Convergence in L^p-norm on the complete product of finite noncommutative groups

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Sep 29, 2008
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Walsh functions

The term of Walsh functions refers to one of three orthonormal systems which differ only from their enumerations.

- The original Walsh system J. L. Walsh (1923)
  was generated recursively, it is the Hadamard transform of the Haar system.
- The Walsh-Paley system R. E. A. C. Paley (1932)
  is the finite products of Rademacher functions.
- The Walsh-Kaczmarz system A. A. Šneider (1948)
  is also the finite products of Rademacher functions, but in different order.

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The Walsh system is an orthonormal and complete system on $L^2([0, 1[)$, *taking on only the values* $+1$ *and* -1.

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The Rademacher functions H. A. Rademacher (1922)

$$r(x) = \mathrm{sgn}(\sin(2^{n+1}\pi x)) \qquad x \in [0,1[.$$

The binary expansion of n: $(n_0, n_1, \dots)$

Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k 2^k, \qquad \text{where } n_k = 0 \text{ or } n_k = 1.$$

The Walsh-Paley system

$$\omega_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (x \in [0,1[).$$

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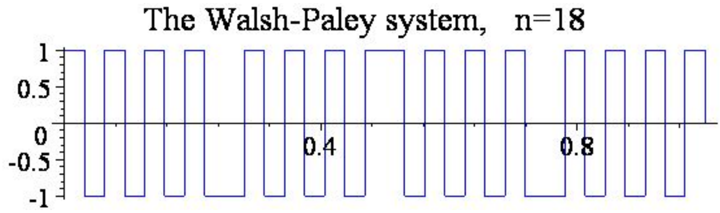
The Walsh-Paley system
Plotted by Maple



The characters of the Dyadic group

The Dyadic group $\left(G := \bigtimes_{k=0}^{\infty} \mathbb{Z}_2\right)$

is the complete product of cyclic groups of order 2, with discrete topology and assign each singleton the measure $\frac{1}{2}$. G has the product topology and measure. (Haar measure)

The system of characters

Define $\varphi(x) = (-1)^x$, $(x \in \mathbb{Z}_2)$. For each $n \in \mathbf{N}$ with binary expansion $(n_0, n_1, \dots)$ let

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi^{n_k}(x_k), \quad (x = (x_0, x_1, \dots) \in G).$$

Theorem

The system of characters is an orthonormal and complete system on $L^2(G)$.

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The representation of the Dyadic group on $[0,1[$

The Fine's map N. J. Fine (1949)

For any $x \in [0,1[$ there exists a sequence of numbers 0 and 1 such that

$$x := \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}} \qquad ((x_0, x_1, \dots) \in G),$$

but only the numbers $p/2^n$ have two expressions of this form. In this case we have the one which terminates in 0's. Define Fine's map by

$$\rho(x) = (x_0, x_1, \dots) \in G.$$

Fine's map gives a natural relation between the new structure of $[0,1[$ and the structure of G (Harmonic analysis).

- The Haar measure corresponds to the Lebesgue measure.
- The characters of G corresponds to the Walsh-Paley system.

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is the complete product of cyclic groups of order m_k ($m_k \geq 2$, $k \in \mathbf{N}$), with discrete topology and assign each singleton the measure $\frac{1}{m_k}$. G has the product topology and measure. (Haar measure)

Bounded Vilenkin group

if the sequence $m = (m_0, m_1, \dots)$ is a bounded sequence.

The generalized Rademacher functions

$$\varphi_k^s(x) = \exp(2\pi\imath sx/m_k) \qquad (s \in \{0, \dots m_k - 1\},\ x \in \mathcal{Z}_{m_k},\ \imath^2 = -1)$$

The generalized Rademacher functions are the characters of cyclic groups.

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Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, $(k \in \mathbf{N})$. Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \le n_k < m_k).$$

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is the product system of φ:

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Theorem

The functions of the Vilenkin system are the characters of the Vilenkin group, thus it is an orthonormal and complete system on $L^2(G)$.

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The group $\left(G := \bigtimes_{k=0}^{\infty} G_k\right)$

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$\varphi_k^s =?, \psi_n =?$ $\longrightarrow$ Harmonic Analysis

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Orthonormal systems on finite groups

The dual object (Σ_k) of the finite group G_k $(k \in \mathbf{N})$

is the set of all continuous irreducible unitary representations of the group G_k which are not equivalents.

