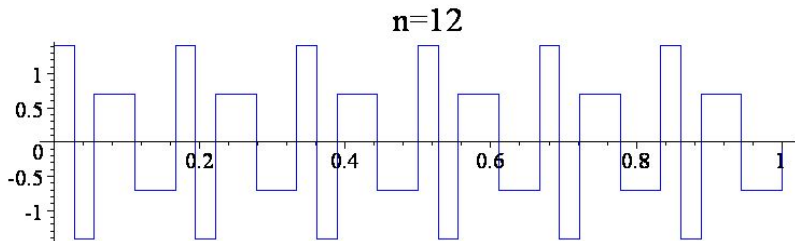
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The complete product of S_3

Plotted by Maple



Divergence in L^p -norm of Fourier series

The n -th partial sums of Fourier series

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \quad (n \in \mathbf{N}), \quad \text{where } \widehat{f}_k := \int_G f \overline{\psi}_k d\mu.$$

The sequence Ψ

$$\Psi_k = \prod_{i=0}^{k-1} \max_{s < m_i} \|\varphi_i^s\|_1 \|\varphi_i^s\|_\infty \quad (k \in \mathbf{N}).$$

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If G is a bounded group with unbounded sequence Ψ , then for all $p \neq 2$ there exists a function $f \in L^p(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^p -norm.

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$$\sigma_n^\alpha f = \frac{1}{A_n^\alpha} \sum_{k=0}^n A_{n-k}^{\alpha-1} S_k f, \quad \text{where } A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!}, \quad (n \in \mathbf{P}).$$

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Let G be a bounded group,

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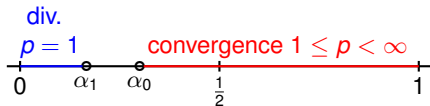
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Summary of results

- G is bounded group
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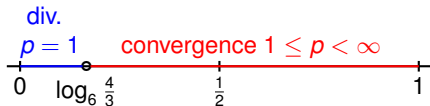


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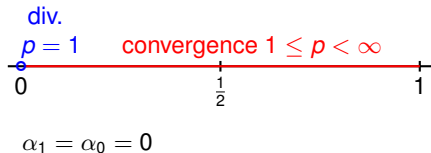
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