The Coordinate functions

For any $\sigma \in \Sigma_k$, let $\{\xi_1, \ldots, \xi_{d_\sigma}\}$ be a fixed basis of the representation space of a representation $U^{(\sigma)}$ in the class σ having the dimension d_σ. The Coordinate functions:

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j \rangle, \quad i, j \in \{1, ..., d_\sigma\}, \sigma \in \Sigma_k$$

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The system φ_k

We order the all normalized coordinate functions of the finite group G_k ($\varphi_k^0(x) = 1$) to obtain exactly m_k number of functions.

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_k,\ s = 0, \ldots, m_k - 1),$$

where $\sigma \in \Sigma_k$, $i, j \in \{1, ..., d_\sigma\}$.

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	e	(12)	(13)	(23)	(123)	(132)	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

$$\max_{s=0\ldots5} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3}$$

	e	(12)	(13)	(23)	(123)	(132)	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1
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Example 1: The permutation group of 3 elements, $\mathcal{S}_3$

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φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
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$$\max_{s=0\ldots5} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3}$$

Example 2: The quaternion group of order 8: $\mathcal{Q}_2 := \{[a,b] : a^4 = e,\ b^2 = a^2,\ bab^{-1} = a^3\}$

	e	a	a^2	a^3	b	ab	a^2b	a^3b	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1	1	1
φ^2	1	-1	1	-1	1	-1	1	-1	1	1
φ^3	1	-1	1	-1	-1	1	-1	1	1	1
φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}\imath$	$\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$

$$\max_{s=0\ldots7} \|\varphi^s\|_1 \|\varphi^s\|_\infty = 1$$

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φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}\imath$	$\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$

$$\max_{s=0\ldots7} \lVert\varphi^s\rVert_1 \lVert\varphi^s\rVert_\infty = 1$$

Example 2: The quaternion group of order 8: $\mathcal{Q}_2 := \{[a,b] : a^4 = e,\ b^2 = a^2,\ bab^{-1} = a^3\}$

	e	a	a^2	a^3	b	ab	a^2b	a^3b	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1	1	1
φ^2	1	-1	1	-1	1	-1	1	-1	1	1
φ^3	1	-1	1	-1	-1	1	-1	1	1	1
φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
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φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
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Representative product systems

The m-adic expansion of n: $(n_0, n_1, \dots)$

Denote $M_0 := 1$ and $M_{k+1} := m_k M_k$, $(k \in \mathbf{N})$. Given $n \in \mathbf{N}$ it is possible to write n uniquely as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad (0 \leq n_k < m_k).$$

A representative product systems G. Gát and R. Toledo (1996)

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A representative product system is an orthonormal and complete system on $L^2(G)$.

Characteristics of the system ψ for noncommutative cases:

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The representation of G on $[0,1[$

It is similar to the dyadic group, but first we need to enumerate the elements of all groups G_k, ($k \in \mathbf{N}$) in an arbitrary way but the first is always their identity.

$$G_k \ni x \overset{\text{bijection}}{\longrightarrow} \overline{x} \in \{0, 1, \ldots, m_k - 1\}, \qquad \overline{e} = 0.$$

Fine's map and norm

With the bijection above we can introduce the Fine's map:

$$\rho(x) = (x_0, x_1, \ldots) \in G.$$

and norm:

$$|x| := \sum_{k=0}^{\infty} \frac{\overline{x_k}}{M_{k+1}} \qquad (x = (x_0, x_1, \ldots) \in G)$$

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Fine's map gives a natural relation between the new structure of $[0, 1[$ and the structure of G (Harmonic analysis).

Theorem (Toledo (2003))

Let ρ denote the Fine's map. If f is integrable on G then $f \circ \rho$ is Lebesgue integrable and

$$\int_G f \, d\mu = \int_0^1 (f \circ \rho)(x) \, dx.$$

Conversely, if g is Lebesgue integrable and f is defined by $f(x) := g(|x|)$ $(x \in G)$ then f is integrable on G and

$$\int_0^1 g(x) \, dx = \int_G f \, d\mu.$$

- The Haar measure corresponds to the Lebesgue measure.
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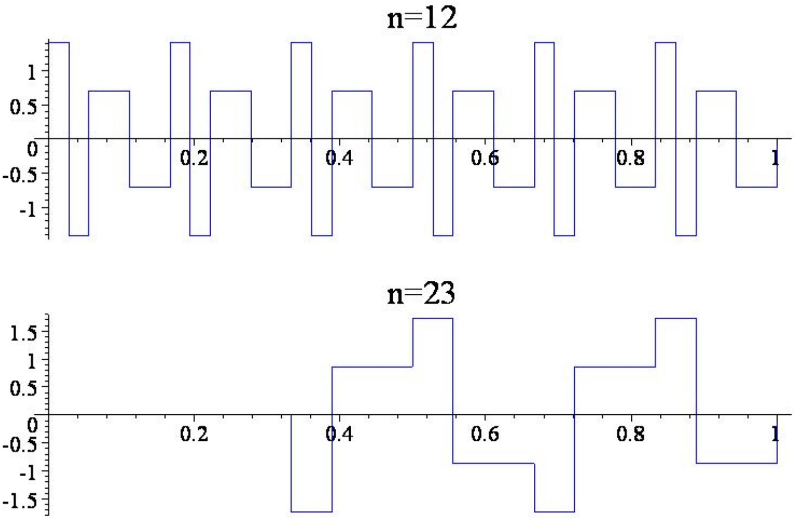
The complete product of $\mathcal{S}_3$

Plotted by Maple



The n-th partial sums of Fourier series

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k \qquad (n \in \mathbf{N}), \qquad \text{where } \widehat{f}_k := \int_G f \overline{\psi}_k d\mu.$$

Theorem (R. Toledo)

For all G groups there exists a function $f \in L^1(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^1-norm.

Divergence in L^p-norm of Fourier series

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$$\Psi_k = \prod_{i=0}^{k-1} \max_{s<m_i} \|\varphi_i^s\|_1 \|\varphi_i^s\|_\infty \qquad (k \in \mathbf{N}).$$

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If G is a bounded group with unbounded sequence Ψ, then for all $p \neq 2$, $1 < p < \infty$ there exists a function $f \in L^p(G)$ such that the sequence of partial sums $S_n f$ of the Fourier series of f does not converge to the function f in L^p-norm.

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Convergence in L^p-norm of Fejér means

The Fejér means of Fourier series

$$\sigma_n f = \frac{1}{n}\sum_{k=1}^{n-1} S_k f \qquad (n \in \mathbf{P})$$

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If G is a bounded group and $f \in L^p(G)$, $1 \le p < \infty$, then $\sigma_n f \to f$ in L^p-norm.

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$$\sigma_n^\alpha f = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f, \quad \text{where } A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!}, \quad (n \in \mathbf{P}).$$

$$\sigma_n^0 f = S_n f, \qquad \sigma_n^1 f = \sigma_n f$$

Theorem

Let G be a bounded group,

$$\alpha_0 := \limsup_{k\to\infty} \log_{m_k} \left(\max_{0\le s<m_k} \|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty \right)$$

$\alpha_0 < \alpha < 1$ *and* $f \in L^p(G)$ *for* $1 \le p < \infty$*. Then* $\sigma_n^\alpha f \to f$ *in* L^p*-norm.*

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Corollary

If G is a bounded monomial group, $0 < \alpha < 1$ *and* $f \in L^p(G)$, $1 \leq p < \infty$, *then* $\sigma_n^\alpha f \to f$ *in* L^p*-norm.*

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and $0 < \alpha < \alpha_1$*. Then there exists an* $f \in L^1(G)$ *such that* $\sigma_n^\alpha f$ *does not converge to the function f in* L^1*-norm.*

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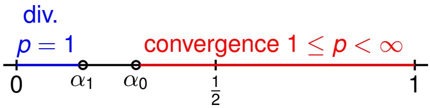
Summary of results

- *G* is bounded group
- *G* is the complete product of $\mathcal{S}_3$
- *G* is the complete product of $\mathcal{Q}_2$



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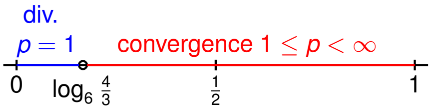

$$\alpha_1 = \alpha_0 = \log_6 \frac{4}{3}$$

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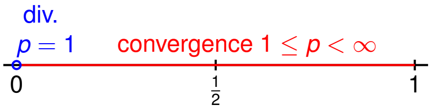

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